OBLIQUE SHOCK WAVES IN A PLASMA WITH FINITE CONDUCTIVITY

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The structure of an oblique shock wave in a plasma with finite conductivity is considered neglecting its viscosity and thermal conductivity. The conditions of applicability of the approximation are obtained. An estimate of the width of the wave front is given. The limiting angle for the propagation of an oblique shock wave is obtained in a plasma of infinite conductivity.

1. BOUNDARY CONDITIONS

THE structure of the front of a normal shock wave in a plasma of finite conductivity has been investigated neglecting viscosity and thermal conductivity in the paper by Golitsyn and Staniukovich.[1] In this paper we consider the problem of the structure of the front of an oblique shock wave for an arbitrary orientation of the field ahead of the front in the same kind of plasma.

We consider a plasma with a constant and isotropic conductivity sufficiently large not to have to take displacement current into account. Let us determine the conditions for neglecting the kinematic viscosity ν and the electronic thermal conductivity κ in comparison with the magnetic viscosity ν_m in the system of equations of magneto-hydrodynamics (cf., for example, Syrovatskii[2]).

As is well known (cf. Chapman and Cowling[3]),

$$\nu_m = c^2/4\pi\sigma = c^2/4\pi e^2 n_e \tau_{ei} = c^2/\omega_0^2\tau_{ei}; \quad (1)$$

$$\nu = 3a^2\tau_{ii}/2\gamma, \quad (2)$$

where $\omega_0^2 = 4\pi n_e e^2/m_e$ is the characteristic plasma frequency, $\gamma = c_p/c_v$ is the ratio of specific heats, a is the velocity of sound, τ_{ei} and τ_{ii} are the times between electron-ion and ion-ion collisions respectively

$$\tau_{ei} = m_e^{1/2}(3kT)^{3/2}/0.7\cdot 4\pi n_e e^4 \ln\lambda_e,$$
$$\tau_{ii} = m_i^{1/2}(3kT)^{3/2}/0.7\cdot 8\pi n_i e^4 \ln\lambda_i. \quad (3)$$

Here

$$\lambda = (3/2e^3)(k^3T^3/\pi n)^{1/2} = h/p_0,$$

where h is the Debye screening radius, p_0 is the impact parameter for which an electron on colliding with an ion is deflected by 90° (for details see Spitzer[4]); n_e and n_i are the electron and ion densities respectively. We assume everywhere in the following that $n_e = n_i = n$.

If the kinetic and magnetic energies satisfy the inequality $\rho v^2 \leq H^2/4\pi$, then the condition $\nu_m \gg \nu$ assumes the form:

$$\frac{S^2}{r_0 v_{av}}\left(\frac{m_i}{m_e}\right)^{1/2} \gg 1, \qquad S = \frac{1}{\pi}\left(\frac{e^2}{kT}\right)^4 \ln\lambda. \quad (4)$$

Here S is the electron-ion collision cross-section; $r_0 = e^2/m_e c^2$ is the classical electron radius, $1/n = v_{av}$ is the average volume per particle. For $n = 10^{16}\,\text{cm}^{-3}$, $T = 10^4\,°K$, $m_i = 3.2\times 10^{-24}$ gm the left-hand side of the inequality is $\sim 10^4$.

A similar condition for neglecting the electronic thermal conductivity $\kappa = 20(2/\pi)^{3/2} kT/m^{1/2} e^4 \ln\lambda$ (cf. Spitzer[4]) can be written in the case $nkT \leq H^2/4\pi$ in the form:

$$S^2/22.5 r_0 v_{av} \gg 1. \quad (5)$$

The system of equations of magneto-hydrodynamics has the following particular integrals of the motion if all the quantities depend on only one coordinate x:

$$[\mathbf{v}\times\mathbf{H}]_{y,z} - \nu_m \mathbf{curl}_{y,z}\mathbf{H} = cE_{y,z} = \text{const}, \quad H_x = \text{const}.$$
$$\rho v_x = \text{const} = m, \quad g_x = \text{const} = \varepsilon, \quad \pi_{ix} = \text{const} = I_i, \quad (6)$$

where g_x is the energy flux density, while π_{ix} is the momentum flux density.

2. THE STRUCTURE OF AN OBLIQUE SHOCK WAVE

We consider an oblique shock wave in a plasma of finite conductivity. The parameters of the medium ahead of the front we shall denote by the subscript 1, while those for the medium inside the transition layer and behind the front we shall denote by 2. If $v_{1z} = 0$, $H_{1z} = 0$ then the system (6) assumes the form:

$$\rho_1 v_{1x} = \rho_2 v_{2x} = m, \qquad H_{1x} = H_{2x} = H_x,$$
$$v_{1x}H_{1y} - v_{1y}H_{1x} = v_{2x}H_{2y} - v_{2y}H_{2x} - \nu_m \frac{dH_{2y}}{dx} = cE_z,$$
$$\rho_1 v_{1x} v_{1y} - H_{1x}H_{1y}/4\pi = \rho_2 v_{2x} v_{2y} - H_{2x}H_{2y}/4\pi = \alpha,$$
$$\rho_1 v_{1x}\left(\frac{v_1^2}{2} + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1}\right) + \frac{cE_{z1}}{4\pi} H_{1y}$$
$$= \rho_2 v_{2x}\left(\frac{v_2^2}{2} + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2}\right) + \frac{cE_{z2}}{4\pi} H_{2y} = \varepsilon,$$
$$p_1 + H_1^2/8\pi + \rho_1 v_{1x}^2 = p_2 + H_2^2/8\pi + \rho_2 v_{2x}^2 = I. \tag{7}$$

By solving the system for dH_{2y}/dx, and by eliminating all the variables except for H_{2y}, we obtain the following equation:

$$\nu_m \frac{dH_{2y}}{dx} = \frac{\gamma}{\gamma-1}\frac{H_{2y}}{m}\left(I - \frac{H_{2y}^2}{8\pi}\right) - \frac{H_x}{m}\left(\alpha + \frac{H_x}{4\pi}H_{2y}\right) - cE_z$$
$$+ H_{2y}\left\{\frac{\gamma^2}{(\gamma-1)^2 m^2}\left(I - \frac{H_{2y}^2}{8\pi}\right)^2\right. \tag{8}$$
$$\left. + 2\frac{\gamma-1}{\gamma+1}\left[\frac{1}{2m^2}\left(\alpha + \frac{H_x H_{2y}}{4\pi}\right)^2 - \frac{\varepsilon}{m} + \frac{cE_z}{4\pi m}H_{2y}\right]\right\}^{1/2} \equiv f(H_{2y}).$$

Hence

$$x - x_0 = \nu_m \int_{H_{2y_0}}^{H_{2y}} dH_{2y}/f(H_{2y}). \tag{9}$$

The integral (9) is obtained by numerical methods.

Let the wave be propagated in monatomic deuterium whose initial parameters are given by:

$$T_1 = 2\cdot 10^4\,{}^\circ\mathrm{K}, \qquad p_1 = 2\cdot 10^4 \text{ g/cm-sec}^2,$$
$$v_{1x} = 2.36\cdot 10^6 \text{ cm/sec}, \quad H_{1y} = 700 \text{ oersted}, \tag{10}$$
$$v_{1y} = v_{1x}\tan\chi, \qquad H_{1y} = H_{1x}\tan\varphi,$$

and let $\varphi = \chi = 45°$ to simplify the calculations. The results are given in graphical form (Fig. 1), where the dimensionless quantity $\xi = xa/\nu_m$ is plotted along the horizontal axis.

The width of the front may be defined by

$$\Delta x = \Delta H_y / (dH_y/dx)_{\max}. \tag{11}$$

In the present case the width of the front is given in dimensionless units by $\Delta\xi = 6$. On calculating the number $k = \Delta x/l$ of mean free paths $l = \tau_{ii}/a$ contained in the width of the wave front we obtain

$$\Delta x = \nu_m \Delta\xi/a = c^2\Delta\xi/a\omega_0^2\tau_{ei}. \tag{12}$$

Finally

$$k = c^2\Delta\xi/\omega_0^2 a^2 \tau_{ei}\tau_{ii} \gg 1 \tag{13}$$

according to the inequality (6). In the case considered above $k \sim 10^4$.

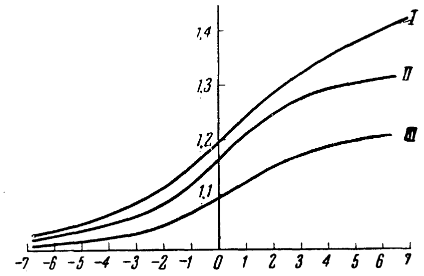

FIG. 1. Curve I shows the dependence of p/p_1 on ξ, curve II shows H_y/H_{1y}, curve III shows ρ/ρ_1. $p_{2\infty}/p_1 = 1.45$, $H_{2y\infty}/H_{1y} = 1.32$; $\rho_{2\infty}/\rho_1 = 1.22$, where the subscript ∞ refers to the limiting values of the quantities behind the wave front.

3. THE LIMITING ANGLE FOR THE PROPAGATION OF AN OBLIQUE SHOCK WAVE IN A PLASMA OF INFINITE CONDUCTIVITY

The following system is equivalent to an equation of the type which holds for a shock wave in an ideally conducting plasma with $H_{1z} = 0$:

$$v_{2x} = \frac{4\pi\,(mcE_z + \alpha H_x) + H_{2y}H_x^2}{4\pi m H_{2y}}, \quad v_{2y} = \frac{(\alpha + H_x H_{2y}/4\pi)}{m}, \tag{14}$$

where the parameter H_{2y} is the first root of the equation

$$\frac{\gamma}{\gamma-1}\left(I - \frac{y^2}{8\pi}\right)\frac{y}{m} + y\left\{\frac{\gamma^2}{(\gamma-1)^2}\left(I - \frac{y^2}{8\pi}\right)^2\right.$$
$$\left. + 2\frac{\gamma-1}{\gamma+1}\left[\frac{1}{2m^2}\left(\alpha + \frac{H_x}{4\pi}y\right) - \frac{\varepsilon}{m} + \frac{cE_z}{4\pi m}y\right]\right\}^{1/2}$$
$$- \frac{H_x}{m}\left(\alpha + \frac{H_x}{4\pi}y\right) - \frac{cE_z}{m} = 0,$$

which satisfies the condition $y \geq H_{1y}$.

In order to find $v_{2y}\,\varphi(v_{2x})$ it is necessary to vary $\mathbf{v}_1$ keeping v_1 constant. Variation of the magnitude of $\mathbf{H}_1$ and of the angle at which $\mathbf{H}_1$ is inclined with respect to the front will yield a two-parametric family of shock waves.

It is possible to obtain an expression for the limiting angle of tilt of an oblique shock wave in a certain special case when $\mathbf{H}_1(0, H_{1y}, H_{1z})$, $\mathbf{v}_1(v_{1x}, 0, 0)$, i.e. when the coordinate system is so chosen that the wave front is at rest in it, and makes a certain angle χ with the YZ plane (cf. Fig. 2).

In order to obtain boundary conditions for this problem we rotate the original system of coordinates by an angle χ about the Z axis. Then

$$\rho_1 v_{1x} = \rho_2 (v_{2x} - v_{2y}\tan\chi); \tag{15}$$

$$H_{2x} = H_{2y}\tan\chi; \tag{16}$$

$$v_{1x}H_{1z} = H_{2z}(v_{2x} - v_{2y}\tan\chi), \quad v_{1x}H_{1y} = H_{2y}(v_{2x} - v_{2y}\tan\chi); \tag{17}$$

$$\frac{v_1^2}{2} + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \frac{H_{1y}^2}{4\pi\rho_1\cos^2\chi} - \frac{H_{1z}^2}{4\pi}$$

$$= \frac{v_2^2}{2} + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} + \frac{H_{1y}H_{2y}}{4\pi\rho_1\cos^2\chi} + \frac{H_{1z}H_{2z}}{4\pi\rho_1}; \tag{18}$$

$$p_1 + \rho_1 v_{1x}^2\cos^2\chi + \frac{H_{1y}^2}{8\pi\cos^2\chi} + \frac{H_{1z}^2}{8\pi}$$

$$= p_2 + \rho_1 v_{1x}\cos^2\chi\,(v_{2x} - v_{2y}\tan\chi) + \frac{H_{2y}^2}{8\pi\cos^2\chi} + \frac{H_{2z}^2}{8\pi}; \tag{19}$$

$$v_{1x}\tan\chi = v_{2x}\tan\chi + v_{2y}, \quad v_{2z} = 0. \tag{20}$$

For $v_{2x} \to v_{1x}$, $v_{2y} \to 0$ we have $H_{2y} \to H_{1y}$, $H_{2z} \to H_{1z}$, $\rho_2 \to \rho_1$, $p_2 \to p_1$ and from (20):

$$(dv_{2y}/dv_{2x})_{v_{2x}\to v_{1x}} \to -\tan\chi_0,$$

where χ_0 is the limiting angle for the propagation of an oblique shock wave. From (15), (17) and (19) we obtain ρ_2, H_{2y}, H_{2z} and p_2 as functions of v_{2y} and v_{2x}. On substituting these values into (18) we obtain the implicit relation $f(v_{2x}, v_{2y}) = 0$. On differentiating it with respect to v_{2x}, and then after setting $v_{2x} \to v_{1x}$, $v_{2y} \to 0$ we obtain

$$\cos^2\chi_0 = \frac{1}{2}\left(\frac{a_1^2}{v_1^2} - \frac{H_{1z}^2}{4\pi\rho_1 v_1^2}\right)$$

$$\pm\frac{1}{2}\left\{\left(\frac{a_1^2}{v_1^2} - \frac{H_{1z}^2}{4\pi\rho_1 v_1^2}\right)^2 - \frac{2\gamma-1}{\pi\rho_1 v_1^2}H_{1y}^2\right\}^{1/2}, \tag{21}$$

where $a_1^2 = \gamma p_1/\rho_1$ is the velocity of sound in the medium ahead of the front. It is necessary to retain the plus sign in formula (21) as is evident from a transition to ordinary hydrodynamics ($H = 0$) where $\cos^2\chi_0 = a_1^2/v_1^2$.

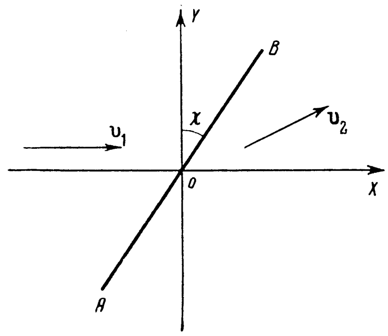

FIG. 2. AB is the front of an oblique shock wave, v_1 is the velocity ahead of the front, v_2 is the velocity behind the front.

It is clear from formula (21) that the limiting angle for the propagation of oblique shock waves in the presence of a magnetic field is greater than in the field-free case.

In conclusion we consider it to be our pleasant duty to express our gratitude to K. P. Staniukovich for suggesting the problem and for constant interest in the work.

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