A SUPERCONDUCTOR IN A HIGH FREQUENCY FIELD

A. A. ABRIKOSOV, L. P. GOR'KOV, and I. M. KHALATNIKOV

Institute for Physical Problems, Academy of Sciences, U.S.S.R.

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We derive an equation describing the behavior of superconductors in a high frequency field. With the aid of this equation the frequency and temperature dependence of the impedance of a bulk superconductor have been evaluated.

As is well known Bardeen, Cooper, and Schrieffer[1] have recently constructed a microscopic theory of superconductivity in which they succeeded in explaining a whole number of properties of superconductors. In particular, in that paper they considered the behavior of superconductors in a constant weak field and obtained a new equation expressing the connection of the current with the field, replacing the equation of the phenomenological theory of F. and H. London. This equation turned out to be non-local; with its help the problem of the penetration of a weak static field into a bulk superconductor was solved in reference 1 and the dependence of the penetration depth on the temperature was found.

In the present paper we consider the behavior of a superconductor in a high frequency field. The calculation given below shows that in an alternating field the character of the connection between the current and the vector potential changes, and depends in an essential way on the frequency. For non-zero temperatures or a sufficiently high frequency this equation, written down for the Fourier components of the current, contains imaginary components, which are connected with the absorption of radiation in the superconductor. The impedance of the bulk superconductor is determined with the aid of the equation obtained.

1. THE EQUATION FOR THE CURRENT IN THE SUPERCONDUCTOR

In the non-stationary problem under consideration, it is most convenient to start directly from the expression for the current operator in second quantization:

$$\hat{\mathbf{j}}(x) = -\frac{ie}{2m}\{\tilde{\psi}^+(x)\nabla\tilde{\psi}(x) - \nabla\tilde{\psi}^+(x)\tilde{\psi}(x)\} - \frac{e^2}{mc}\tilde{\psi}^+(x)\tilde{\psi}(x)\mathbf{A}(x) \qquad (1)$$

The operators $\tilde{\psi}^+$ and $\tilde{\psi}$ are here written down in the Heisenberg representation and depend on the field $\mathbf{A}$. This dependence has the usual form

$$\tilde{\psi}(x) = \hat{S}^{-1}(t)\psi(x)\hat{S}(t), \qquad (2)$$

where $\psi(x)$ is the second quantization operator in the absence of the field, and $\hat{S}(t)$ the S-matrix satisfying the equation

$$i\frac{\partial}{\partial t}\hat{S}(t) = -\frac{1}{c}\left(\int \hat{j}(x)\mathbf{A}(x)\,d^3x\right)\hat{S}(t). \qquad (3)$$

The field $\mathbf{A}(x)$ is, as usual, assumed to be adiabatically switched on at $t = -\infty$. The value of the current in the superconductor at a given point and at a given instant can clearly be obtained by taking the average of the operator (1) over the Heisenberg state of the system coinciding with the state of the system in the absence of the field:

$$\mathbf{j}(x) = -\frac{ie}{2m}\langle\tilde{\psi}^+\nabla\tilde{\psi} - \nabla\tilde{\psi}^+\tilde{\psi}\rangle - \frac{e^2}{mc}\langle\tilde{\psi}^+\tilde{\psi}\rangle\mathbf{A}(x).$$

In a weak field it is sufficient to perform the calculations up to terms linear in $\mathbf{A}$. We can therefore in the second term at once put $\tilde{\psi} = \psi$; this is the usual "London" term

$$-(e^2N/mc)\mathbf{A}(x). \qquad (4)$$

In the first term it is necessary to expand the operators $\tilde{\psi}$ and $\tilde{\psi}^+$ according to (2) and (3) up to terms of the first order in $\mathbf{A}$:

$$\tilde{\psi}(x) = \psi(x) - \frac{i}{c}\int_{-\infty}^{t}[\hat{j}_\alpha(y), \psi(y)]A_\alpha(y)\,d^4y \qquad (5)$$

with a similar formula for $\tilde{\psi}^+$. It is sufficient in turn to retain in formula (5) for $\hat{\mathbf{j}}(y)$ only the first part of expression (1)

$$\hat{\mathbf{j}}_1(x) = -\frac{ie}{2m}(\nabla_x - \nabla_{x'})\psi^+(x')\psi(x)\Big|_{x'\to x}. \qquad (6)$$

After substituting (5) into (6) we obtain the term in $\mathbf{j}_1(x) \equiv \langle \hat{\mathbf{j}}_1(x) \rangle$ which is linear in the field:

$$\mathbf{j}_1(x) = \frac{ie^2}{4m^2c}(\nabla_x - \nabla_{x'}) \int_{-\infty}^{t} (\mathbf{A}(y)(\nabla_y - \nabla_{y'})) \langle \{\psi^+(y')\psi(y)\psi^+(x')\psi(x) - \psi^+(x')\psi(x)\psi^+(y')\psi(y)\}\rangle d^4y. \quad (7)$$

Here and henceforth the primed coordinates differ only in the space variables, which must be put equal after performing the differentiation.

The evaluation of the averages of the products of the four ψ and ψ^+ operators in (7) can most conveniently be performed by using the method developed by one of the authors in reference 2. These averages are expressed in terms of pair averages:

$$G_{\alpha\beta}(x - x') = -i\langle T(\psi_\alpha(x)\psi_\beta^+(x'))\rangle; \quad F_{\alpha\beta}^+(x - x') = \langle T(\psi_\alpha^+(x)\psi_\beta^+(x'))\rangle; \quad F_{\alpha\beta}(x - x') = \langle T(\psi_\alpha(x)\psi_\beta(x')\rangle.$$

The dependence of these quantities on the spinor indices is as follows:

$$G_{\alpha\beta}(x - x') = \delta_{\alpha\beta}G(x - x'); \qquad F_{\alpha\beta}^+(x - x') = -F_{\alpha\beta}(x - x') = F(x - x')\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{\alpha\beta}.$$

Performing the averaging in (7) we get

$$\mathbf{j}_1(x) = \frac{ie^2}{2m^2c}(\hat{\mathbf{p}}_x - \hat{\mathbf{p}}_{x'}) \int_{-\infty}^{t} (\mathbf{A}(y)(\hat{\mathbf{p}}_y - \hat{\mathbf{p}}_{y'})) \{[G(y - x')G(x - y') - F(x' - y')F(x - y)] - \mathrm{C.C.}\} d^4y. \quad (8)$$

Going over to Fourier components we find:

$$\mathbf{j}_1(\mathbf{k}, \omega) = \frac{2e^2}{m^2c}\int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \mathbf{p}(\mathbf{A}(\mathbf{k}, \omega)\mathbf{p})$$

$$\times \left[G\left(\mathbf{p} - \frac{\mathbf{k}}{2}, \omega_1\right)G\left(\mathbf{p} + \frac{\mathbf{k}}{2}, \omega_2\right) - F\left(\mathbf{p} - \frac{\mathbf{k}}{2}, \omega_1\right)F\left(\mathbf{p} + \frac{\mathbf{k}}{2}, \omega_2\right) - \mathrm{C.C.}\right]\frac{1}{\omega_1 - \omega_2 - \omega - i\delta}.$$

The Fourier components of the functions $G(x - x')$ and $F(x - x')$ were found in reference 2. It is convenient to write them in the form

$$G(\mathbf{p}, \omega) = \frac{u_p^2(1 - n_p)}{\omega - \varepsilon_p + i\delta} + \frac{v_p^2 n_p}{\omega + \varepsilon_p + i\delta} + \frac{v_p^2(1 - n_p)}{\omega + \varepsilon_p - i\delta} + \frac{u_p^2 n_p}{\omega - \varepsilon_p - i\delta};$$

$$F(\mathbf{p}, \omega) = iu_p v_p\left[\frac{1 - n_p}{\omega - \varepsilon_p + i\delta} - \frac{n_p}{\omega + \varepsilon_p + i\delta} - \frac{1 - n_p}{\omega + \varepsilon_p - i\delta} + \frac{n_p}{\omega - \varepsilon_p - i\delta}\right], \quad (9)$$

where

$$u_p^2 = {}^1\!/_2(1 + \xi_p/\varepsilon_p); \quad v_p^2 = {}^1\!/_2(1 - \xi_p/\varepsilon_p); \quad \xi_p = v(p - p_0); \quad \varepsilon_p = \sqrt{\xi_p^2 + \Delta^2}; \quad n_p = (e^{\varepsilon_p/T} + 1)^{-1}.$$

The quantity Δ depends on the temperature in the way found in the paper by Bardeen et al.

Substituting equations (9) into the expression for the current and integrating we find for the Fourier components of the current:

$$\mathbf{j}(\mathbf{k}, \omega) = \frac{2e^2}{(2\pi)^3 m^2c}\int d^3p\, \mathbf{p}(\mathbf{p}\mathbf{A}(\mathbf{k}, \omega)) \left[(v_1u_2 - u_1v_2)\left(\frac{v_1u_2}{\varepsilon_1 + \varepsilon_2 + \omega + i\delta} + \frac{u_1v_2}{\varepsilon_1 + \varepsilon_2 - \omega - i\delta}\right)\right.$$

$$\left.\times(1 - n_1 - n_2) + (u_1u_2 + v_1v_2)\left(\frac{u_1u_2}{\varepsilon_2 - \varepsilon_1 + \omega + i\delta} + \frac{v_1v_2}{\varepsilon_2 - \varepsilon_1 - \omega - i\delta}\right)(n_1 - n_2)\right] - \frac{e^2}{mc}N\mathbf{A}(\mathbf{k}, \omega), \quad (10)$$

when the index 1 corresponds to the momentum $\mathbf{p} + \frac{1}{2}\mathbf{k}$, and the index 2 to the momentum $\mathbf{p} - \frac{1}{2}\mathbf{k}$. Substituting into (7), assuming the vector $\mathbf{A}$ to lie in the surface plane of the specimen and performing the integration over φ we get

$$\mathbf{j}(\mathbf{k}, \omega) = \frac{3e^2N\mathbf{A}(\mathbf{k}, \omega)}{32mc}\int_{-1}^{1} d\cos\theta\sin^2\theta \int_{-\xi_0}^{\xi_0} d\xi \left[\left(1 - \frac{\xi_1\xi_2 + \Delta^2}{\varepsilon_1\varepsilon_2}\right)\left(\tanh\frac{\varepsilon_1}{2T} + \tanh\frac{\varepsilon_2}{2T}\right)\left(\frac{1}{\varepsilon_1 + \varepsilon_2 + \omega + i\delta} + \frac{1}{\varepsilon_1 + \varepsilon_2 - \omega - i\delta}\right)\right.$$

$$\left. + \left(1 + \frac{\xi_1\xi_2 + \Delta^2}{\varepsilon_1\varepsilon_2}\right)\left(\tanh\frac{\varepsilon_1}{2T} - \tanh\frac{\varepsilon_2}{2T}\right)\left(\frac{1}{\varepsilon_1 - \varepsilon_2 + \omega + i\delta} + \frac{1}{\varepsilon_1 - \varepsilon_2 - \omega - i\delta}\right)\right] - \frac{e^2}{mc}N\mathbf{A}(\mathbf{k}, \omega). \quad (11)$$

In this expression we used the fact that terms containing one ξ give zero on integration. Indeed, we have

$$\xi_{1,2} = \xi \pm {}^1\!/_2\, vk \cos\theta, \tag{12}$$

and the integrations over $\cos\theta$ and ξ are performed over a symmetric domain. The quantity ξ_0 determines the upper limit of integration over ξ.

2. PIPPARD'S LIMITING CASE

Further evaluations are impossible to perform without more concrete assumptions about the magnitude of the parameters important for the problem considered. Of the greatest importance for us is the fact that, apparently, the penetration depth of the field for the majority of superconductors is much less than the quantity v/Δ, i.e., $vk \gg \Delta$ (See reference 1).* This inequality is violated only in the neighborhood of the transition point T_c, where the region with $vk \ll \Delta$ occurs. As was shown in reference 1, the basic domain of electrodynamics is founded upon the equation proposed earlier by Pippard.[3] We shall call this region the Pippard region. In the second region the applicability of the London electrodynamics[4] is conserved (London region).†

In the case of a variable field the London region gets narrowed. This is clear, though, from the fact that a normal metal is a Pippard one, since $\Delta = 0$ for it, whence follows that the London region cannot continue up to the transition point itself. Leaving a detailed analysis of this problem until the next section, we shall consider here how Eq. (11) simplifies in the Pippard region. We shall at the same time also assume $vk \gg \omega$. As we shall see in the following, this is always correct in the range of frequencies of most interest.

Equation (11) can be transformed in the following way. In view of the fact that ω only enters into denominators such as $\epsilon_1 + \epsilon_2 + \omega + i\delta$, we can subtract from the coefficient for $A(k, \omega)$ which we denote as $-(3e^2N\Delta/4mcvk)\, Q(\omega)$ its value in the static case. The remainder will be the integral over ξ and $\cos\theta$ where the important region of integration is the region $\xi \ll vk$, $\cos\theta \ll 1$. We introduce now as new variables ξ_1 and ξ_2. If we put $vk \gg T$ (this will be shown below) we can assume that the integrations over ξ_1 and ξ_2 proceed independently with limits from $-\infty$ to $+\infty$. Terms with the product $\xi_1\xi_2$ will thus drop out of the integral. Apart from that, the factor $\sin^2\theta$ can be replaced by unity. Finally we go over to the variables ϵ_1/Δ and ϵ_2/Δ (denoting them again by ϵ_1 and ϵ_2). As a result we get:

$$\begin{aligned} Q(\omega) - Q(0) = \frac{1}{2}\int_1^\infty \frac{d\varepsilon_1}{\sqrt{\varepsilon_1^2-1}} \int_1^\infty \frac{d\varepsilon_2}{\sqrt{\varepsilon_2^2-1}} \Big[(1-\varepsilon_1\varepsilon_2)\left(\tanh\frac{\varepsilon_1\Delta}{2T} + \tanh\frac{\varepsilon_2\Delta}{2T}\right)\Big(\frac{1}{\varepsilon_1+\varepsilon_2+\omega/\Delta+i\delta} \\ + \frac{1}{\varepsilon_1+\varepsilon_2-\omega/\Delta-i\delta} - \frac{2}{\varepsilon_1+\varepsilon_2}\Big) - (1+\varepsilon_1\varepsilon_2)\left(\tanh\frac{\varepsilon_1\Delta}{2T} - \tanh\frac{\varepsilon_2\Delta}{2T}\right)\left(\frac{1}{\varepsilon_1-\varepsilon_2+\omega/\Delta+i\delta} + \frac{1}{\varepsilon_1-\varepsilon_2-\omega/\Delta-i\delta} - \frac{2}{\varepsilon_1-\varepsilon_2}\right)\Big]. \end{aligned} \tag{13}$$

To fix ideas we shall now assume $\omega > 0$ and isolate the imaginary part from the integral (13). By means of slight transformations one can convince oneself that the real part of expression (13) is always of the same form as at $T = 0$, but with Δ depending on the temperature. The final expression is of the form

$$\begin{aligned} Q(\omega) - Q(0) &= \int_1^\infty \frac{d\varepsilon_1}{\sqrt{\varepsilon_1^2-1}} \int_1^\infty \frac{d\varepsilon_2}{\sqrt{\varepsilon_2^2-1}} \left[(1-\varepsilon_1\varepsilon_2)\left(\frac{1}{\varepsilon_1+\varepsilon_2+\omega/\Delta} + \frac{1}{\varepsilon_1+\varepsilon_2-\omega/\Delta} - \frac{2}{\varepsilon_1+\varepsilon_2}\right)\right] \\ &- \frac{i\pi}{2}\theta\left(\frac{\omega}{2\Delta}-1\right)\int_1^{\omega/\Delta-1} d\varepsilon\, \frac{[\varepsilon(\omega/\Delta-\varepsilon)-1]\,[\tanh(\omega/2T-\varepsilon\Delta/2T)+\tanh(\varepsilon\Delta/2T)]}{\sqrt{\varepsilon^2-1}\sqrt{(\omega/\Delta-\varepsilon)^2-1}} \\ &- i\pi\int_1^\infty d\varepsilon\, \frac{[\varepsilon(\omega/\Delta+\varepsilon)+1]\,[\tanh(\omega/2T+\varepsilon\Delta/2T)-\tanh(\varepsilon\Delta/2T)]}{\sqrt{\varepsilon^2-1}\sqrt{(\varepsilon+\omega/\Delta)^2-1}}, \qquad \text{where } \theta(x) = \begin{cases}1 & x>0\\ 0 & x<0.\end{cases} \end{aligned} \tag{14}$$

*Strictly speaking, the strong inequality $vk \gg \Delta$ is not valid. But the applicability of the formulae obtained in this limiting case is provided by the insertion of significant numerical factors.

†Such a family of superconductors we shall call Pippard superconductors. The opposite case (which can be called the London case) when the superconductor for $\omega = 0$, is described by the equations of F and H. London in the whole temperature range is, apparently, less common, and we shall not consider it.

One integration in the real part of $Q(\omega)$ can be performed. After this there remains

$$\operatorname{Re}[Q(\omega)-Q(0)]=\mathcal{J}(\omega/\Delta)+\mathcal{J}(-\omega/\Delta)-2\mathcal{J}(0), \tag{15}$$

where

$$\mathcal{J}\left(\frac{\omega}{\Delta}\right)=\int_1^\infty \frac{d\varepsilon\,[\varepsilon(\varepsilon+\omega/\Delta)+1]}{\sqrt{\varepsilon^2-1}\sqrt{(\varepsilon+\omega/\Delta)^2-1}}\cosh^{-1}\left(\varepsilon+\frac{\omega}{\Delta}\right). \tag{16}$$

Although each of the integrals in (15) diverges, together they give a finite expression. In the integral $\mathcal{J}(-\omega/\Delta)$ in the region where $1>\epsilon-\omega/\Delta>0$, we must replace

$$\cosh^{-1}\left(\varepsilon-\frac{\omega}{\Delta}\right)\Big/\sqrt{\left(\varepsilon-\frac{\omega}{\Delta}\right)^2-1}\rightarrow\cos^{-1}\left(\varepsilon-\frac{\omega}{\Delta}\right)\Big/\sqrt{1-\left(\varepsilon-\frac{\omega}{\Delta}\right)^2},$$

and in the region where $\epsilon-\omega/\Delta$ is negative we must take $-\arccos(\omega/\Delta-\epsilon)$ or, respectively, $\frac{1}{2}\pi+\sin^{-1}(\omega/\Delta-\epsilon)$. The limiting value of this expression in the region of small frequencies is of the form

$$\operatorname{Re}[Q(\omega)-Q(0)]\approx(5\pi^2/12)(\omega/\Delta)^2. \tag{17}$$

In the region $\omega/\Delta\gg1$ we get

$$\operatorname{Re}[Q(\omega)-Q(0)]=2\ln(2\omega/\Delta)-\pi^2. \tag{18}$$

An expression for $Q(0)$ was found in Bardeen, Cooper, and Schrieffer's paper.[1] It is of the form

$$Q(0)=\pi^2\tanh(\Delta/2T). \tag{19}$$

The integration (16) can be carried out in the case $\omega=2\Delta$. In view of the fact that according to (14) the imaginary part of $Q(\omega)$ is equal to zero for $\omega<2\Delta$, we get

$$Q(2\Delta)-Q(0)=2\pi+2\sqrt{3}+4\ln(\sqrt{3}+1)$$
$$-2\ln2-\pi^2\approx2.5.$$

Hence it is clear that for $T=0$ the magnitude of the current, and consequently the penetration depth which is proportional to $Q(\omega)^{1/3}$ (see Sec. 4), changes little when the frequency is changed from 0 to 2Δ:

$$\delta(2\Delta)/\delta(0)|_{T=0}\approx0.93.$$

The imaginary part of formula (13) depends in an essential way on the relations between Δ, ω, and T. We consider first of all the case $T=0$. Then there remains only the first term of the imaginary part in equation (14) (we shall call it the "pair" term) which is different from zero only for $\omega>2\Delta$. Physically this is completely clear. As is well known, the imaginary part of the current determines the absorption and in the absence of an excitation such an absorption can only take place thanks to the destruction of a pair. The quantity 2Δ determines the energy necessary for this, when the statistics are taken into account.

Performing some simple transformations of the integral we find

$$\operatorname{Im}Q(\omega)=-2\pi\theta\left(\frac{\omega}{2\Delta}-1\right)$$
$$\times\int_0^{\omega/2\Delta-1}d\varepsilon\,\frac{[(\omega/2\Delta)^2-\varepsilon^2-1]}{\{[(\omega/2\Delta+\varepsilon)^2-1][(\omega/2\Delta-\varepsilon)^2-1]\}^{1/2}}. \tag{20}$$

This integral can be reduced to an elliptical one. Near the threshold, i.e., for $\omega/2\Delta-1\ll1$ it has the form

$$\operatorname{Im}Q(\omega)=-\pi^2(\omega/2\Delta-1). \tag{21}$$

In the limiting case of large frequencies $\omega\gg\Delta$ it becomes equal to

$$\operatorname{Im}Q(\omega)=-\pi\omega/\Delta. \tag{22}$$

In the case of temperatures different from zero, of most interest are some limiting cases, as we shall see in the next section.

(a) The case $T\sim\omega\ll\Delta$. Here, of course, only the second term of the imaginary part of Eq. (14) (which we shall call the "electron" term) will take part. Transforming it, we find

$$\operatorname{Im}Q(\omega)=-4\pi\sinh\frac{\omega}{2T}K_0\left(\frac{\omega}{2T}\right)e^{-\Delta/T}, \tag{23}$$

where $K_0(x)$ is a Hankel function of an imaginary argument. In particular, we have for $T\ll\omega\ll\Delta$

$$\operatorname{Im}Q(\omega)=-2\pi\sqrt{\pi T/\omega}\,e^{-\Delta/T}, \tag{24}$$

and for $\omega\ll T\ll\Delta$

$$\operatorname{Im}Q(\omega)=-2\pi\frac{\omega}{T}\ln\frac{4T}{\gamma\omega}e^{-\frac{\Delta}{T}};\quad(\gamma=e^C\approx1.78). \tag{25}$$

(b) The case $\omega\ll T\sim\Delta$. Again only the electron term takes part. By means of a series of transformations we find

$$\operatorname{Im}Q(\omega)=-\pi\left[\frac{\omega}{T}\cosh^{-2}\frac{\Delta}{2T}\ln2\sqrt{\frac{2\Delta}{\omega}}\right.$$
$$\left.+\frac{\omega}{\Delta}\left(1-\tanh\frac{\Delta}{2T}\right)-2\frac{\omega}{T}P\left(\frac{\Delta}{T}\right)\right], \tag{26}$$

where the function $P(x)$ is the integral

$$P(x)=\int_1^\infty\frac{d\varepsilon}{\varepsilon^2-1}\,\frac{\cosh x\varepsilon-\cosh x}{(\cosh x\varepsilon+1)(\cosh x+1)}. \tag{27}$$

The limiting values of this function are as follows:

$$x \gg 1 \quad P(x) = e^{-x} \ln 2\gamma x,$$
$$x \ll 1 \quad P(x) = 4x\pi^{-2}\zeta(3) \approx 0.485x. \tag{28}$$

From expression (26) we find that for $\omega \ll T \ll \Delta$ the result coincides with (25) while for $\omega \ll \Delta \ll T$

$$\operatorname{Im} Q(\omega) = -\pi\omega/\Delta. \tag{29}$$

(c) The case $\omega \sim \Delta \ll T$. Although the pair term in principle takes part in Im Q, its contribution turns out to be small. As to the electron term, we get from it again equation (29).

(d) The case $T \ll \omega \sim \Delta$. Here there are two possibilities. If $\omega < 2\Delta$, only the electron term will take part which gives

$$\operatorname{Im} Q(\omega) = -2\pi\sqrt{\pi}\sqrt{\frac{T}{2\Delta} + \frac{T}{\omega}}\, e^{-\frac{\Delta}{T}}. \tag{30}$$

In particular, Eq. (24) is obtained for $T \ll \omega \ll \Delta$. If $\omega > 2\Delta$, expression (30) is exponentially small compared to the contribution from the pair term (of course slightly away from the threshold). In view of the smallness of the temperature we can then use Eqs. (20) to (21).

(e) The case $\Delta \ll T \sim \omega$. Here both the pair term and the electron term take part to an equal degree. The calculation leads to the result

$$\operatorname{Im} Q(\omega) = -\pi\left(\frac{\omega}{\Delta} - 2\tanh\frac{\omega}{2T}\right). \tag{31}$$

Although in this formula the second term is much smaller than the first one, we have retained it for a reason which will be given below.

(f) The case $T \sim \Delta \ll \omega$. Here the most important contribution is given by the pair term, and it is sufficient to consider only the main expression which is the same as Eq. (29).

At the end of this section we must note that in all cases when $\omega \gg \Delta$ the real part of $Q(\omega)$ is small compared to the imaginary part which is, independently of the temperature, equal to $-\pi\omega/\Delta$. In this way the relation between the current and the vector potential is in the case $\omega \gg \Delta$ of the form

$$j(\mathbf{k}, \omega) = i\,(3\pi e^2 N\omega / 4mcvk)\, A(\mathbf{k}, \omega). \tag{32}$$

This relation does not contain Δ and is exactly the same as the one obtained from the theory of the anomalous skin effect in a normal metal. This result is natural.

However, in the cases $\omega \gg \Delta$ we can not restrict ourselves only to the main term. In order that the difference between the superconducting state and the normal state can be revealed, it is necessary to take into account also terms of the next order. In the case $\Delta \ll T \sim \omega$ these terms come both from the real and from the imaginary part of $Q(\omega)$. Just for that reason was the main expression in Eq. (31) supplemented by a small term. In the last case $T \sim \Delta \ll \omega$ the main correction gives only the real part of $Q(\omega)$.

3. THE LONDON REGION

We shall now elucidate under what conditions the London region $vk \ll \Delta$ occurs. We shall not consider, as already stated, the possible, but apparently very uncommon, case when under static conditions the superconductor is a London one for the whole temperature range.

As was already noted in the preceding section the London region can appear only near T_c. On the other hand near T_c the gap width Δ becomes very small and since for $\omega \gg \Delta$ the metal differs little from a normal conductor, the London region cannot extend to the critical temperature itself and in any case has an upper limit through the condition $\omega \ll \Delta$.

To find the more exact location of the London region we find first of all the connection between the current and the vector potential in the case $\Delta \gg vk$. To do this we take into account the earlier noted fact that for the whole range of frequencies of interest we can assume $\omega \ll vk$ and also that the London region occurs only in the immediate neighborhood of T_c, i.e., that we can assume that $\Delta \ll T$. The necessary equation is obtained from Eq. (14). Only the integral with the difference of the tangents is then essential and the last term in that equation.

We find in that way

$$\mathbf{j}(\mathbf{k}, \omega) = -\frac{e^2 N\mathbf{A}(\mathbf{k}, \omega)}{mc}\left\{\alpha\frac{\Delta^2}{T_c^2} - \frac{3i\pi\omega}{4vk}\right\}; \tag{33}$$

where

$$\alpha = \frac{1}{4}\int_0^\infty \frac{\sinh x\,dx}{x\cosh^3 x} = \frac{7}{4\pi^2}\zeta(3) = 0.21.$$

If we restrict ourselves to the real part of this relation, we obtain the equation of the Londons, where the expression within the braces plays the role of the ratio of the number of superconducting electrons N_s to the total number of electrons. The imaginary part corresponds to absorption. We note that this expression is obtained under the assumption $\omega \ll \Delta \ll T$ and $\omega \ll vk$ but the relation between vk and Δ and between vk and T can be arbitrary. This explains indeed also the fact that this expression coincides within a small correction term with the corresponding equation for the Pippard case.

As a criterion for the transition from the Pippard region to the London region one can take the equality of the coefficients of **A** in the expression for the current. For the Pippard case, assuming $\omega \ll \Delta \ll T$, we find from (17), (19), and (29)

$$\mathbf{j}(\mathbf{k}, \omega) = -\frac{e^2N}{mc}\mathbf{A}(\mathbf{k}, \omega)\left\{\frac{3\pi^2}{8}\frac{\Delta^2}{vkT_c} - i\frac{3}{4}\pi\frac{\omega}{vk}\right\}. \quad (34)$$

We shall compare this formula with Eq. (33). We assume that the imaginary parts in both equations are small compared to the real parts. In that case the criterion coincides with the criterion for the static case. The London region occurs for

$$vk \ll vk_1 = 3\pi^2 T_c / 8\alpha \approx 17.7 T_c. \quad (35)$$

Assuming $k \sim 1/\delta$ where δ is the penetration depth, one can show, using the well-known London equation for the penetration depth $\delta_L = (mc^2/4\pi N_s e^2)^{1/2}$ and Eq. (33), that this corresponds to the condition

$$\Delta \ll \frac{3}{16}\left(\frac{\pi}{\alpha}\right)^{3/2}\left(\frac{mc^2}{Ne^2v^2}\right)^{1/2} T_c^2. \quad (36)$$

From this inequality it follows that for instance for aluminium the London region occurs for $(T_c - T)/T_c \sim 4 \times 10^{-4}$, for tin for $(T_c - T)/T_c \sim 3 \times 10^{-2}$.*

When the temperature is increased the real part in Eq. (33) becomes small compared to the imaginary part. But since in the latter case the superconductor differs little from a normal metal, we obtain necessarily the Pippard case (we remember that the imaginary part in (33) refers to that case). In that way we can assume that the London region is bounded by the condition

$$vk \gg vk_2 = (3\pi\omega/4\alpha)(T_c/\Delta)^2 \quad (37)$$

or, after substituting for the London penetration depth,

$$\Delta \gg \frac{3^{1/3}\pi^{1/6}}{(4\alpha)^{1/2}}\left(\frac{mc^2}{Ne^2v^2}\right)^{1/6} T_c \omega^{1/3}. \quad (38)$$

Comparing Eqs. (34) and (35) we get the condition for the occurrence of the London region

$$\omega \ll \omega_c = \frac{9\pi^4}{(8\alpha)^3}\left(\frac{mc^2}{Ne^2v^2}\right) T_c^3. \quad (39)$$

*Tin is in fact not a genuine Pippard metal and is on the boundary between the London and the Pippard situations, since for tin $vk \approx vk_1$ already at $T = 0$. But since far from T_c the temperature dependence and the penetration depth are very weak, essential departures from the Pippard equations occur only in the neighborhood of T_c.

Such a limitation can mean completely different frequencies for different substances. For aluminium, for instance, this frequency corresponds to $2 \times 10^{-2} T_c$ or $\sim 3 \times 10^9 \text{ sec}^{-1}$. For tin it is equal to $0.5\, T_c \approx 2 \times 10^{11} \text{ sec}^{-1}$.

In view of the fact that of the greatest interest is the range of frequencies not too far from $\Delta(0)$ which for a Pippard metal is practically always beyond the limits of the region bounded by the inequality (39) we shall in general not consider the London region.

It is appropriate to note here also that apart from a limitation as to frequencies, the applicability of F. and H. London's equations in the form (33) is also limited from the high temperature side by the condition that the dimensions of pairs, which is of the order of v/Δ, must be small compared to the mean free path. In the opposite case the constant coefficient occurring in F. and H. London's equation can no longer be described by the first term within the brackets of Eq. (33). Since the mean free path can on the average be assumed to be of the order of 10^{-3} cm, and $v/\Delta(0) \sim 10^{-4}$ cm, the region of applicability of Eq. (33) is generally speaking small. As to the case $v/\Delta \gg l$, the electrodynamics for it has not been formulated and we shall not discuss it.

To conclude this section, we note that, as can be easily seen, for frequencies larger than ω_c a direct estimate gives $vk \gg T_c$ for the whole range of temperatures, including the neighborhood of T_c where the penetration depth is maximum. Such a relation between vk and T was essential for the conclusion reached in Sec. 2.

4. THE IMPEDANCE

The relation between **j** and **A** can be substituted into the Maxwell equations and one can obtain an expression for the dependence of the vector potential on the coordinates.

In the Pippard case considered by us the corresponding calculations are not different from the static case 1. Assuming the reflection of the electrons from the surface to be diffuse, we get for the penetration depth:

$$\delta = \frac{1}{H(0)}\operatorname{Re}\int_0^\infty H\,dz = \frac{\operatorname{Re} A(0)}{H(0)} = \frac{\sqrt{3}}{2\pi}\left(\frac{c^2mv}{3\pi e^2N\Delta}\right)^{1/3}\operatorname{Re}[Q(\omega)]^{-1/3}. \quad (40)$$

The impedance is determined as follows:

$$Z = R + iX = E(0) \Big/ \int_0^\infty j\,dz = -\frac{4\pi}{c}\frac{E(0)}{H(0)} = -\frac{4\pi i\omega}{c^2}\frac{A(0)}{H(0)} = -\frac{2\sqrt{3}\,\pi i\omega}{c^2}\left(\frac{mc^2 v}{3\pi e^2 N\Delta Q(\omega)}\right)^{1/3} \tag{41}$$

For complex $Q(\omega)$ the value of the root is determined as the analytical continuation of the real root for real $Q(\omega)$.

It is convenient to relate the magnitude of the impedance to the value of the active resistivity in the normal state which is equal to

$$R_n = \sqrt{3}\,(mvc^2\pi / 3Ne^2)^{1/3}\,\omega^{2/3} / c^2. \tag{42}$$

Such an expression is obtained by substituting $Q(\omega) = -i\pi\omega/\Delta$ into (41) (see also reference 5). The ratio Z/R_n is determined by the equation

$$Z(\omega) / R_n = -2i\,(\pi\omega / \Delta Q(\omega))^{1/3}. \tag{43}$$

For the case $T = 0$ the frequency dependence of the impedance is obtained from Eqs. (43), (15), (19), and (20).

In the case of non-zero temperatures we carry out an analysis of the temperature dependence of the impedance for different frequencies.

(A) The case $\omega \ll \Delta(0)$

(1) For the lowest temperatures there occurs the region $T \ll \omega \ll \Delta$. Then T becomes of the order of ω and finally we go over to the region $\omega \ll T \ll \Delta$. A description of that transition is given by Eqs. (19), (23). Substituting into (43) and taking into account that $\mathrm{Re}\,Q(\omega) \gg \mathrm{Im}\,Q(\omega)$ we get

$$\frac{Z(\omega)}{R_n} = 2\left(\frac{\omega}{\pi\Delta}\right)^{1/3}\left[\frac{4}{3\pi}\sinh\frac{\omega}{2T}K_0\left(\frac{\omega}{2T}\right)e^{-\Delta/T} - i\right]. \tag{44}$$

(2) When the temperature is raised further, we get into the range $\omega \ll \Delta \sim T$ which then goes over into the range $\omega \ll \Delta \ll T$. This transition is described by Eqs. (19) and (26). As long as $\Delta/T \gg \omega/\Delta$, the imaginary part of $Q(\omega)$ will be small compared to the real part, as before, and we get in that way

$$\frac{Z(\omega)}{R_n} = 2\left[\frac{\omega}{\pi\Delta\tanh(\Delta/2T)}\right]^{1/3}\left[\frac{2\omega}{3\pi T}\frac{1}{\sinh(\Delta/T)}\ln 2\sqrt{\frac{2\Delta}{\omega}} + \frac{1}{3\pi}\frac{\omega}{\Delta}\left(\coth\frac{\Delta}{2T} - 1\right) - \frac{2}{3\pi}P\left(\frac{\Delta}{T}\right)\coth\frac{\Delta}{2T} - i\right], \tag{45}$$

where $P(\Delta/T)$ is given by expression (27).

(3) For still higher temperatures, Δ decreases so much that it becomes comparable with and then less than ω. This region is described by Eqs. (19), (15), and (29); for $\omega \sim \Delta$ the real and imaginary parts of $Q(\omega)$ are of the same order of magnitude, and for $\omega \gg \Delta$ the real part of $Q(\omega)$ turns out to be small compared to the imaginary part. In that limiting case the equation for the impedance is of the form

$$\frac{Z(\omega)}{R_n} = 1 + \frac{1}{\sqrt{3}\pi}\frac{\Delta}{\omega}\left(2\ln\frac{2\omega}{\Delta} + \pi^2\right) - i\sqrt{3}\left[1 - \frac{1}{3\sqrt{3}\pi}\frac{\Delta}{\omega}\left(2\ln\frac{2\omega}{\Delta} + \pi^2\right)\right]. \tag{46}$$

(B) The case $\omega \sim \Delta(0)$

(1) At low temperatures we have $T \ll \omega \sim \Delta$. In this region one applies Eqs. (15), (19), and (30) in the case when $\omega < 2\Delta(0)$, or Eq. (20) if $\omega > 2\Delta(0)$. In the first case the imaginary part of $Q(\omega)$ will be small compared to the real part, and the expression for the impedance will be of the form

$$\frac{Z(\omega)}{R_n} = 2\sqrt{\frac{\omega\pi}{\Delta\,\mathrm{Re}\,Q(\omega)}} \times \left[\frac{2\pi\sqrt{\pi}}{3\,\mathrm{Re}\,Q(\omega)}\sqrt{\frac{T}{2\Delta} + \frac{T}{\omega}}\;e^{-\Delta/T} - i\right], \tag{47}$$

where $\mathrm{Re}\,Q(\omega)$ is given by Eq. (15) with an additional π^2. In the case $\omega > 2\Delta(0)$ the equations for the case $T = 0$ can be used.

(2) When the temperature is raised we go over to the range $\omega \sim \Delta \sim T$ and finally we get into the range $\Delta \ll T \sim \omega$. Here we can use equations (18) and (31). Inasfar as the real part of $Q(\omega)$ is less than the imaginary part we can again use an expansion. As a result we find:

$$\frac{Z(\omega)}{R_n} = 1 + \frac{1}{\sqrt{3}\,\pi}\frac{\Delta}{\omega}\left(2\ln\frac{2\omega}{\Delta} + \pi^2 + \frac{2\pi}{\sqrt{3}}\tanh\frac{\omega}{2T}\right) - i\sqrt{3}\left[1 - \frac{1}{3\sqrt{3}\,\pi}\frac{\Delta}{\omega}\left(2\ln\frac{2\omega}{\Delta} + \pi^2 - 2\pi\sqrt{3}\tanh\frac{\omega}{2T}\right)\right]. \tag{48}$$

It is interesting to note that when the temperature is lowered starting from T_c, the real part of the impedance initially does not decrease, but slightly increases.

(C) The case $\omega \gg \Delta(0)$

In this case only the relation between T and Δ changes, but ω is all the time large compared with them. Here we apply Eqs. (18), (19), and (29). Taking into account again that the real part is small, we find:

$$\frac{Z(\omega)}{R_n} = 1 + \frac{1}{\sqrt{3}\,\pi}\frac{\Delta}{\omega}\left(2\ln\frac{2\omega}{\Delta} + \pi^2\left(1 - \tanh\frac{\Delta}{2T}\right)\right) - i\sqrt{3}\left[1 - \frac{1}{3\sqrt{3}\,\pi}\frac{\Delta}{\omega}\left(2\ln\frac{2\omega}{\Delta} + \pi^2\left(1 - \tanh\frac{\Delta}{2T}\right)\right)\right]. \tag{49}$$

In view of the fact that a detailed comparison of the theory with experimental data requires large numerical calculations (first of all a tabulation of the functions (15) and (20), which at this moment has not yet been concluded) such a comparison will be given in a later paper.

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