PRODUCTION OF TWO π MESONS IN COLLISIONS BETWEEN π MESONS OR γ QUANTA WITH NUCLEONS OR DEUTERONS

N. V. DUSHIN

Leningrad Polytechnic Institute

Submitted to JETP editor March 3, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 401-407 (August, 1958)

The production of two π mesons in collisions between π mesons or γ quanta with nucleons or deuterons is studied by a phenomenological method for small angles of emission of the π mesons produced.

1. Let us consider the reaction $\pi^+ + p \rightarrow n + \pi^+ + \pi^+$ and let us denote the amplitude of that process for the pseudopotential given in references 1 and 2 by A. We have

$$V = A\delta(\mathbf{r}_n - \mathbf{r})\tau_n. \tag{1}$$

where τ_n is an operator acting on the isotopic variable of the nucleon.

The π mesons are pseudoscalar particles. The amplitude A, corresponding to the transformation of one π meson into two π mesons in collision with nucleons, should therefore be a pseudoscalar. In the following, we shall limit ourselves to reactions in which the produced π mesons are emitted at small angles. The expression $\mathbf{A}\boldsymbol{\sigma}$, where $\mathbf{A} = a\mathbf{k}_1 + b\mathbf{k}_2 + c\mathbf{k}_0$, represents then the only pseudoscalar which can be constructed from the given $\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2$ (wave vectors of the incident and the produced mesons, respectively) and nucleon spin vector $\boldsymbol{\sigma}$. In consequence, the amplitude should be of the form

$$A = \mathbf{A}\boldsymbol{\sigma} = (a\mathbf{k}_1 + b\mathbf{k}_2 + c\mathbf{k}_0)\boldsymbol{\sigma}, \tag{2}$$

where a, b, c are invariant functions of the vectors $\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2$. The differential cross-section for the reaction $\pi^+ + p \rightarrow n + \pi^+ + \pi^+$ can be expressed in terms of the amplitude $\mathbf{A}$ of the reaction:

$$d\sigma = \frac{2\pi}{k_0/E_0}|\mathbf{A}_p|^2\delta(E_0 - E_f)\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{k}_2}{(2\pi)^3}. \tag{3}$$

where E_0 is the energy of the incident meson. The vector $\mathbf{A}$ in Eq. (3) has an index p indicating that the reaction goes on a proton.

2. Let us consider the transformation reaction of one π meson into two in collisions with deuterons. The interaction between the incident π meson and a deuteron can be described by a pseudopotential which is a sum of pseudopotentials (1):

$$V = V(1) + V(2). \tag{4}$$

where 1 and 2 denote the coordinates of the first and second nucleons in the deuteron.

The reaction $\pi^- + d \rightarrow n + n + \pi^- + \pi^+$. The pseudopotential is then

$$V = \mathbf{A}_p\sigma_1\delta(\mathbf{r}_1 - \mathbf{r})\tau_1^+ + \mathbf{A}_p\sigma_2\delta(\mathbf{r}_2 - \mathbf{r})\tau_2^+. \tag{5}$$

For the differential cross-section of the reaction $\pi^- + d \rightarrow n + n + \pi^- + \pi^+$ we obtain (after some intermediate calculations):

$$d\sigma(\mathbf{k}_1\mathbf{k}_2\mathbf{q}) = \frac{2\pi}{k_0/E_0}\frac{2}{3}|\mathbf{A}_p|^2\{2|I^-|^2 + |I^+|^2\} \times \delta(E_0 - E_f)\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{k}_2}{(2\pi)^3}\frac{d\mathbf{q}}{(2\pi)^3}. \tag{6}$$

Here $\mathbf{q}$ is the momentum of the relative motion of produced nucleons and where

$$I^{\pm} = \int \varphi_{\mathbf{q}}^{\pm}(\mathbf{r})e^{i\boldsymbol{\varkappa}\cdot\mathbf{r}}\varphi_0(r)d\mathbf{r}. \tag{7}$$

In the above integrals $\varphi_0(\mathbf{r})$ denotes the coordinate part of the deuteron wave function (we assume that the ground state of the deuteron is a S state), $\varphi_{\mathbf{q}}^{\pm}(\mathbf{r})$ are the symmetric and antisymmetric coordinate parts, respectively, of the wave function of the relative motion of two nucleons with relative momentum $\mathbf{q}$. We have, furthermore,

$$\varkappa = \tfrac{1}{2}(\mathbf{k}_0 - \mathbf{k}_1 - \mathbf{k}_2),$$
$$E_0 - E_f = \sqrt{m^2 + k_0^2} - \sqrt{m^2 + k_1^2} - \sqrt{m^2 + k_2^2} - \frac{\varkappa^2}{M} - \frac{q^2}{M} + \varepsilon = 0; \tag{8}$$

where κ^2/M and q^2/M are, respectively, the energy of motion of the center of mass of two nucleons and the energy of the relative motion of the nucleons, and ϵ is the deuteron binding energy. In Eq. (8) the difference between the mass of charged and neutral π mesons and between the mass of protons and neutrons h has been neglected.

The reaction $\pi^- + d \to p + n + \pi^- + \pi^0$. For the pseudopotential we have:

$$V = \frac{1}{2}\{\mathbf{A}_p\sigma_1(1+\tau_3^{(1)}) + \mathbf{A}_n\sigma_1(1-\tau_3^{(1)})\}\,\delta(\mathbf{r}_1-\mathbf{r}) + \frac{1}{2}\{\mathbf{A}_p\sigma_2(1+\tau_3^{(2)}) + \mathbf{A}_n\sigma_2(1-\tau_3^{(2)})\}\,\delta(\mathbf{r}_2-\mathbf{r}). \tag{9}$$

For the differential cross section of the reaction $\pi^- + d \to p + n + \pi^- + \pi^0$ we obtain:

$$d\sigma(\mathbf{k}_1\mathbf{k}_2\mathbf{q}) = \frac{2\pi}{k_0/E_0}\frac{1}{3}\{|I^-|^2(2|\mathbf{A}_p-\mathbf{A}_n|^2+|\mathbf{A}_p+\mathbf{A}_n|^2) + |I^+|^2(2|\mathbf{A}_p+\mathbf{A}_n|^2+|\mathbf{A}_p-\mathbf{A}_n|^2) \tag{10}$$
$$+ (I^{+*}I^-[2(\mathbf{A}_p+\mathbf{A}_n)^*(\mathbf{A}_p-\mathbf{A}_n)+(\mathbf{A}_p-\mathbf{A}_n)^*(\mathbf{A}_p+\mathbf{A}_n)] + \text{compl. conj.})\}\,\delta(E_0-E_f)\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{k}_2}{(2\pi)^3}\frac{d\mathbf{q}}{(2\pi)^3}.$$

The integrals $I^{\pm}$ of Eq. (10) are given by Eq. (7).

In the case when the reaction goes without deuteron disintegration, i.e., $\pi^- + d \to d + \pi^- + \pi^0$, the differential cross section is given by the expression

$$d\sigma(\mathbf{k}_1\mathbf{k}_2) = \frac{2\pi}{k_0/E_0}\frac{2}{3}|\mathbf{A}_p+\mathbf{A}_n|^2|I_0|^2\,\delta(E_0-E_f)\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{k}_2}{(2\pi)^3}, \tag{11}$$

where

$$I_0 = \int \varphi_0^2(r)\,e^{i\varkappa\mathbf{r}}\,d\mathbf{r}, \tag{12}$$

$$E_0 - E_f = \sqrt{m^2+k_0^2} - \sqrt{m^2+k_1^2} - \sqrt{m^2+k_2^2} - \varkappa^2/M. \tag{13}$$

3. Let us determine the differential cross section $d\sigma(\mathbf{k}_1\mathbf{n}_2)$, corresponding to a given momentum of one meson $\mathbf{k}_1$ and to a given angle of emission of the other $\mathbf{n}_2 = \mathbf{k}_2/k_2$ independently from the state of the nucleons produced. For that purpose we shall integrate Eq. (6) and (10) over all possible values of $\mathbf{q}$. The values of $\mathbf{q}$ are determined by the law of conservation of energy [Eq. (8)]. The integration over $d\mathbf{q}$ can be carried out easily taking into account that the integrals of $I^{\pm}$ are markedly different from zero only for a small region of the values of q permitted by the conservation law (8).[1,2]

Taking into account that the functions $\varphi_{\mathbf{q}}^{\pm}(\mathbf{r})$ form a complete system, we obtain, approximately,

$$\int |I^{\pm}|^2\frac{d\mathbf{q}}{(2\pi)^3} = \frac{1}{2}\left(1 \pm \frac{\alpha}{\varkappa}\tan^{-1}\frac{\varkappa}{\alpha}\right). \tag{14}$$

For the differential cross-section $d\sigma(\mathbf{k}_1\mathbf{n}_2)$ for the reaction $\pi + d \to n + n + \pi' + \pi''$ (two neutrons in the final state) we find

$$d\sigma(\mathbf{k}_1\mathbf{n}_2) = \frac{2\pi}{k_0/E_0}|\mathbf{A}_p|^2\left(1-\frac{\alpha}{3\varkappa}\tan^{-1}\frac{\varkappa}{\alpha}\right) \times (E_0-E_1)\sqrt{(E_0-E_1)^2-m^2}\,\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\Omega_2}{(2\pi)^3}. \tag{15}$$

For the reaction $\pi + d \to p + n + \pi' + \pi''$ (a neutron and a proton are present in the final state) we have:

$$d\sigma(\mathbf{k}_1\mathbf{n}_2) = \frac{2\pi}{k_0/E_0}\Big\{|\mathbf{A}_p|^2+|\mathbf{A}_n|^2 + \frac{\alpha}{6\varkappa}\tan^{-1}\frac{\varkappa}{\alpha}(|\mathbf{A}_p+\mathbf{A}_n|^2-|\mathbf{A}_p-\mathbf{A}_n|^2)\Big\} \tag{16}$$
$$\times (E_0-E_1)\sqrt{(E_0-E_1)^2-m^2}\,\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\Omega_2}{(2\pi)^3}.$$

A correction to the differential cross sections (15) and (16) for the existence of deuterons in the D state contributes about 10% to $d\sigma(\mathbf{k}_1\mathbf{n}_2)$ (the correction is less for $\kappa \ll \alpha$ and for large κ).

4. It is easy to show on the basis of the isotopic-invariance hypothesis that the amplitudes A_p and A_n are equal and have opposite signs near the thresholds of the reactions $\pi^- + p \to p + \pi^- + \pi^0$ and $\pi^- + n \to n + \pi^- + \pi^0$, as well as in the production of π mesons with equal momentum $\mathbf{k}_1 = \mathbf{k}_2$.

In fact, the initial state is a superposition of states with total isotopic spin T equal to ½ and 3/2. The system of two π mesons can be found in states with total isotopic spin t equal to 0, 1, and 2.

Let us introduce amplitudes A_t^T corresponding to a given value of the isotopic spin of the whole system T and a given value of the isotopic spin of the produced mesons t. For the amplitude of the process of transmutation of one π meson into two π mesons $A(m_1m_2s'; sm)$ we have the following expression

$$A(m_1m_2s';\,sm) = \sum_{T,t} A_t^T C_{1m^1/_2s}^{Tm+s}\, C_{1m_11m_2}^{tm_1+m_2}C_{tm_1+m_2{}^1/_2s'}^{Tm+s}, \tag{17}$$

where $C_{j_1m_1j_2m_2}^{jm}$ are the Clebsch-Gordan coefficients; m, m_1, and m_2 are the projections of the isotopic spin of the incident and of the produced π mesons, and s and s′ are the projections of the isotopic spin of the nucleon in the initial and final state.

The amplitudes $A(m_1m_2s'; sm)$ satisfy the following symmetry relations:

$$A(m_1m_2s';\,sm) = -A(-m_1-m_2-s';\,-s-m),$$
$$A(m_1m_2s';\,s0) = -A(m_2m_1-s';\,-s0). \tag{18}$$

Near the reaction threshold, the π mesons are produced in the s state. Since the wave function of the π meson system would be symmetric with respect to a permutation of the mesons, production of two π mesons with $t = 1$ is forbidden near the threshold of the reaction $\pi + n \to n' + \pi' + \pi''$.[3] It follows then from Eq. (17) that

$$A(m_1m_2s,\,sm) = -A(m_1m_2-s;\,-sm). \tag{19}$$

A similar symmetry relation holds, apart from the reaction threshold, when the two π mesons are produced in course of the considered reactions with identical momentum, $\mathbf{k}_1 = \mathbf{k}_2$.

Taking the above into account we find that the differential cross-section for the reaction $\pi + d \rightarrow p + n + \pi' + \pi''$ for small angles of emission of π -mesons and equal momenta $\mathbf{k}_1 = \mathbf{k}_2$ can be written in the form

$$d\sigma(\mathbf{k}_1\mathbf{k}_2\mathbf{q}) = \frac{2\pi}{k_0/E_0}\frac{4}{3}|\mathbf{A}_p|^2\{2|I^-|^2 + |I^+|^2\} \times \delta(E_0 - E_f)\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{k}_2}{(2\pi)^3}\frac{d\mathbf{q}}{(2\pi)^3}. \tag{20}$$

For the same conditions the transmutation reaction of one π meson into two without deuteron disintegration $\pi + d \rightarrow d + \pi' + \pi''$ is forbidden.

5. The reactions of photoproduction of π mesons can be treated by an analogous method, using a pseudopotential.[2]

The amplitude of photoproduction of two π mesons should be a scalar. Limiting ourselves to the reactions in which the produced π mesons are emitted at small angles, we shall construct the following scalar expression containing $\mathbf{k}_1$, $\mathbf{k}_2$, σ, and the polarization vector ϵ of the γ quantum:

$$[(a_1\mathbf{k}_1 + a_2\mathbf{k}_2) \times \varepsilon]\sigma, \tag{21}$$

where a_1 and a_2 are invariant functions of the vectors $\mathbf{k}_0$, $\mathbf{k}_1$, and $\mathbf{k}_2$. The expression (21) represents the most general scalar containing $\mathbf{k}_1$, $\mathbf{k}_2$, and σ, and linear with respect to ϵ.

The pseudopotential of photoproduction of two π mesons on a nucleon is given by the expression

$$V = [(a_1\mathbf{k}_1 + a_2\mathbf{k}_2) \times \varepsilon]\sigma\delta(\mathbf{r}_n - \mathbf{r})\tau_n. \tag{22}$$

Since Eq. (22) is analogous to Eq. (1), the formulae for differential cross sections of photoproduction can be obtained directly from the analogous expressions, (3) to (16).

For small angles of emission of π mesons, the momentum of these mesons is approximately perpendicular to the polarization vector ϵ. The cross sections of the studied processes for unpolarized nucleons and deuterons are, therefore, independent of the direction of polarization of the γ quanta.

6. We shall determine now the momentum and angle distributions of nucleons in the deuteron reactions under consideration.

In formulae (6) and (10) for the differential cross sections we eliminated momentum of the center of mass of the system consisting of two nucleons by using the law of conservation of momentum. It is convenient to make use of this law for the elimination of other variables.

We shall determine the momentum distribution function of the nucleons separately for three different specifications of the π-meson variable in the final state.

(a) Let the meson momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ be given. Out of six other variables $\mathbf{p}_1$, and $\mathbf{p}_2$, the three variables of the vector $\mathbf{p}_2$ are eliminated by the momentum-conservation law. The energy-conservation law yields:

$$\sqrt{m^2 + k_0^2} - \sqrt{m^2 + k_1^2} - \sqrt{m^2 + k_2^2} - p_1^2/M - 2\varkappa^2/M + 2\varkappa p_1/M + \varepsilon = 0. \tag{23}$$

Since $\kappa = (\mathbf{k}_0 - \mathbf{k}_1 - \mathbf{k}_2)/2 = \text{const}$, we shall choose the z axis as the direction of the vector κ and integrate, in the differential cross section, over the angle variables of the vector $\mathbf{p}_1$.

The integration over the azimuth $d\varphi_1$ is carried out from 0 to 2π. The integration over the polar angle $d\vartheta_1$ can be carried out easily thanks to the existence of the energy δ-function.

We finally obtain for the momentum distribution function $W(p_1, \mathbf{k}_1, \mathbf{k}_2)$ of the nucleons in the reaction $\pi + d \rightarrow n + n + \pi' + \pi''$

$$W(p_1, \mathbf{k}_1\mathbf{k}_2)\,dp_1\,d\mathbf{k}_1\,d\mathbf{k}_2 = \frac{|\mathbf{A}_p|^2}{3k_0/E_0}\frac{M}{(2\pi)^7\varkappa}\{2|I^-|^2 + |I^+|^2\}\,p_1\,dp_1\,d\mathbf{k}_1\,d\mathbf{k}_2. \tag{24}$$

where the integrals $I^{\pm}$ are given by the expressions:

$$I^- = -\sqrt{4\pi\alpha}\left(\frac{1}{\alpha^2 + p_1^2} - \frac{1}{\alpha^2 + 2M\Delta - p_1^2}\right), \tag{25}$$

$$I^+ = \sqrt{4\pi\alpha}\left(\frac{1}{\alpha^2 + p_1^2} + \frac{1}{\alpha^2 + 2M\Delta - p_1^2} + \frac{1 - e^{2i\delta_0}}{2\varkappa\sqrt{M\Delta - \varkappa^2}}\left[\frac{1}{2}\ln\frac{\alpha^2 + (\sqrt{M\Delta - \varkappa^2} - \varkappa)^2}{\alpha^2 + (\sqrt{M\Delta - \varkappa^2} + \varkappa)^2} + i\tan^{-1}\frac{\sqrt{M\Delta - \varkappa^2} + \varkappa}{\alpha} - i\tan^{-1}\frac{\sqrt{M\Delta - \varkappa^2} - \varkappa}{\alpha}\right]\right), \tag{26}$$

$$\Delta = \sqrt{m^2 + k_0^2} - \sqrt{m^2 + k_1^2} - \sqrt{m^2 + k_2^2}. \tag{27}$$

(b) The momentum of one meson $\mathbf{k}_1$ and the direction of emission of the other, $\mathbf{n}_2 = \mathbf{k}_2/k_2$, are given. The momentum-conservation law yields:

$$\mathbf{k}_0 - \mathbf{k}_1 - \mathbf{n}_2\left[\left(E_0 - E_1 - \frac{p_1^2 + p_2^2}{2M} + \varepsilon\right)^2 - m^2\right]^{1/2} - \mathbf{p}_1 - \mathbf{p}_2 = 0. \tag{28}$$

Integrating over the angle variables ϑ_1, φ_1, ϑ_2, and φ_2 of the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ we obtain the following expression for the distribution function $W(p_1p_2, \mathbf{k}_1\mathbf{n}_2)$:

$$W(p_1 p_2, \mathbf{k}_1 \mathbf{n}_2)\, dp_1\, dp_2\, d\mathbf{k}_1\, d\Omega_{2k} = \frac{|\mathbf{A}_p|^2}{3k_0/E_0} \frac{\Delta\sqrt{\Delta^2 - m^2}}{(2\pi)^7 \varkappa} \times \{2|I^-|^2 + |I^+|^2\}\, p_1 p_2\, dp_1\, dp_2\, d\mathbf{k}_1\, d\Omega_{2k}. \quad (29)$$

where I^- and I^+ are expressed in terms of the variables $\mathbf{p}_1$ and $\mathbf{p}_2$:

$$I^- = -\sqrt{4\pi\alpha}\left(\frac{1}{\alpha^2 + p_1^2} - \frac{1}{\alpha^2 + p_2^2}\right), \quad (30)$$

$$I^+ = \sqrt{4\pi\alpha}\left(\frac{1}{\alpha^2 + p_1^2} + \frac{1}{\alpha^2 + p_2^2} + \frac{1 - e^{2i\delta_0}}{2q\varkappa}\left[\frac{1}{2}\ln\frac{\alpha^2 + (q-\varkappa)^2}{\alpha^2 + (q+\varkappa)^2} + i\tan^{-1}\frac{q+\varkappa}{\alpha} - i\tan^{-1}\frac{q-\varkappa}{\alpha}\right]\right), \quad (31)$$

where

$$q^2 + \varkappa^2 = {}^1/_2(p_1^2 + p_2^2), \quad \varkappa = {}^1/_2\left|\mathbf{k}_0 - \mathbf{k}_1 - \mathbf{n}_2\sqrt{\Delta^2 - m^2}\right|,$$

$$\Delta = E_0 - E_1 - (p_1^2 + p_2^2)/2M + \varepsilon \approx E_0 - E_1.$$

(c) Let the momentum $\mathbf{k}_1$ of one meson be given. It follows from the energy-conversation law that

$$\sqrt{m^2 + k_0^2} - \sqrt{m^2 + k_1^2} - \sqrt{m^2 + (\mathbf{k}_0 - \mathbf{k}_1 - \mathbf{p}_1 - \mathbf{p}_2)^2} - (p_1^2 + p_2^2)/2M + \varepsilon = 0. \quad (32)$$

Taking into account that we deal with reactions in which fast mesons are emitted at small angles, we have, approximately

$$\sqrt{m^2 + k_0^2} - \sqrt{m^2 + k_1^2} - \sqrt{m^2 + (\mathbf{k}_0 - \mathbf{k}_1)^2 - 2(\mathbf{k}_0 - \mathbf{k}_1)\cdot(\mathbf{p}_1 + \mathbf{p}_2)} = 0. \quad (33)$$

The integration over the angular variables of the vector $\mathbf{p}_1$ can be carried out easily thanks to the existence of the energy δ-function. Integrating over the angular variables of the vector $\mathbf{p}_2$ we take it into account that the integrals $I^\pm$ depend on these variables through the variables q and κ. Since we have

$$q^2 + \varkappa^2 = {}^1/_2(p_1^2 + p_2^2),$$

$$\varkappa = (\mathbf{k}_0 - \mathbf{k}_1)\cdot(\mathbf{p}_1 + \mathbf{p}_2)/2|\mathbf{k}_0 - \mathbf{k}_1| + O(p^2/k^2),$$

then the integrals $I^\pm$ contain the variables ϑ_1 and ϑ_2 in the form $p_1\cos\vartheta_1 + p_2\cos\vartheta_2$ only.

The result of integration of the differential cross-section over $d\vartheta_1$ depends therefore only on p_1 and p_2. The limits of integration over $d\vartheta_2$ are determined by the energy conservation law (33).

Finally, we obtain for the distribution function $W(p_1 p_2, \mathbf{k}_1)$:

$$W(p_1 p_2, \mathbf{k}_1)\, dp_1\, dp_2\, d\mathbf{k}_1 = \frac{2|\mathbf{A}_p|^2}{3k_0/E_0} \frac{E_0 - E_1}{(2\pi)^6|\mathbf{k}_0 - \mathbf{k}_1|} \times \{2|I^-|^2 + |I^+|^2\}(p_1 + p_2 - 2\varkappa)\, p_1 p_2\, dp_1\, dp_2\, d\mathbf{k}_1. \quad (34)$$

In Eq. (34), $I^\pm$ have their previous meanings [cf. Eqs. (30) and (31)] and

$$\varkappa = \frac{1}{4}\left(|\mathbf{k}_0 - \mathbf{k}_1| - \frac{(E_0 - E_1)^2 - m^2}{|\mathbf{k}_0 - \mathbf{k}_1|}\right).$$

We consider now the angular distribution of nucleons in two cases. In the first we assume that $\mathbf{k}_1$ and $\mathbf{k}_2$ are given, and denote the corresponding distribution function by $W(\vartheta_1, \mathbf{k}_1, \mathbf{k}_2)$. In the second, we assume that $\mathbf{k}_1$ and $\mathbf{n}_2$ are given, and denote the corresponding distribution function by $W(\vartheta_1\vartheta_2, \mathbf{k}_1\mathbf{n}_2)$.

(a) Given $\mathbf{k}_1$ and $\mathbf{k}_2$, we express Eq. (6) in terms of $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{p}_1$. Integrating over dp_1 and $d\varphi_1$, we obtain the distribution function:

$$W(\vartheta_1, \mathbf{k}_1\mathbf{k}_2)\sin\vartheta_1\, d\vartheta_1\, d\mathbf{k}_1\, d\mathbf{k}_2 = \frac{|\mathbf{A}_p|^2}{3k_0/E_0} \frac{Mp_1^2}{(2\pi)^7(\varkappa^2\cos^2\vartheta_1 - 2\varkappa^2 + M\Delta)^{1/2}} \times \{2|I^-|^2 + |I^+|^2\}\sin\vartheta_1\, d\vartheta_1\, d\mathbf{k}_1\, d\mathbf{k}_2. \quad (35)$$

where $I^\pm$ are given, as before, by Eqs. (25) and (26), and are expressed in terms of ϑ_1. It follows from Eq. (23) that

$$p_1 = \varkappa\cos\vartheta_1 + \sqrt{\varkappa^2\cos^2\vartheta_1 - 2\varkappa^2 + M\Delta}. \quad (36)$$

(c) Given $\mathbf{k}_1$ and $\mathbf{n}_2$, we choose the z axis in direction of the vector $\mathbf{n}_2$. Expressing $\mathbf{p}_1$, $\mathbf{p}_2$, and the azimuth φ_2 of the vector $\mathbf{p}_2$ through the variables k_2, ϑ_1, ϑ_2, φ_1 (using the momentum-conservation law) and integrating over dk_2 we find:

$$W(\vartheta_1\vartheta_2\varphi_1, \mathbf{k}_1\mathbf{n}_2)\sin\vartheta_1\sin\vartheta_2\, d\vartheta_1\, d\vartheta_2\, d\varphi_1\, d\mathbf{k}_1\, d\Omega_{2k} = \frac{2|\mathbf{A}_p|^2}{3k_0/E_0}\frac{(E_0 - E_1)\sqrt{(E_0 - E_1)^2 - m^2}}{(2\pi)^8} \times \frac{p_1^4\sin\varphi_1\sin\vartheta_1/\sin\varphi_2\sin^3\vartheta_2}{|[\mathbf{n}_2\times(\mathbf{k}_0 - \mathbf{k}_1)]|\cos\varphi_2\cot\vartheta_2 - \mathbf{n}_2(\mathbf{k}_0 - \mathbf{k}_1)} \times \{2|I^-|^2 + |I^+|^2\}\sin\vartheta_1\sin\vartheta_2\, d\vartheta_1\, d\vartheta_2\, d\varphi_1\, d\mathbf{k}_1\, d\Omega_{2k}. \quad (37)$$

where $I^\pm$ are given by Eqs. (30) and (31) and are expressed in terms of ϑ_1, ϑ_2, and φ_1:

$$p_2 = -p_1\sin\vartheta_1\sin\varphi_1/\sin\vartheta_2\sin\varphi_2, \quad (38)$$

$$\mathbf{n}_2(\mathbf{k}_0 - \mathbf{k}_1) - p_1\cos\vartheta_1 + p_1\sin\vartheta_1\sin\varphi_1\cos\vartheta_2/\sin\vartheta_2\sin\varphi_2 = 0. \quad (39)$$

The angle φ_2 is determined by the equation

$$\cot\varphi_2 = \cot\varphi_1 - |[\mathbf{n}_2\times(\mathbf{k}_0 - \mathbf{k}_1)]|/p_1\sin\vartheta_1\sin\varphi_1. \quad (40)$$

For $[\mathbf{n}_2\times(\mathbf{k}_0 - \mathbf{k}_1)] = 0$, we have $\varphi_2 = \varphi_1 \pm \pi$. For this condition we find the following distribution function:

$$W(\vartheta_1 \vartheta_2, \mathbf{k}_1 \mathbf{n}_2) \sin\vartheta_1 \sin\vartheta_2 \, d\vartheta_1 \, d\vartheta_2 \, d\mathbf{k}_1 \, d\Omega_{2k} = \frac{2 \, |\mathbf{A}_p|^2}{3k_0 / E_0} \frac{(E_0 - E_1) \, |\mathbf{k}_0 - \mathbf{k}_1|^3 \sqrt{(E_0 - E_1)^2 - m^2}}{(2\pi)^7} \{2 \, |I^-|^2 + |I^+|^2\}$$

$$\times \frac{\sin \vartheta_1 \sin \vartheta_2}{(\cos \vartheta_1 + \cos \vartheta_2)^4} \sin \vartheta_1 \sin \vartheta_2 \, d\vartheta_1 \, d\vartheta_2 \, d\mathbf{k}_1 \, d\Omega_{2k}, \quad (41)$$

$$p_{1,2} = |\mathbf{k}_0 - \mathbf{k}_1| \sin \vartheta_{2,1} / (\cos \vartheta_1 + \cos \vartheta_2). \quad (42)$$

The author would like to thank I. M. Shmushkevich for suggesting the present work and for advice in the course of its execution.

[1] I. Ia. Pomerȧnchuk, J. Exptl. Theoret. Phys. (U.S.S.R.) **21**, 1113 (1951); **23**, 129 (1952); V. B. Berestetskii and I. Ia. Pomeranchuk, Dokl. Akad. Nauk SSSR **81**, 1019 (1951).

[2] B. L. Ioffe and I. M. Shmushkevich, Dokl. Akad. Nauk SSSR **82**, 869 (1952).

[3] N. V. Dushin, Dokl. Akad. Nauk SSSR **106**, 977 (1956), Soviet Phys. "Doklady" **1**, 128 (1956).

Translated by H. Kasha
72

SOVIET PHYSICS JETP VOLUME 35(8), NUMBER 2 FEBRUARY, 1959

PHENOMENOLOGICAL THEORY OF SUPERFLUIDITY NEAR THE λ POINT

L. P. PITAEVSKII

Institute for Physical Problems, Academy of Sciences, U.S.S.R.

Submitted to JETP editor, March 12, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 408-415 (August, 1958)

A complete set of phenomenological equations is derived to describe the behavior of superfluid helium near the λ point. The normal part of the liquid is described in the usual way, while the superfluid part is described by an "effective" wave function.

IN the present work, equations obtained earlier by V. L. Ginzburg and the author to describe the behavior of helium II in the immediate vicinity of the λ point[1] are extended to include the nonstationary case. In contrast with the ordinary hydrodynamics of helium II, in the scheme considered here the density of the superfluid part ρ_s is not assumed a given function of p and T, but is determined from the equations; these equations characterize the approach of ρ_s to its equilibrium value. As in reference 1, the superfluid part of the liquid is described by a complex function $\psi(x, y, z, t) = \eta e^{i\varphi}$ which is so defined that

$$\rho_s = m |\psi|^2, \quad \mathbf{v}_s = \frac{\hbar}{m} \nabla\varphi \quad (1)$$

(m is the mass of the helium atom). This function is introduced to take account of the quantum nature of the effect; its role in this scheme is the same as that of the expansion parameter in the usual theory of second-order phase changes.[2] The helium state is characterized by the density of the liquid ρ, the velocity of the normal part $\mathbf{v}_n$, and the entropy per unit volume S in addition to ψ.

1. BASIC EQUATIONS

In reference 1 an equation was obtained to determine the equilibrium values of ψ for $\mathbf{v}_n = 0$. Here, we first examine the equilibrium condition for the case $\mathbf{v}_n \neq 0$. At low values of the velocities $\mathbf{v}_n$ and $\mathbf{v}_s$, the energy per unit volume of the liquid can be written in the form

$$E = (\rho - m|\psi|^2) \frac{v_n^2}{2} + \frac{\hbar^2}{2m} |\nabla\psi|^2 + \varepsilon(\rho, S, |\psi|^2). \quad (1.1)$$

This expression is the first term in the energy expansion in terms of the velocity $\mathbf{v}_n$ and the