Here

$$d_{SV} = -2\mathrm{Im}\,(g_S g_V^* - g_S' g_V'^*); \quad d_{ST} = 2\mathrm{Im}\,(g_T g_S^* + g_T' g_S'^*); \quad d_{VT} = -2\mathrm{Im}\,(g_V g_T^* - g_V' g_T'^*).$$

$$\Phi_S = E_\mu\left(ME_\pi - \frac{\Delta}{2}\right) + m_\mu^2 (M - E_\pi); \quad \Phi_{SV} = E_\mu^2 m_\mu + E_\mu (E_\pi - M)\, m_\mu + m_\mu \left(\frac{\Delta}{2} - ME_\pi - m_\mu^2\right);$$

$$\Phi_{ST} = E_\mu^2\left(\frac{m_\pi^2}{M} - 2E_\pi + M\right) + E_\mu\left\{-m_\mu^2 + \frac{E_\pi m_\mu^2}{M} + (E_\pi - M)\left(\frac{\Delta}{2M} - E_\pi\right)\right\} + M^{-1}\left\{m_\mu^2\left(\frac{\Delta}{2} - m_\pi^2\right) + (E_\pi - M)\, E_\pi m_\mu^2\right\};$$

$$\Phi_V = 2E_\mu^3 + E_\mu^2(-2M + 2E_\pi) + E_\mu\left\{\left(\frac{\Delta}{2} - ME_\pi\right) - 2m_\mu^2\right\} + m_\mu^2 (M - E_\pi);$$

$$\Phi_{VT} = E_\mu^2 M^{-1} m_\mu (E_\pi - M) + E_\mu M^{-1}\left\{\frac{\Delta}{2} m_\mu + E_\pi^2 m_\mu + M m_\mu (M - 3E_\pi)\right\} + M^{-1}\left\{m_\mu\left(ME_\pi - \frac{\Delta}{2}\right)(M - E_\pi)\right\};$$

$$\Phi_T = M^{-2}E_\mu^3\,\{4E_\pi M - 2m_\pi^2 - 2M^2\} + E_\mu^2 M^{-1} 4\left(\frac{\Delta}{2} - \frac{E_\pi\Delta}{2M} - ME_\pi + E_\pi^2\right) + E_\mu M^{-2}\left\{2m_\pi^2 m_\mu^2 - 2E_\pi^2 m_\mu^2\right.$$

$$\left. + \left(\frac{\Delta}{2} - ME_\pi\right)(2ME_\pi - \Delta - E_\pi^2 + m_\pi^2)\right\} + (M - E_\pi)(E_\pi^2 - m_\pi^2)\, m_\mu^2 M^{-2}.$$

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THEORY OF EXCITATION OF HYDROMAGNETIC WAVES

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Excitation of hydromagnetic and magnetoacoustic waves by external currents is investigated. Damping of the waves as a result of conductivity and viscosity is taken into account. The intensity of excitation by currents is compared with the intensity of excitation by mechanical means.

1. As is well known, propagation of hydromagnetic and magnetoacoustic waves is possible in a conducting liquid located in an external magnetic field.[1] In the experiments of Lundquist,[2] hydromagnetic waves were excited in liquid mercury by mechanical means, with the use of a rotating disk equipped with blades. Excitation of hydromagnetic waves is also possible by means of external variable currents. It is therefore of interest to determine the intensity of the excitation of hydromagnetic waves by this method and to compare this intensity with that of the excitation of hydromagnetic waves by mechanical means. The present paper is devoted to a consideration of this problem.

2. Let us consider an ideal compressible conducting liquid located in an external magnetic field $\mathbf{H}_0$. The motion of the liquid in the present of external currents $\mathbf{j}_0$ will be determined by the equations of hydrodynamics:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v}\nabla)\mathbf{v} = -\nabla p + \frac{\mu}{c}[\mathbf{j}\times\mathbf{H}], \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0$$

and Maxwell's equations

$$\operatorname{curl}\mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t}, \quad \operatorname{div}\mathbf{H} = 0, \operatorname{curl}\mathbf{H} = \frac{4\pi}{c}(\mathbf{j} + \mathbf{j}_0), \quad (2)$$

where $\mathbf{v}$ and ρ are the velocity and density of the liquid, p is the pressure, $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields arising in the liquid, $\mathbf{j}$ is the current density defined by the relation

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{\mu}{c}[\mathbf{v}\times\mathbf{H}]\right),$$

and σ and μ are the conductivity and magnetic susceptibility of the liquid.

In the case of a liquid of infinite conductivity, which we shall consider first, it follows from the last equation that

$$\mathbf{E} = -\frac{\mu}{c}[\mathbf{v}\times\mathbf{H}]. \quad (3)$$

If the current $\mathbf{j}_0$ is sufficiently small, then we can linearize the set of equations (1). In such a case, we obtain the following equation for the determination of the velocity of the liquid $\mathbf{v}$:

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} - S^2 \operatorname{grad}\operatorname{div}\mathbf{v} \quad (4)$$

$$-[\operatorname{curl}\operatorname{curl}[\mathbf{v}\times\mathbf{V}_0]\times\mathbf{V}_0] = \frac{\mu}{c\rho_0}\left[\mathbf{H}_0\times\frac{\partial \mathbf{j}_0}{\partial t}\right],$$

where ρ_0 is the equilibrium liquid density, S the velocity of sound, and $\mathbf{V}_0 = \sqrt{\mu/4\pi\rho_0}\,\mathbf{H}_0$ the Alfven velocity.

The variable magnetic field $\mathbf{h} = \mathbf{H} - \mathbf{H}_0$ and the change in the density associated with the wave are determined by the equations:

$$\frac{\partial \mathbf{h}}{\partial t} = \operatorname{curl}[\mathbf{v}\times\mathbf{H}_0], \quad \frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div}\mathbf{v} = 0. \quad (5)$$

By means of Eqs. (1) and (2), we can show that the increase in the total energy of the medium per unit time, including the kinetic energy, the energy of the sound waves, and the energy of the magnetic field is determined by the following formula:

$$I = \frac{\partial}{\partial t}\int\left\{\frac{\rho_0 v^2}{2} + \frac{S^2\rho^2}{2\rho_0} + \frac{\mu H^2}{8\pi}\right\}dv = \frac{\mu}{c}\int \mathbf{v}\cdot[\mathbf{H}_0\times\mathbf{j}_0]\,dv. \quad (6)$$

The intensity of radiation of the hydromagnetic and magnetoacoustic waves is also determined by this equation.

3. We shall look for $\mathbf{v}$ in the form of a Fourier integral

$$\mathbf{v}(\mathbf{r}, t) = \int \mathbf{v}(\mathbf{k}, \omega)\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\, d\mathbf{k}\,d\omega.$$

Substituting this expression in (4), we get

$$\{\omega^2 - (\mathbf{k}\mathbf{V}_0)^2\}\mathbf{v} - \{(S^2 + \mathbf{V}_0^2)\mathbf{k} - (\mathbf{k}\mathbf{V}_0)\mathbf{V}_0\}(\mathbf{k}\mathbf{v}) \quad (7)$$

$$+ \mathbf{k}(\mathbf{k}\mathbf{V}_0)(\mathbf{V}_0\mathbf{v}) = i\omega\frac{\mu}{c\rho_0}[\mathbf{H}_0\times\mathbf{j}_0],$$

where $\mathbf{j}_0(\mathbf{k}, \omega)$ is the Fourier component of the external current density

$$\mathbf{j}_0(\mathbf{k}, \omega) = (2\pi)^{-4}\int \mathbf{j}_0(\mathbf{r}, t)\, e^{-i\mathbf{k}\cdot\mathbf{r} + i\omega t}\, d\mathbf{r}\,dt.$$

Setting the determinant of (7), equal to zero, we obtain the dispersion equation for free waves in an ideally conducting medium in the presence of an external magnetic field:

$$[\omega^2 - (\mathbf{k}\mathbf{V}_0)^2]\{\omega^2[\omega^2 - (S^2 + V_0^2)k^2] + S^2k^2(\mathbf{k}\mathbf{V}_0)^2 = 0. \quad (8)$$

This equation has three different solutions, corresponding to the three different types of waves that can be propagated in an ideally conducting, compressible liquid located in an external magnetic field. The squares of the phase velocities of the hydromagnetic and the two magnetoacoustic waves are respectively equal to

$$u_1^2 = V_0^2\cos^2\vartheta,$$

$$u_{2,3}^2 = {}^1\!/_2\{(S^2 + V_0^2) \pm \sqrt{(S^2 + V_0^2)^2 - 4S^2V_0^2\cos^2\vartheta}\},$$

where ϑ is the angle between the direction of wave propagation and the magnetic field $\mathbf{H}_0$. All three types of waves are excited in the medium in the general case in the presence of an external magnetic field.

Starting out with $\mathbf{v}$ from (7) and using Eq. (6), we get the following general expression for the intensity of radiation of the three types of waves per element of solid angle do:

$$dI = 8\pi^5\mu\frac{V_0^2}{c^2}\omega_0^2\left\{\left|j_\perp\left(\frac{\omega_0}{u_1}, \vartheta, \varphi\right)\right|^2\frac{\cos^2\varphi}{u_1^3}\right.$$

$$+ \frac{u_2^2 - S^2\cos^2\vartheta}{u_2^2 - u_3^2}\left|j_\perp\left(\frac{\omega_0}{u_2}, \vartheta, \varphi\right)\right|^2\frac{\sin^2\vartheta}{u_2^3} \quad (9)$$

$$\left.+ \frac{S^2\cos^2\vartheta - u_3^2}{u_2^2 - u_3^2}\left|j_\perp\left(\frac{\omega_0}{u_3}, \vartheta, \varphi\right)\right|^2\frac{\sin^2\varphi}{u_3^3}\right\}do,$$

where $\mathbf{j}_\perp(\mathbf{k}, \vartheta, \varphi)$ is the Fourier component of the component of the current density in the plane perpendicular to the direction of the magnetic field $\mathbf{H}_0$; φ is the angle between two planes, one defined by $\mathbf{j}_0$ and $\mathbf{H}_0$, and the other by the direction of wave propagation and $\mathbf{H}_0$; ω_0 is the frequency of the external current (we consider the external current $\mathbf{j}_0$ to be a harmonic function of the time).

The first component in (9) determines the intensity of excitation of hydromagnetic waves with phase velocity u_1, the second and third components the intensities of excitation of the magnetoacoustic waves with phase velocities u_2 and u_3.

We note that only the excitation of hydromagnetic waves is possible in an incompressible fluid, wherein the intensity of radiation is equal to

$$dI = 8\pi^5 \mu \frac{\omega_0^2 V_0^2}{c^2 u_1^3} \left| j_\perp\left(\frac{\omega_0}{u_1}, \vartheta, \varphi\right)\right|^2 do. \quad (10)$$

4. Let us now consider several special cases.

(a) Surface current. If the surface current

$$\mathbf{j}_0 = \mathbf{j}_S \delta(z) e^{-i\omega_0 t}, \quad (11)$$

exists in a plane perpendicular to the magnetic field $\mathbf{H}_0$, then only hydromagnetic waves will be propagated. These travel along the field. The total radiation intensity of these waves for a unit surface current is equal to

$$I_S = \pi\mu V_0 c^{-2} j_S^2. \quad (12)$$

We note that the radiation intensity does not depend on the frequency of excitation of the current.

(b) Line current. In this case,

$$\mathbf{j}_0 = \mathbf{j}_L \delta(x)\delta(z) e^{-i\omega_0 t},$$

and the total intensity of radiation of the hydromagnetic waves per unit length is determined by the expression

$$I_L = (\pi\mu\omega_0 / 2c^2) j_L^2. \quad (13)$$

The radiation intensity is proportional to the frequency of the current.

(c) Current loop. In this case,

$$j_0 = j_C \frac{\delta(\rho - a)}{2\pi\rho} \delta(z) e^{-i\omega_0 t},$$

and only the magnetoacoustic waves are excited. In such a case the radiation intensity is equal to

$$dI_C = \frac{\mu\omega_0^2}{8\pi c^2} V_0^2 j_C^2 \left\{ J_1^2\left(\frac{a\omega_0}{u_2}\sin\vartheta\right) \frac{u_2^2 - S^2\cos^2\vartheta}{u_2^3(u_2^2 - u_3^2)} + J_1^2\left(\frac{a\omega_0}{u_2}\sin\vartheta\right) \frac{S^2\cos^2\vartheta - u_3^2}{u_3^3(u_2^2 - u_3^2)}\right\} do. \quad (14)$$

We can find the total intensity in two limiting special cases: if $V_0^2 \gg S^2$, then

$$u_2^2 = V_0^2 + S^2\sin^2\vartheta, \quad u_3^2 = S^2\cos^2\vartheta.$$

Here only the magnetoacoustic wave with phase velocity u_2 is excited. The angular distribution of the radiation is determined from the formula

$$dI_C = \frac{\mu\omega_0^2}{8\pi V_0 c^2} j_C^2 J_1^2\left(\frac{a\omega_0}{V_0}\sin\vartheta\right) do.$$

If the inequality $a\omega_0/V_0 \ll 1$ is satisfied, the total radiation intensity becomes

$$I_C = (\mu a^2\omega_0^4 / 12c^2V_0^3) j_C^2. \quad (15)$$

If $V_0^2 \ll S^2$, then

$$u_2^2 = S^2 + V_0^2\sin^2\vartheta, \quad u_3^2 = V_0^2\cos^2\vartheta.$$

Here only waves with the phase velocity u_3 are excited. The angular distribution of the radiation of these waves is determined from the relation

$$dI_C = \frac{\mu\omega_0^2}{8\pi c^2 V_0} j_C^2 J_1^2\left(\frac{a\omega_0}{V_0}\tan\vartheta\right)\tan\vartheta\, do.$$

The total radiation intensity is equal to

$$I_C = \frac{\mu\omega_0^2}{2c^2V_0} j_C^2 I_1\left(\frac{a\omega_0}{V_0}\right) K_1\left(\frac{a\omega_0}{V_0}\right).$$

If $a\omega_0/V_0 \ll 1$, then

$$I_C = (\mu\omega_0^2 / 4c^2 V_0) j_C^2. \quad (16)$$

5. We now consider the effect of finite conductivity and the viscosity of the fluid on the excitation of hydromagnetic waves. Limiting ourselves for simplicity to the case of an incompressible fluid, we have the following system of equations for the velocity $\mathbf{v}$ and magnetic field $\mathbf{h}$:

$$\frac{\partial \mathbf{v}}{\partial t} = \nu\nabla\nabla\cdot\mathbf{v} + \frac{\mu}{4\pi\rho_0}[\mathrm{curl}\,\mathbf{h} \times \mathbf{H}_0] - \frac{\mu}{c\rho_0}[\mathbf{j}_0 \times \mathbf{H}_0],$$
$$\frac{\partial \mathbf{h}}{\partial t} = \mathrm{curl}\,[\mathbf{v}\times\mathbf{H}_0] - \frac{c^2}{4\pi\mu\sigma}\mathrm{curl}\,\mathrm{curl}\,\mathbf{h} + \frac{c}{\mu\sigma}\mathrm{curl}\,\mathbf{j}, \quad (17)$$

where ν is the viscosity of the liquid.

The radiation intensity can be found from the formula

$$I = -\int \mathbf{E}\cdot\mathbf{j}_0 dv,$$

where the intensity of the electric field $\mathbf{E}$ is equal to

$$\mathbf{E} = -\frac{\mu}{c}[\mathbf{v}\times\mathbf{H}_0] + \frac{c}{4\pi\sigma}\mathrm{curl}\,\mathbf{h} - \frac{\mathbf{j}_0}{\sigma}.$$

We can show that the intensity of the radiation of the hydromagnetic waves will be determined by the expression

$$dI = 16\pi^4 \int_0^\infty \left\{ \frac{\mu c^{-2}\omega_0^2 k^2(\nu + c^2/4\pi\mu\sigma)}{[\omega^2 - (\mathbf{k}\cdot\mathbf{V}_0)^2 - c^2k^4\nu/4\pi\mu\sigma]^2 + \omega_0^2 k^4(\nu + c^2/4\pi\mu\sigma)^2} |[\mathbf{V}_0 \times \mathbf{j}_0(\mathbf{k})]|^2 \right. \quad (18)$$
$$\left. + \frac{\omega_0^2(\mathbf{k}\cdot\mathbf{V}_0)(1/4\pi\sigma)[\omega_0^2 - (\mathbf{k}\cdot\mathbf{V}_0)^2 - c^2k^4\nu/2\pi\mu\sigma - (c^2k^2/4\pi\mu\sigma)^2]}{[\omega_0^2 + (c^2k^2/4\pi\mu\sigma)^2]\{[\omega^2 - (\mathbf{k}\cdot\mathbf{V}_0)^2 - c^2k^4\nu/4\pi\mu\sigma]^2 + \omega_0^2k^4(\nu + c^2/4\pi\mu\sigma)^2\}} |\mathbf{j}_0(\mathbf{k})|^2 + \frac{\omega_0^2/4\pi\sigma}{\omega_0^2 + (c^2k^2/4\pi\mu\sigma)^2}|\mathbf{j}_0(\mathbf{k})|^2 \right\} k^2\, dk\, do.$$

In a viscous liquid with finite conductivity, the harmonic current with frequency ω_0 [in contrast to (1)] excites hydromagnetic waves with different frequencies $\omega = ku_1$. The spectral distribution of the hydromagnetic waves radiated is determined from (18).

Making use of the formula

$$\delta(x) = \lim \frac{1}{\pi} \frac{\alpha}{\alpha^2 + x^2},$$

it is easy to show that for $\nu \rightarrow 0$ and $\sigma \rightarrow \infty$, Eq. (18) goes over into (10).

6. We now compare the intensity of the excitation of hydromagnetic waves by currents with the intensity of excitation of hydromagnetic waves by mechanical means, in which case, for a certain plane perpendicular to the magnetic field $\mathbf{H}_0$ (the plane $z = 0$), the velocity of the liquid perpendicular to the magnetic field is given by

$$\mathbf{v} = \mathbf{v}_0 e^{-i\omega_0 t}, \quad \mathbf{v}_0 \perp \mathbf{H}_0, \quad z = 0.$$

Assuming $\mathbf{v}, \mathbf{h} \sim \exp\{-i\omega_0 (t - z/V_0)\}$, and taking the liquid to be incompressible, we get, by (3) and (5),

$$\mathbf{E} = -\frac{\mu}{c}[\mathbf{v} \times \mathbf{H}_0], \quad \mathbf{h} = -\sqrt{\frac{4\pi\rho_0}{\mu}}\mathbf{v}. \tag{19}$$

The energy flow is determined primarily by the flow of electromagnetic energy

$$I = \frac{c}{8\pi} \operatorname{Re}\,[\mathbf{E} \times \mathbf{h}^*].$$

Substituting this expression in (19), we get

$$I = {}^1/_2\, \rho_0 V_0 v_0^2. \tag{20}$$

A comparison of (20) with Eq. (12) shows that the surface current j_S is equivalent to a velocity

$$v_0 = \sqrt{2\pi\mu / \rho_0}\, j_S / c$$

from the viewpoint of the excitation of hydromagnetic waves.

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QUANTUM-MECHANICAL PROBABILITIES AS SUMS OVER PATHS

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A new formulation of nonrelativistic quantum mechanics is proposed, namely a general definition of the probability of any event. The physical content of quantum mechanics is reduced to a single principle similar to the principle of Gibbs; this makes it possible to solve problems without resorting to the use of wave functions and operators.

THE idea that there may exist in quantum mechanics a general expression for the probability amplitude of any event is due to Feynman.[1] These amplitudes are multiplied and combined like classical probabilities; this leads to the idea of constructing quantum mechanics according to the model of classical statistical physics. In statistical physics the probability of finding a system to have some given property is equal to the sum over all configurations having this property; each configuration is used with the weight assigned by Gibbs. In quantum mechanics the role of the configurations is played by the paths of the particle; according to Feynman's idea the probabilities are replaced by amplitudes. A simple and complete "atomistic" description is obtained (see Sec. 8).

This program has not, however, been completely carried out. According to Feynman, the amplitude of any state must be the sum over all paths consistent with the conditions of the experiment, but,