The solution may be given in parametric form (parameter R_2):

$$\frac{1}{h_1} = \frac{[1-(2-\varkappa)R_2^2(1-R_2)][(\varkappa/2)(1-R_2^2)+\varkappa/(\varkappa-1)]}{2R_2[\varkappa(1-R_2)+1]} + \frac{2-\varkappa}{4}R_2(1-R_2^2) - \frac{2-\varkappa}{2(\varkappa-1)}R_2 - 1; \quad (19)$$

$$z_2 = h_1[1-(2-\varkappa)R_2^2(1-R_2)]/2R_2[\varkappa(1-R_2)+1];$$

$$V_2^2 = \varkappa z_2 + \frac{2-\varkappa}{2}h_1R_2, \quad V_1 = R_2V_2,$$

where κ is the adiabatic coefficient. This solution is given in Fig. 7 for $\kappa = 5/3$.

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A NONLINEAR THEORY OF VECTOR FIELDS

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In analogy with Einstein's theory of gravitation, the existence of a physical vector field is treated as the curvature in a "non-Pythagorean" space whose metric is $ds = \gamma_i dx^i$. This leads to nonlinear field equations which in the linear approximation (for weak fields) become the usual equations. In spite of its potential being singular, the total energy in the field of a point charge is finite.

1. STATEMENT OF THE PROBLEM

As a rule, the nonlinear generalizations of field theories are constructed in an attempt to eliminate divergences in ordinary field theory. It is known, however, that this can be done also in other ways (nonlocal interactions, the use of higher drivatives, space quantization, etc.). Furthermore, one likes to hope that it is possible to go on without essentially leaving the framework of the existing theory. This would seem to be why the nonlinear theories do not receive wide recognition at present.

We wish to emphasize that a nonlinear generalization of field theory is necessary, independently of the divergences in the linear theory and independently of the possible necessity for other generalizations. Our argument is based on the following considerations.

(a) In principle, nonlinearity in field theory follows unavoidably from the fact of pair creation and annihilation, which leads to nonlinear effects such as scattering of light by light. It is not, therefore, merely accidental that when dealing with interactions in modern field theory we arrive at nonlinear equations. Nonlinearity whould therefore not be considered simply one of many possible methods for eliminating the difficulties of the theory, but as a reflection of the objective properties of the field.

(b) It is well known that in all nonlinear theories there appears a characteristic length which, although it has different meanings in different versions, is always of the order of magnitude of the "classical radius" $r_0 = e^2/m_0c^2$ of the field's source. This parameter serves as a criterion that can be used to define weak fields (with $r \gg r_0$) for which nonlinear effects can be neglected. It turns out that in electrodynamics all real processes take place in a region for which $r \gg r_0$ (weak fields and low energies), and the linear theory is insufficient only when one needs to consider virtual processes with arbitrary energies. The situation is different, however, where in mesodynamcs, weak fields (in the above sense of the word) practically never exist. In this case the critical parameter will be $R_0 = g^2/M_0c^2$, where M_0 is the nucleon mass. If we take $g^2/hc \approx 14$ (as is presently believed), we obtain $R_0 \approx 3 \times 10^{-13}$ cm. But because the range of nuclear forces is so short that they practically vanish at a distance $\Lambda_\mu = h/\mu c \approx 1.4 \times 10^{-13}$ cm (where μ is the meson mass), there is essentially no field at $r > R_0$. Therefore the values of r equal to and less than Λ_μ, at which the field is measurably different from zero, lie entirely within a region whose radius is less than R_0, a region in which the linear theory is no longer applicable. This means that the Yukawa potential cannot serve as a first approximation, and this may be the cause of many of the difficulties in the meson theory of nuclear forces. It is important to see that it is not so much a matter of g^2/hc being greater than 1, but of R_0 being greater than Λ_μ (which is determined by the value of μ). It is not accidental, therefore, that an expansion in inverse powers of g^2/hc also fails. The difficulties in linear mesodynamics lie not in the fact that the nuclear interaction is strong, but that it is of such short range.

(c) It is thought that an important deficiency of nonlinear theories is that on the one hand, they are difficult to quantize, while on the other hand an unquantized (classical) nonlinear theory can hardly be of interest. This is because much before the nonlinear deviations become of importance (at distances of the order of the classical radius of the particle which is the source of the field), quantum effects begin to predominate (at distances of the order of the Compton wavelength of this particle). But this also is true only for electrodynamics, where $r_0/\lambda_0 = (e^2/m_0c^2)/(h/m_0c) = e^2/hc = 1/137$. In mesodynamics, on the other hand, $R_0/\Lambda_0 = (g^2/M_0c^2)(h/M_0c) = g^2/hc \approx 14$, so that progress in classical mesodynamics should indeed begin with the nonlinearities.

(d) In the special theory of relativity it becomes clear that Newtonian mechanics is good only if $v \ll c$ and must be generalized at high velocities. The general theory of relativity shows that mechanics must be generalized also for high accelerations, i.e., for large forces or intense fields (and this generalization leads to a nonlinear theory). But it has been shown by Fock[1] that Einstein's theory does not describe arbitrary accelerated motion, but only motion in the gravitational field. The theory therefore refers only to the gravitational field, and is not a general theory of noninertial motion. The problem then arises of performing analogous generalizations for other fields as well, in particular for the practically most important cases of the electromagnetic and meson fields.

At first glance such a statement of the problem (to the extent, at any rate, that it refers to the electromagnetic field) may seem confusing, since there already exists a very well known general relativistic covariant formulation of electrodynamics. This generalization, however, deals only with the gravitational field associated with the electromagnetic one, and leads, in particular, to a criterion of nonlinearity which is not e^2/m_0c^2, but $(e^2/m_0c^2)(\xi/e)$, where $\xi = m_0\sqrt{\kappa}$ is the gravitational charge. For this reason this generalization has little practical meaning. What we are discussing, on the other hand, is a generalization that makes electrodynamics itself nonlinear, without considering any other kinds of fields (even the gravitational). From this point of view, Einstein's theory (and, in particular, the geometrical methods he uses) should be thought of as a model for the construction of the nonlinear field theory, a model which, although it has been applied to a tensor (specifically, the gravitational) field, is one that can be used in attempts to construct a nonlinear theory also for vector, spinor, and other fields. In the present work, this will be done for a vector field,* and the first field treated will be the simple electromagnetic one (with zero rest mass, in quantum terms).

It may seem that a geometric treatment of the electromagnetic field is impossible even if only because this field produces in general different accelerations in different bodies, so that one cannot formulate an equivalence principle. Also, one may point out that the four components of the electromagnetic potential are insufficient for a generalization of the Pythagorean theorem, since they cannot be identified with the components of a metric tensor.

These difficulties can, however, be overcome.

(a) If we admit to "Einstein's elevator" only those bodies whose charge-to-mass ratio (specific charge) e/m_0 has a given fixed value, the situation arising for these bodies is the same as in the theory of gravitation, since by a suitable coordinate change the external field can be eliminated and the desired relations obtained. For any other group of bodies with another specific charge, we obtain (formally) the same relations. The specific charge will enter only in the form of a parameter (replacing the gravitational constant in the gravitational equations). Such a device has

*A preliminary version has already been published.[2]

already been used by other authors, for instance by Rumer.[3]

(b) The four-component electromagnetic field can be treated geometrically by associating with it a "non-Pythagorean" four-space, whose metric is given by four functions making up a four-vector, so that the line element in this space is given in terms of increments in the coordinates by the equation

$$ds = \gamma_i \, dx^i, \; i = 1, 2, 3, 4, \tag{1}$$

where $dx^4 = ic\,dt$. This "space" is simply a model; so far it is in no way related to any physical space, and is merely an accessory similar to the n-dimensional space of statistical mechanics or "isotopic spin space."

A non-Pythagorean metric such as (1) has been used before. It was first introduced by Fock and Ivanenko[4] and was later analyzed in detail by Mirura et al.[5] More recently it has been studied by Flint and Williamson.[6] In all these works, however, the γ_i were treated as modifications of the Dirac matrices and were used to generalize the Dirac equation, mostly in order to account for the gravitational field, so that it was assumed that $\gamma_i\gamma_k + \gamma_k\gamma_i = 2g_{ik}$.

Here we propose an entirely different problem, namely that of studying the field γ_i itself. With this field we associate a definite physical field whose equations we wish to find.

2. CURVED "NON-PYTHAGOREAN" SPACE

A construction of a curved non-Pythagorean space whose metric is given by $ds = \gamma_i \, dx^i$ has already been given.[2] In particular, the following expressions were obtained.

(a) The equation of a geodesic is

$$d^2x^i / ds^2 + \Gamma^i_h dx^k / ds = 0; \tag{2}$$

(b) The action function is

$$\mathscr{L} = \text{const}\, \Gamma_{im} \Gamma^{im}, \tag{3}$$

(c) The energy-momentum tensor is

$$\mathscr{T}^{ik} = \mathscr{T}^i_s b^{sk} = \text{const}' (\Gamma_{sn}\Gamma^{in} - \tfrac{1}{4} b^i_s \Gamma_{mn}\Gamma^{mn})\, b^{sk}, \tag{4}$$

with

$$\Gamma^i_k = \Gamma_{km} b^{mi}, \quad \Gamma^{im} = \Gamma_{kn} b^{ik} b^{mn},$$

$$\Gamma_{ik} = \partial\gamma_k / \partial x^i - \partial\gamma_i / \partial x^k = -\Gamma_{hi},$$

$$b_{ik} = {}^1\!/_2 (\gamma_i\gamma_k + \gamma_k\gamma_i) = b_{ki}, \quad b^{ik} = {}^1\!/_2 (\gamma^i\gamma^k + \gamma^k\gamma^i) = b^{ki},$$

$$\gamma^i = b^{ik}\gamma_k, \quad \gamma_i = \alpha_i + h_i,$$

where the α_i are Dirac matrices, and the h_i are c-number increments which give the deviation of the space from Euclidean.

In addition, we have the relations

$$b^{il} b_{kl} = b^i_k = \delta^i_k; \tag{5}$$

$$\partial\Gamma_{kl} / \partial x^n + \partial\Gamma_{ln} / \partial x^k + \partial\Gamma_{nk} / \partial x^l = 0. \tag{6}$$

The generalization given by Eq. (3) of the Maxwell Lagrangian

$$L_0 = -F_{im}F_{im} / 16\pi \tag{7}$$

follows obviously from the concepts described at the start: the Lagrangian should be invariant in the non-Euclidean space defined by the metric (1).

Thus the geometric interpretation can be used to unite the formalism of electrodynamics in the "general theory of relativity" with the well known idea of Mie. In the expression $F^{ik} = F_{mn}g^{im}g^{kn}$ we make the substitution $g_{ik} \rightarrow b_{ik} = \tfrac{1}{2}(\gamma_i\gamma_k + \gamma_k\gamma_i)$, where $\gamma_i = \alpha_i + \xi A_i$, so that we obtain, in agreement with Mie, a tensor of the form $H^{ik} = u^{ik}(F_{mn}, A_l)$, and, as must occur in his scheme,

$$\Gamma^{im} = \text{const}\,(\partial\mathscr{L} / \partial\Gamma_{im}).$$

Roughly speaking, what happens is that the action of the gravitational field on the electromagnetic one is replaced by the electromagnetic field acting on itself.

The fact that the Lagrangian of Eq. (3) contains the field potentials explicitly (in terms of the b^{ik}) might seem to be a contradiction in our theory, which then is no longer gauge invariant. But it is known that Einstein's theory of gravitation also has this property, and this is never considered a defect. The fact is that a direct relation between gauge invariance of the field equations and the zero rest mass of the quanta associated with this field is necessary only in a linear theory. Thus all we can demand from a nonlinear theory is that the equations be gauge invariant in linear approximation. Our theory satisfies this condition, since in the linear approximation the Lagrangian of Eq. (3) becomes the Maxwell Lagrangian of Eq. (7).

3. THE ELECTROMAGNETIC FIELD

Let us base the theory of the electromagnetic field on Eqs. (2), (6), (3), and (4). It is clear that the expressions entering into these equations are matrix expressions. They can therefore be treated as operators in some function space, and the "ordinary" expressions (for instance the Lagrangian and the components of the energy momentum tensor) can be treated as sort of "mean values" given by relations of the form

$$L = {}'\text{Sp}\,\mathscr{L}. \tag{8}$$

These then establish the relation between subsidiary ("matrix") space of Eq. (1) and "ordinary" Riemannian space.*

In addition, we must yet establish the relation between the Γ_{im} from the nonlinear theory and the F_{im} from the linear one, as well as find the constants in Eqs. (3) and (4).

For a weak field, the geodesic equation (2) gives

$$\dot{u}_i \approx c\Gamma_{ik}u_k,$$

since $b^{mi} \approx b_0^{mi} = \delta^{mi}$, and $ds = c\,d\tau$. Comparing this with the Lorentz formula

$$m_0\dot{u}_i = \frac{e}{c}F_{ik}u_k$$

and noting that $\gamma_i = \gamma_i^0 + h_i = \alpha_i + h_i$, we have

$$h_i = (e/m_0c^2)A_i. \tag{9}$$

Similarly

$$L \approx \text{const}\,\Gamma_{im}\Gamma_{im} = \text{const}\,(e/m_0c^2)^2 F_{im}F_{im},$$

so that comparing with (7), we have

$$\mathcal{L} = -(1/16\pi)(m_0c^2/e)^2\Gamma_{im}\Gamma_{kn}b^{ik}b^{mn}. \tag{10}$$

Similarly, we obtain

$$\mathscr{T}^{ik} = -\frac{1}{4\pi}\left(\frac{m_0c^2}{e}\right)^2\left(\Gamma_{sn}\Gamma^{in} - \frac{1}{4}\Gamma_{mn}\Gamma^{mn}\right)b^{sk}. \tag{11}$$

Here e and m_0 are the charge and rest mass of a test body. It should be noted that the properties of the test body enter into our theory in a very important way, since they give the criterion of smallness (and therefore nonlinearity) of the field. According to this criterion the field may be considered weak (and described with sufficient accuracy by the linear theory) if the energy of the charge in this field is much less than the self-energy of the charged body, that is if $e\varphi \ll m_0c^2$ or $h \ll 1$.

It is a well known fact that no such test body enters into the theory of gravitation, and that the limit of applicability of the linear approximation to the gravitational field is the same for all test bodies and is uniquely determined by the field "strength" at the given point (for a static spherically symmetric field this is the ratio of the gravitational radius $\kappa M/c^2$ of a source of mass M to the length of the radius vector at the given point). In other words, no test-body parameters enter into the solution of the field equations. This is related to the fact that the gravitational charge of any body is proportional to its mass (with the same coefficient of proportionality for all bodies), so that $(\xi/m_0c^2)^2 = (m_0\sqrt{\kappa}/m_0c^2)^2 = \kappa/c^4$, which does not involve the test body. The fact that the test-body parameters enter into the solution of the field equations in the nonlinear theory of the electromagnetic field was pointed out at the very start when we spoke of using an equivalence principle in this theory.

To obtain and solve the field equations for the general case involves great mathematical difficulties. It is therefore useful to consider the simple but important special case of a static spherically-symmetric field. In this case we may attempt to find a solution of the form $h_\mu = 0$ (for $\mu = 1, 2$, or 3), and $h_4 = ih \neq 0$. Then all the $\Gamma_{\mu\nu} = 0$, and $\Gamma_{\mu 4} = -\Gamma_{4\mu} = \partial h_4/\partial x^\mu$, so that

$$\mathcal{L} = \Gamma_{im}\Gamma^{im} = \Gamma_{im}\Gamma_{kn}b^{ik}b^{mn} = 2\frac{\partial h_4}{\partial x^\mu}\frac{\partial h_4}{\partial x^\nu}(b^{\mu\nu}b^{44} - b^{\mu 4}b^{\nu 4}). \tag{12}$$

Since

$$b_{ik} = \delta_{ik} + \alpha_i h_k + \alpha_k h_i + h_i h_k,$$

we have

$$b_{\mu\nu} = \delta_{\mu\nu},\ b_{\mu 4} = \alpha_\mu h_4,\ b_{44} = 1 + 2\alpha_4 h_4 + h_4^2.$$

Using the relation $b^{ik}b_{kl} = \delta_l^i$, which gives ten equations for the ten desired b^{ik}, we arrive at

$$\begin{gathered} b^{\mu\mu} = 1 + h_4^2\frac{g}{H} + \frac{2h_4^3}{H}\alpha_4,\ b^{44} = \frac{g}{H} - \frac{2h_4}{H}\alpha_4, \\ b^{\mu\nu} = h_4^2\frac{g}{H}\alpha_\mu\alpha_\nu + (-1)^\lambda\frac{2h_4^3}{H}\alpha_5\alpha_\lambda,\ \nu \neq \mu,\ \lambda \neq \mu,\ \nu, \\ b^{\mu 4} = -h_4\frac{g}{H}\alpha_\mu + \frac{2h_4^2}{H}\alpha_\mu\alpha_4, \end{gathered} \tag{13}$$

where $g = 1 - 2h_4^2$, and $H = 1 - 8h_4^2 + 4h_4^4$.

From this it follows immediately that

$$\operatorname{Sp}(b^{\mu\nu}b^{44} - b^{\mu 4}b^{\nu 4}) = 0\ (\mu \neq \nu), \tag{14}$$

and

$$\operatorname{Sp}(b^{\mu\mu}b^{44} - b^{\mu 4}b^{\mu 4}) = 4g/H, \tag{15}$$

since the trace of all the α_i and products of different α_i vanishes.

Thus

$$L = \text{const}\,(\nabla h_4)^2 g/H, \tag{16}$$

so that in agreement with the usual expressions

$$\frac{\delta L}{\delta h_i} = \frac{\partial L}{\partial h_i} - \frac{\partial}{\partial x^k}\frac{\partial L}{\partial(\partial h_i/\partial x^k)} = 0$$

we obtain the field equation in the form

$$\nabla^2 h_4 + \frac{1}{2}(\nabla h_4)^2\frac{(\partial/\partial h_4)(g/H)}{g/H} = 0$$

or

$$h'' + \frac{2}{r}h' + \frac{1}{2}(h')^2\frac{\partial \ln(g/H)}{\partial h} = 0, \tag{17}$$

*This idea is due to E. S. Fradkin.

where $g = 1 + 2h^2$, and $H = 1 + 8h^2 + 4h^4$, since $h_4 = ih$.

Integration gives

$$\int\left[\frac{1+2h^2}{1+8h^2+4h^4}\right]^{1/2} dh = \frac{C}{r} + C_1. \qquad (18)$$

Further calculations are difficult, since it is necessary not only to write the elliptical integral on the left as an explicit function of h, but to find the inverse function $h = h(r)$.

It is therefore convenient to obtain an approximate but simple solution, writing

$$\left[\frac{1+2h^2}{1+8h^2+4h^4}\right]^{1/2} \approx \left[\frac{1+2h^2}{1+4h^2+4h^4}\right]^{1/2} = (1+2h^2)^{-1/2}. \qquad (19)$$

It should be noted that the difference between the approximate answer and the exact one increases neither as $h \to 0$, nor as $h \to \infty$. On the contrary, in these limits this difference vanishes, and is as large as about 20 percent only in a small interval about $h \approx 1$.

With (19), the solution of (18) becomes

$$\int dh / \sqrt{1+2h^2} = \frac{C}{r} + C_1$$

or

$$\frac{1}{\sqrt{2}} \sinh^{-1}(\sqrt{2}\, h) = \frac{C}{r} + C_1.$$

Since in the linear approximation we must have $h = r_0/r$, where $r_0 = e^2/m_0c^2$, we have $C = r_0$ and $C_1 = 0$, so that finally

$$\sinh^{-1}(\sqrt{2}\, h) = \sqrt{2}\, r_0/r,$$

whence

$$h = \sinh(\sqrt{2}\, r_0/r)/\sqrt{2}, \qquad (20)$$

$$\varphi = (e/r_0\sqrt{2}) \sinh(r_0\sqrt{2}/r). \qquad (21)$$

As r approaches 0, the potential starts increasing very rapidly at a distance of the order of r_0, so that one may say that two colliding electrons behave like solid spheres of radius r_0. For $r \gg r_0$, we obtain the usual Coulomb potential. In the second approximation, we have

$$\varphi \approx (e/r)(1 + r_0^2/3r^2);$$

so that the correction for the first Bohr orbit ($r = a_0 = h^2/me^2$) is of the order of α^4, where $\alpha = e^2/hc$.

Although the potential of (21) has a high singularity at $r = 0$, for $r > r_0$ stable orbits are possible. This can be seen using the virial theorem, according to which if periodic motion is to take place, then $\overline{E} < 0$, where $\overline{E} = \overline{T} + \overline{V} = \frac{1}{2} r\overline{V'} + \overline{V}$.

In our case we have

$$V = -\frac{e}{r_0\sqrt{2}} \sinh\frac{r_0\sqrt{2}}{r}, \quad V' = \frac{e}{r^2}\cosh\frac{r_0\sqrt{2}}{r},$$

so that

$$E = \frac{e}{r_0\sqrt{2}} \cosh\frac{r_0\sqrt{2}}{r}\left(\frac{1}{2}\frac{r_0\sqrt{2}}{r} - \tanh\frac{r_0\sqrt{2}}{r}\right)$$

$$= \frac{e}{r_0\sqrt{2}} \cosh x\left(\frac{x}{2} - \tanh x\right),$$

where $x = r_0\sqrt{2}/r$. Since $\tanh x > x/2$ for $x \lesssim 2$, and since $(e/r_0)\cosh x > 0$, it follows that $\overline{E} < 0$ for all $r > \sqrt{2}\, r_0/2$, or almost always.

The potential of (21) can now be used to calculate the total energy of the electromagnetic field of the electron. In our case of a static spherically symmetric potential ($h_\alpha = 0$, $\partial h_i/\partial x^4 = 0$), we obtain

$$T^{44} = -\frac{1}{4\pi}\left(\frac{m_0c^2}{e}\right)^2 \frac{1}{2}\Gamma_{\mu 4}\Gamma_{\nu 4}\frac{1}{4}\operatorname{Sp}\{(b^{\mu\nu}b^{44} - b^{\mu 4}b^{\nu 4})\, b^{44}\}$$

$$= -(e^2/8\pi r_0^2)(h')^2(1+4h^4)/(1+8h^2+4h^4)^2;$$

so that

$$m = -\frac{1}{ic}P^4 = \frac{1}{c^2}\int_0^\infty T^{44} d\tau = \frac{e^2}{2\sqrt{2}\, r_0c^2} J,$$

where

$$J = \int_0^\infty \frac{\cosh^2 x\,(\cosh^4 x - 2\sinh^2 bx)}{(\cosh^4 x + 2\sinh^2 x)^2} dx \approx 1,$$

or

$$m \approx m_0/2\sqrt{2}. \qquad (22)$$

In spite of the highly singular potential given by (21), the total energy of the field is finite.

It follows from (22) that in the given theory the classical electromagnetic electron mass is only part of the total electron mass (the question of the "bare" mass remains open) so that the condition of Laue's theorem ($\int T^{ik} d\tau = 0$ for all components except T^{44}) is naturally not fulfilled in this case. This is not a defect in the theory, but speaks in its favor, since certain phenomena (such as electron production in μ-meson decay) indicate that the electron has some elements in common with other "elementary" particles. In addition, we must yet "leave room" for the quantum ("transverse") mass (this quantity will be calculated later; it is also found to be finite and of the correct order of magnitude αm_0).

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THEORY OF THE ANISOTROPY OF THE WIDTH OF FERROMAGNETIC RESONANCE ABSORPTION LINE

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The dependence of the width of a resonance absorption line on the internal field is found from the Landau-Lifshitz equations. Specific examples of ferrites with single-axis and cubic symmetry are considered.

1. The width of a radio-frequency resonance absorption line, considering only spin-spin relaxation, can be described with the equation[1,2]

$$\dot{\mathbf{M}} = \gamma[\mathbf{M}\times\mathbf{H}] - \frac{1}{\tau}(\mathbf{M} - \chi_0\mathbf{H}), \qquad (1)$$

where χ_0 is the equilibrium susceptibility, and $M = \chi_0 H_M$. In the case of weak radio-frequency fields, when $|h| \ll H_M$, this equation leads to a Lorentzian line shape. When applied to ferromagnets, however, χ_0 is no longer constant. The magnitude of χ_0 can be deduced from Eq. (1), assuming a constant magnitude for the vector $\mathbf{M}$. Then

$$\chi_0 = M^2/(\mathbf{M}\cdot\mathbf{H})$$

and

$$\dot{\mathbf{M}} = \gamma[\mathbf{M}\times\mathbf{H}] - \lambda M^{-2}[\mathbf{M}\times[\mathbf{M}\times\mathbf{H}]], \qquad (2)$$

where $\lambda = \chi_0/\tau$.

The Landau-Lifshitz equation (2), in which the magnitude of the magnetization vector $\mathbf{M}$ is constant, is conveniently expressed in polar coordinates, where the orientation of the vector $\mathbf{M}$ is given by its polar angle ϑ and its azimuthal angle φ. Introducing the radial, polar, and azimuthal components of the field, H_M, H_ϑ, and H_φ, Eq. (2) becomes

$$\dot{\vartheta} = -\gamma(H_\varphi - \alpha H_\vartheta), \quad \dot{\varphi}\sin\vartheta = \gamma(H_\vartheta + \alpha H_\varphi), \qquad (3)$$

where a dimensionless attenuation parameter $\alpha = \lambda/\gamma M$ has been introduced.

Analogously, Eq. (1) becomes in spherical coordinates

$$\dot{M} = \frac{1}{\tau}(\chi_0 H_M - M), \qquad (4)$$

$$\dot{\vartheta} = \frac{\chi_0}{M\tau}H_\vartheta - \gamma H_\varphi, \quad \dot{\varphi}\sin\vartheta = \frac{\chi_0}{M\tau}H_\varphi + \gamma H_\vartheta. \qquad (5)$$

When $M = \text{const.}$, with $\chi_0 = \alpha\gamma M\tau$, these equations reduce to Eqs. (3).

2. In a state of thermodynamic equilibrium the direction of the magnetization vector $\mathbf{M}$ in a ferromagnet coincides with the direction of the effective internal field $\mathbf{H}_M$, whose magnitude in turn can be found using the free energy F:

$$H_M = -F_M \equiv -\partial F/\partial M. \qquad (6)$$

The equilibrium orientation of the vector $\mathbf{M}$, given by the angles ϑ_0 and φ_0, is found from the conditions

$$F_\vartheta \equiv \partial F/\partial\vartheta = 0, \quad F_\varphi \equiv \partial F/\partial\varphi = 0. \qquad (7)$$