Letters to the Editor

EXACT NONLINEAR GRAVITATIONAL EQUATIONS FOR A SPECIAL CASE ON THE BASIS OF BIRKOFF'S THEORY

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BIRKHOFF's linear theory of gravitation,[1-4] unlike the general theory of relativity, is based on the general premises of modern field theory. This theory, like the general theory of relativity, predicts a number of observable effects (the deflection of light rays in a gravitational field, the red shift, and the advance of the perihelion) in good agreement with experiment.* We select as initial equations

$$\partial h_{\mu[\nu\lambda]} / \partial x_\lambda = -\alpha T_{\mu\nu}, \quad (\alpha = G/c^4); \tag{1}$$

$$\partial h_{\lambda[\nu\rho]} / \partial x_\mu + \partial h_{\lambda[\rho\mu]} / \partial x_\nu + \partial h_{\lambda[\mu\nu]} / \partial x_\rho = 0, \tag{2}$$

where $T_{\mu\nu}$ is the symmetrized momentum-energy tensor of gravitating matter.

Equation (2) can be replaced by the equivalent definitions of $h_{\mu[\nu\rho]}$ in terms of the potentials $h_{\mu\nu}$:

$$h_{\mu[\nu\rho]} = \partial h_{\mu\rho} / \partial x_\nu - \partial h_{\mu\nu} / \partial x_\rho. \tag{3}$$

Equations (1) and (3) lead to the potential equation

$$\Box^2 h_{\mu\nu} = \alpha T_{\mu\nu}, \tag{4}$$

if we take into account the additional Lorentz-type condition that follows from conservation of $T_{\mu\nu}$:

$$\partial h_{\mu\lambda} / \partial x_\lambda = 0. \tag{5}$$

This theory is a good approximation under ordinary conditions because of the smallness of α, but it is still only an approximation because the field $h_{\mu\nu}$ itself possesses energy and momentum that produce a gravitational effect. This inherent nonlinearity of the gravitational field can be taken into account only by perturbation theory, since the exact nonlinear equations are unknown. We shall not attempt here to treat Birkhoff's theory as a limiting form of the general theory of relativity for slightly curved space (which is shown to be justified by subsequent calculation). We shall show that in the special case of static fields of masses the infinite expansions of perturbation theory can be put into closed form. This procedure results in exact nonlinear equations of the generalized Birkhoff theory.

In the absence of an external $T_{\mu\nu}$, Eq. (4) is obtained from the linear Lagrangian

$$\mathscr{L}_L = -{}^1/_4 \delta_{\lambda\sigma} (\partial h_{\rho\tau} / \partial x_\lambda)(\partial h_{\rho\tau} / \partial x_\sigma). \tag{6}$$

The Lagrangian of the nonlinear field, with the self-force taken into account, must be

$$\mathscr{L}_{NL} = \mathscr{L}_L + (\alpha/2) h_{\lambda\sigma} T^{(NL)}_{\lambda\sigma}, \tag{7}$$

where $T^{(NL)}_{\mu\nu}$ is the unknown momentum-energy tensor of the nonlinear gravitational field. This can be defined in terms of the Lagrangian $\mathscr{L}_{NL}$, which is also unknown, in the usual form

$$T^{(NL)}_{\mu\nu} = (\partial h_{\rho\tau} / \partial x_\mu)(\partial \mathscr{L}_{NL} / \partial(\partial h_{\rho\tau} / \partial x_\nu)) - \delta_{\mu\nu} \mathscr{L}_{NL}, \tag{8}$$

thus giving an equation for $\mathscr{L}_{NL}$:

$$\mathscr{L}_{NL} = \mathscr{L}_L + (\alpha/2) h_{\lambda\sigma} ((\partial h_{\rho\tau} / \partial x_\lambda) \partial \mathscr{L}_{NL} / \partial(\partial h_{\rho\tau} / \partial x_\sigma) - \delta_{\lambda\sigma} \mathscr{L}_{NL}). \tag{9}$$

This equation can be solved by representing $\mathscr{L}_{NL}$ as a power series in $\alpha/2$:

$$\mathscr{L}_{NL} = \mathscr{L}_L + (\alpha/2) \mathscr{L}_1 + (\alpha/2)^2 \mathscr{L}_2 + \cdots, \tag{10}$$

from which we obtain for $\mathscr{L}_k$:

$$\mathscr{L}_k = -{}^1/_4 a^{(k)}_{\lambda\sigma} (\partial h_{\rho\tau} / \partial x_\lambda)(\partial h_{\rho\tau} / \partial x_\sigma), \tag{11}$$

$$a^{(k)}_{\mu\nu} = h_{\mu\lambda} a^{(k-1)}_{\lambda\nu} + a^{(k-1)}_{\mu\lambda} h_{\lambda\nu} - h_{\lambda\lambda} a^{(k-1)}_{\mu\nu}. \tag{12}$$

Thus

$$\mathscr{L}_{NL} = -{}^1/_4 a_{\lambda\sigma}(h_{\mu\nu})(\partial h_{\rho\tau} / \partial x_\lambda)(\partial h_{\rho\tau} / \partial x_\sigma), \tag{13}$$

where $a_{\mu\nu}$ is a series like (10), with the coefficients $a^{(k)}_{\mu\nu}$ ($a^{(0)}_{\mu\nu} \equiv \delta_{\mu\nu}$).

For fields produced by masses

$$h_{\mu\nu} = h\delta_{\mu\nu}, \quad h_{\mu[\nu\rho]} = (\delta_{\mu\rho}\delta_{\nu\lambda} - \delta_{\mu\nu}\delta_{\rho\lambda}) \partial h / \partial x_\lambda \tag{14}$$

and all the series can be summed. For a static gravitational field that takes account of the nonlinear self-force we obtain

$$\nabla^2 h(\mathbf{x}) + (\alpha/2)(\nabla h(\mathbf{x}))^2 / (1 - \alpha h(\mathbf{x})) = 0. \tag{15}$$

The solution will always be a function of the linear solution $h_L(\mathbf{x}) = MG/c^2|\mathbf{x}|$ in the form

$$h(\mathbf{x}) = (c^4/G)(1 - (MG^2/2c^6|\mathbf{x}| - 1)^2). \tag{16}$$

For $|\mathbf{x}| \to \infty, h \to h_L$ with $x_0' = MG^2/2c^6$ the potential is at its maximum and equal to c^4/G; $h = 0$ when $x_0'' = MG^2/4c^6$ and $h \to -\infty$ when $|\mathbf{x}| \to 0$. The phase plane of Eq. (15) contains no solutions of another kind. It is easily seen that the solution (16) cannot be reduced to Schwarzschild's solution, which follows from the general theory of relativity.

*There is an evident illusoriness in the currently widely discussed "test" of the general theory of relativity through measurement of the perihelion shifts of artificial satellites.[5] Such a test can provide no basis for a choice between the general theory of relativity and Birkhoff's theory.

[1] G. Birkhoff, Proc. Nat. Acad. Sci. **29**, 231 (1943); **30**, 324 (1944).

[2] Barajas, Birkhoff, Graef and Vallarta, Phys. Rev. **66**, 138 (1944).

[3] A. Barajas, Proc. Nat. Acad. Sci. **30**, 54 (1944).

[4] G. Birkhoff, Bull, Soc. Mat. Mexicana **1**, 1 (1944).

[5] V. Ginzburg, Usp. Fiz. Nauk **63**, 119 (1957).

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ON THE THEORY OF THE POSITIVE COLUMN IN AN ELECTRONEGATIVE GAS

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1. In a positive column where negative ions are produced spatially and disappear at the wall, their relative concentration, which was obtained in reference 1, satisfies the condition $\kappa > D_e/2D_p - 1$,* where D_e and D_p are the diffusion coefficients of electrons and positive ions. This condition is required for the flow of negative ions to the wall, where they recombine. The wall is a surface sink for the negative ions produced within the volume.

2. When decay of the negative ions through collisions with neutral atoms[2] is included, the situation becomes somewhat more complicated, but we still have a linear problem which can be completely solved. For ambipolar diffusion, subject to the assumptions $D_e \gg D_p$, $D_p \approx D_n$, $b_e \gg b_p$ ahd $b_p \approx b_n$, we obtain an equation† for the concentration of negative ions in the column:

$$\frac{2\kappa(1+\kappa)D_p/D_e - \kappa}{1+2\kappa} = \frac{\beta - \gamma\kappa}{Z-(\beta-\gamma\kappa)},$$

where γ is the rate of decay of negative ions per ion and the rest of the notation is that used in reference 1. This is a cubic equation in κ which can be solved as follows:

(a) $\kappa > D_e/2D_p - 1$. With this concentration more negative ions are created per unit volume than decay, so that the column is a spatial source of negative ions. The negative ions produced in the column diffuse to the wall, which serves as a surface sink.

(b) $\kappa = D_e/2D_p - 1$. In this case the negative ions created by electrons adhering to neutral atoms equals those vanishing through decay. There is no effective resulting creation or disappearance of negative ions in the column. Their radial flow is zero and the total number of negative ions in the column is determined only by processes in the space.

(c) $\kappa < D_e/2D_p - 1$. Here the number of negative ions disappearing from the column exceeds the number produced, so that the column is a spatial sink for negative ions. Their radial flow is directed toward the axis and a stationary state is possible only in the presence of a surface source at the wall.‡

Thus in a positive column, where the disappearance of negative ions obeys a linear law, small values of κ are possible when there is near the surface of the wall** a layer that produces a flow of negative ions into the column, where they disappear through decay as a result of collisions with neutral particles. Our conclusion that a surface source exists agrees with Günterschulze's hypothesis[4] of a layer of negative ions at the wall.

3. When we take into account the spatial recombination of positive and negative ions, we can in general distinguish two regions of the column, an inner region where recombination predominates over creation, and an outer region where creation is stronger than recombination. We are here concerned with effective spatial sources and sinks.†† Since the equations of balance are nonlinear the problem can be solved more or less simply only for regions close to the axis of the discharge.[1]

*In reference 3 it was assumed that $\varkappa \ll 1$ in the absence of spatial disappearance of negative ions. This is unacceptable to us, since $\varkappa$ is a solution of the system and is fully determined by the kinetics of the column. The analysis carried out in reference 1 and in the present note shows that $\varkappa$ can be small only when a spatial loss occurs.

†The solution is obtained as in reference 1.

‡The strength of the surface source per unit length of the column is given by the integral $2\pi \int_0^R (\beta - \gamma\varkappa) N_e r dr$, where N_e is the electron density.