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THE CAUSALITY CONDITION AND SPECTRAL REPRESENTATIONS OF GREEN'S FUNCTIONS

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By means of the causality condition in the form of the requirement that field operators commute on a space-like surface, spectral representations are obtained for the vacuum expectation values of T-products of three Heisenberg operators. The analytic properties of these functions in the complex plane are discussed.

THE present paper presents a method for obtaining spectral representations for the vacuum expectation values of T-products of Heisenberg operators (Green's functions).

These representations [Eqs. (8), (9), (18), and (19)], being natural extensions of the Källén-Lehmann formulas[1,2] for the vacuum expectation values of T-products of two operators, provide a convenient means for investigating the analytic properties of these functions in the complex plane.

In the present paper, which is the first installment of this work, spectral representations are obtained for vacuum expectation values of T-products of three operators, and their analytic properties are studied.

I. THE CASE OF THE SCALAR FIELD

1. Derivation of the Spectral Representation

For simplicity we first consider the Green's function constructed from three scalar operators $\varphi(x_1)$, $\varphi(x_2)$, $\varphi(x_3)$,

$$\langle T\varphi(x_1)\varphi(x_2)\varphi(x_3)\rangle.$$

The vacuum expectation value of the simple (unordered) product of these operators can be written in the form

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\rangle$$

$$= \frac{1}{(2\pi)^9}\int d^4p_1 d^4p_3 e^{ip_1(x_1-x_2)+ip_3(x_2-x_3)}\vartheta(p_{10})\vartheta(p_{30})\rho(p_1^2, q^2, p_3^2),$$

$$\vartheta(p_{10})\vartheta(p_{30})\rho(p_1^2, q^2, p_3^2) = (2\pi)^3\sum\varphi_{0p_1}\varphi_{p_1p_3}\varphi_{p_30}, \quad (1)$$

$$\varphi_{p_1p_3} = \langle p_1|\varphi(0)|p_3\rangle, \quad q^2 = (p_1-p_3)^2, \; \vartheta(x) = \begin{cases}1 & x>0\\ 0 & x<0\end{cases}$$

The summation is taken over all states with definite values of p_1 and p_3. If $x_{10} = x_{20}$, then the requirement

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\rangle = \langle\varphi(x_2)\varphi(x_1)\varphi(x_3)\rangle \quad (2)$$

imposes the following condition on the function $\rho(p_1^2, q^2, p_3^2)$:

$$\int dp_{10}\vartheta(p_{10})\rho(p_1^2, q^2, p_3^2)$$

$$= \int dp_{10}\vartheta(p_{10})\rho((\mathbf{p}_3-\mathbf{p}_1)^2 - p_{10}^2, \mathbf{p}_1^2 - (p_{30}-p_{10})^2, p_3^2). \quad (3)$$

Equation (3) is easily obtained from Eqs. (1) and (2) if one notes that for $x_{10} = x_{20}$ interchange of x_1 and x_2 is equivalent to replacement of the space components of the vector p_1 by $\mathbf{p}_3 - \mathbf{p}_1$.

A similar condition is obtained by considering the case $x_{20} = x_{30}$. These conditions are satisfied if we write $\rho(p_1^2, q^2, p_3^2)$ in the form

$$\vartheta(p_{10})\vartheta(p_{20})\rho(p_1^2, q^2, p_3^2)$$

$$= \int\vartheta(k_{10})\vartheta(k_{20})\vartheta(k_{30})f(-k_1^2, -k_2^2, -k_3^2)\,d^4l. \quad (4)$$

$$k_1 = {}^1/_2(-p_1+l+p_3), \quad k_2 = {}^1/_2(p_1-l+p_3),$$

$$k_3 = {}^1/_2(p_1+l-p_3)$$

and postulate that $f(-k_1^2, -k_2^2, -k_3^2)$ is a symmetric function of its arguments which vanishes if any of them is less than zero.

In fact, if we substitute Eq. (4) into Eq. (3) and in the resulting integral over l and p_{10} make the change of variables $\mathbf{l} = \mathbf{p}_3 - \mathbf{l}'$, $p_{10} = l'_{10}$, $l_{10} = p'_{10}$, then instead of Eq. (3) we get

$$\int dl\,dp_{10}[f(-k_1^2, -k_2^2, -k_3^2) - f(-k_2^2, -k_1^2, -k_3^2)]$$
$$\times\vartheta(k_{10})\vartheta(k_{20})\vartheta(k_{30}) = 0. \quad (5)$$

The condition that $f(-k_1^2, -k_2^2, -k_3^2)$ vanish for positive k_1^2, k_2^2, k_3^2 is necessary in order that the integrals containing $\vartheta(k_{10})$, $\vartheta(k_{20})$, $\vartheta(k_{30})$ be relativistically invariant.

Somewhat later (Sec. 2) we shall see that Eq. (4), regarded as an equation for $f(-k_1^2, -k_2^2, -k_3^2)$ for prescribed $\rho(p_1^2, q^2, p_3^2)$, has a very simple structure and determines f under sufficiently general assumptions regarding ρ. At present we shall assume that Eq. (4) is satisfied, and by considering the causality condition in invariant form we shall show that the symmetry of f is not only a sufficient but also a necessary condition for this equation to be true.

Substituting Eq. (4) into Eq. (1) and replacing $f(-k_1^2, -k_2^2, -k_3^2)$ by $\int d\kappa_1^2 d\kappa_2^2 d\kappa_3^2 f(\kappa_1^2, \kappa_2^2, \kappa_3^2) \times \delta(k_1^2+\kappa_1^2)\delta(k_2^2+\kappa_2^2)\delta(k_3^2+\kappa_3^2)$, we get

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\rangle = \int d\varkappa_1^2 d\varkappa_2^2 d\varkappa_3^2 f(\varkappa_1^2, \varkappa_2^2, \varkappa_3^2)$$
$$\times\Delta^+(x_{12}, \varkappa_3)\Delta^+(x_{13}, \varkappa_2)\Delta^+(x_{23}, \varkappa_1),$$
$$\Delta^+(x, \varkappa) = \frac{1}{(2\pi)^3}\int d^4k\vartheta(k)\delta(k^2+\varkappa^2)e^{ikx}. \quad (6)$$

If the interval x_{12}^2 is space-like, then by Eqs. (2) and (6) we have $\Delta^+(x_{12}, \kappa_3) = \Delta^+(x_{21}, \kappa_3)$ and we get

$$\int d\varkappa_1^2 d\varkappa_2^2 d\varkappa_3^2\Delta^+(x_{12}, \varkappa_3)\Delta^+(x_{13}, \varkappa_2)\Delta^+(x_{23}, \varkappa_1)[f(\varkappa_1^2, \varkappa_2^2, \varkappa_3^2) - f(\varkappa_2^2, \varkappa_1^2, \varkappa_3^2)] = 0. \quad (7)$$

Since the expression (7) must vanish for arbitrary x_{12}^2, x_{23}^2, x_{13}^2, restricted only by weak inequalities (for example $x_{12}^2 > 0$, $x_{13}^2 < 0$, $x_{23}^2 < 0$), $f(\kappa_1^2, \kappa_2^2, \kappa_3^2)$ is a symmetric function.

Possessing the representation (6) and the symmetry property of $f(\kappa_1^2, \kappa_2^2, \kappa_3^2)$, we can easily write out the representation for $\langle T\varphi(x_1)\varphi(x_2) \times \varphi(x_3)\rangle$. Using the definition of the T-product and the relation

$$^1/_2\Delta_F(x, \varkappa) = \vartheta(x)\Delta^+(x, \varkappa) + \vartheta(-x)\Delta^+(-x, \varkappa),$$

we get

$$\langle T\varphi(x_1)\varphi(x_2)\varphi(x_3)\rangle \quad (8)$$
$$= \int d\varkappa_1^2 d\varkappa_2^2 d\varkappa_3^2\Delta_F(x_{12}, \varkappa_3)\Delta_F(x_{13}, \varkappa_2)\Delta_F(x_{23}, \varkappa_1)f(\varkappa_1^2, \varkappa_2^2, \varkappa_3^2).$$

Equation (8) is the desired spectral representation of the Green's function in coordinate space.

To obtain the corresponding representation in momentum space it is necessary to calculate the integral

$$\int e^{il_1x_1+il_2x_2+il_3x_3}\Delta_F(x_{12}, \varkappa_3)\Delta_F(x_{13}, \varkappa_2)\Delta_F(x_{23}, \varkappa_1)\,d^4x_1 d^4x_2 d^4x_3,$$

which is calculated in the usual way and can be written in the following symmetrical form

$$\int_0^1\int_0^1\int_0^1 d\alpha\, d\beta\, d\gamma\, \delta(\alpha+\beta+\gamma-1) \times \frac{\delta(l_1+l_2+l_3)}{l_1^2\beta\gamma + l_2^2\alpha\gamma + l_3^2\alpha\beta + \alpha\varkappa_1^2 + \beta\varkappa_2^2 + \gamma\varkappa_3^2 - i\varepsilon}.$$

Consequently the desired representation has the form

$$\tau(l_1, l_2, l_3) = \delta(l_1+l_2+l_3)\int d\varkappa_1^2\, d\varkappa_2^2\, d\varkappa_3^2 \int d\alpha\, d\beta\, d\gamma \times \frac{\delta(\alpha+\beta+\gamma-1)\, f(\varkappa_1^2, \varkappa_2^2, \varkappa_3^2)}{l_1^2\beta\gamma + l_2^2\alpha\gamma + l_3^2\alpha\beta + \alpha\varkappa_1^2 + \beta\varkappa_2^2 + \gamma\varkappa_3^2 - i\varepsilon} \quad (9)$$

A more detailed study of the properties of $f(\kappa_1^2, \kappa_2^2, \kappa_3^2)$ is necessary for the further analysis of $\tau(l_1, l_2, l_3)$.

2. Properties of $f(\kappa_1^2, \kappa_2^2, \kappa_3^2)$

(a) We shall show that f is a real function. Using the Hermitian character of $\varphi(x_i)$, we have

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\rangle^* = \langle\varphi(x_3)\varphi(x_2)\varphi(x_1)\rangle$$

and consequently

$$\rho^*(p_1^2, q^2, p_3^2) = \rho(p_3^2, q, p_1^2).$$

But according to Eq. (4) it follows from the symmetry of f that

$$\rho(p_1^2, q^2, p_3^2) = \rho(p_3^2, q^2, p_1^2).$$

Consequently ρ is real, and therefore f can also be taken to be a real function.

(b) Let us now determine more precisely the region of values of κ_i^2 for which $f(\kappa_1^2, \kappa_2^2, \kappa_3^2) \neq 0$. For this purpose we write Eq. (4) in the form

$$\rho(p_1^2, q^2, p_3^2) = \int d\varkappa_1^2\, d\varkappa_2^2\, d\varkappa_3^2\, f(\varkappa_1^2, \varkappa_2^2, \varkappa_3^2) \int d^4 l\, \vartheta(k_{10})\,\vartheta(k_{20})\,\vartheta(k_{30}) \times \delta(k_1^2+\varkappa_1^2)\,\delta(k_2^2+\varkappa_2^2)\,\delta(k_3^2+\varkappa_3^2) \quad (10)$$

Calculating the integral over l, we get (see Appendix):

$$\rho(-m_1^2, q^2, -m_3^2) = \frac{\pi}{2} S^{-1}(q^2, m_1, m_3)\int d\varkappa_1^2\, d\varkappa_2^2\, d\varkappa_3^2 f(\varkappa_1^2, \varkappa_2^2, \varkappa_3^2) \times \vartheta(m_1 - \varkappa_2 - \varkappa_3)\,\vartheta(m_3 - \varkappa_1 - \varkappa_3)\,\vartheta(\xi), \quad (10')$$

$$m_1 = \sqrt{-p_1^2}, \quad m_3 = \sqrt{-p_3^2},$$

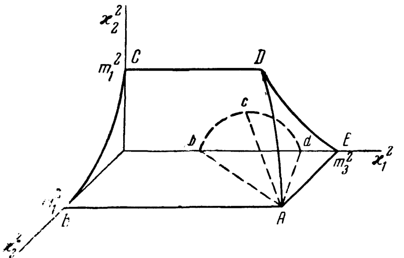
$$S(q^2, m_1, m_3) = [q^2 + (m_1+m_3)^2]^{1/2}[q^2 + (m_1-m_3)^2]^{1/2},$$

$$\xi = S^2(q^2, m_1, m_3) - [(m_1^2 + m_3^2 + 2\varkappa_2^2 - \varkappa_1^2 - \varkappa_3^2)^2 - 4(q^2 + 2m_1^2 + 2m_3^2)\varkappa_2^2] - [(q^2 + 2m_1^2 + 2m_3^2)(m_1^2 + \varkappa_1^2 - m_3^2 - \varkappa_3^2) + (m_3^2 - m_1^2)(m_1^2 + m_3^2 + 2\varkappa_2^2 - \varkappa_1^2 - \varkappa_3^2)].$$

Equation (10) has the following simple structure:

$$\frac{2}{\pi} S(q^2, m_1, m_3)\,\rho(-m_1^2, q^2, -m_3^2) = \int_V d\varkappa_1^2\, d\varkappa_2^2\, d\varkappa_3^2\, f(\varkappa_1^2, \varkappa_2^2, \varkappa_3^2); \quad (11)$$

V is a volume in the space of $\kappa_1^2, \kappa_2^2, \kappa_3^2$ whose shape depends on m_1, m_3, and q^2. It can easily be shown that for given m_1, m_3, and q^2 the volume V defined by the conditions $\kappa_1 + \kappa_2 < m_3$, $\kappa_3 + \kappa_2 < m_1$, and $\xi > 0$ has the form shown in the diagram.



The volume of integration is bounded by the surface Abcd. The surface ABCDE bounds the volume obtained if we do not include the condition $\xi > 0$ ($OB = OC = m_1^2$, $OE = BA = m_3^2$, $m_1 < m_3$). For fixed values of m_1 and m_3, q^2 ranges from $-(m_3 - m_1)^2$ to ∞. For $q^2 = -(m_3 - m_1)^2$ the straight lines Ab and Ad coincide, and the curve Ac coincides with the curve AD. In this limit the volume of integration goes to zero, in accordance with the fact that the factor $S(q^2, m_1, m_3)$ in the left member of Eq. (11) goes to zero. For $q^2 \to \infty$ the straight line Ab coincides with the line AB, the line Ad with AE, and the point c lies on the plane $\kappa_2^2 = 0$. In this limit, the volume of integration also goes to zero. But unlike the case $q^2 = -(m_3 - m_1)^2$, the left member is not necessarily equal to zero for $q^2 \to \infty$, since $f(\kappa_1^2, \kappa_2^2, \kappa_3^2)$ can have a δ-function singularity on the plane $\kappa_2^2 = 0$. An equation of such simple structure as Eq. (11) possesses a solution under very general assumptions regarding the function $\rho(-m_1^2, q^2, -m_3^2)$, if this latter function satisfies the conditions

$$q^2\rho(-m_1^2, q^2, -m_3^2) < \infty \text{ for } q^2 \to \infty,$$
$$\rho(-m_1^2, -(m_3-m_1)^2, -m_3^2) < \infty \quad (12)$$

which we shall assume are satisfied.

To establish more precisely the properties of the function $f(\kappa_1^2, \kappa_2^2, \kappa_3^2)$ we use the fact that $\rho(-m_1^2, q^2, -m_3^2) = 0$ if either m_1 or m_3 is less than m, where m is the mass for the state with the smallest energy for which $\varphi_{0p} \neq 0$. Thus we get

$$\int\limits_V d\varkappa_1^2 d\varkappa_2^2 d\varkappa_3^2 f(\varkappa_1^2, \varkappa_2^2, \varkappa_3^2) = 0 \quad (13)$$

for arbitrary q^2 and m_3 if $m_1 < m$, and for arbitrary q^2 and m_1 if $m_3 < m$.

Since the surface Abcd for various values of q^2 and m_3 contains inside it all parts of the volume bounded by the surface $\kappa_3 + \kappa_2 = m_1$, it follows from Eq. (13) that $f(\kappa_1^2, \kappa_2^2, \kappa_3^2) = 0$ if $\kappa_1 + \kappa_2 < m$ or $\kappa_3 + \kappa_2 < m$. In virtue of its symmetry, the function $f(\kappa_1^2, \kappa_2^2, \kappa_3^2)$ must also equal zero for $\kappa_1 + \kappa_3 < m$.

Thus we have for the T-product of three scalar operators the spectral representations (8) and (9), where $f(\kappa_1^2, \kappa_2^2, \kappa_3^2)$ is a real symmetric function which is nonvanishing if the conditions

$$\varkappa_1 + \varkappa_2 > m, \quad \varkappa_1 + \varkappa_3 > m, \quad \varkappa_2 + \varkappa_3 > m$$

are fulfilled.

3. Analytical Properties

Assuming that the integral over $\kappa_1^2, \kappa_2^2, \kappa_3^2$ in Eq. (9) converges for real l_1^2, l_2^2, and l_3^2 [the function $\tau(l_1^2, l_2^2, l_3^2)$ exists], we find that it defines an analytic function off the real axis for any one of the complex variables l_1^2, l_2^2, l_3^2 if the two others are real. This function can have singularities only at values of the variables for which the denominator in Eq. (9) can vanish. In such cases the integration must be carried out by using the stipulation that the denominator with which we are concerned has an infinitely small negative imaginary part. The imaginary part of $\tau(l_1^2, l_2^2, l_3^2)$ will also be different from zero. A simple analysis of the denominator (which we shall denote by $\square$) shows that if even a single one of the arguments, for example l_1^2, is greater than zero, then $\square > 0$ for $l_2^2 > -(\kappa_1 + \kappa_3)^2$ and $l_3^2 > -(\kappa_1 + \kappa_2)^2$. That is, if we recall the properties of the function $f(\kappa_1^2, \kappa_2^2, \kappa_3^2)$, we get $\square > 0$ for $l_2^2, l_3^2 > -m^2$. If all the $l_i^2 < 0$, then $\square$ can equal zero also for $l_i^2 > -m^2$; for example, for $\kappa_1 = \kappa_2 = \kappa_3 = m/2$ and $\alpha = \beta = \gamma = 1/3$, $\square = 0$ for $l_1^2 = l_2^2 = l_3^2 = -3m^2/4$. But in this deduction we have not taken into account the condition $l_1 + l_2 + l_3 = 0$. If we include this, then we come to the conclusion that also in the case in which all $l_i^2 < 0$, $\square > 0$ if $l_i^2 > -m^2$.

Consequently, Im $\tau(l_1, l_2, l_3) = 0$ if all l_i^2 $-m^2$ and these quantities satisfy the inequalities arising from the condition* $l_1 + l_2 + l_3 = 0$.

II. THE GREEN'S FUNCTION IN THE PSEUDOSCALAR MESON THEORY

In this chapter we shall obtain the spectral representation for

$$\langle T\psi(x_1)\varphi_i(x_2)\overline{\psi}(x_3)\rangle.$$

Just as in the preceding case, we consider first the simple products

$$\langle\psi(x_1)\varphi_i(x_2)\overline{\psi}(x_3)\rangle, \quad \langle\varphi_i(x_2)\psi(x_1)\overline{\psi}(x_3)\rangle,$$
$$\langle\psi(x_1)\overline{\psi}(x_3)\varphi_i(x_2)\rangle \text{ etc.}$$

For example, the first two can be written in the forms

$$\langle\psi(x_1)\varphi_i(x_2)\overline{\psi}(x_3)\rangle = \frac{1}{(2\pi)^9}\int d^4p_1 d^4p_3 e^{ip_1(x_1-x_2)+ip_3(x_2-x_3)}\vartheta(p_{10})\vartheta(p_{30})\gamma_5\tau_i$$
$$\times\Big\{\rho_0^{\psi\varphi\overline{\psi}}(p_1^2, q^2, p_3^2) + \hat{p}_1\rho_1^{\psi\varphi\overline{\psi}}(p_1^2, q^2, p_3^2) + \hat{p}_3\rho_3^{\psi\varphi\overline{\psi}}(p_1^2, q^2, p_3^2)$$
$$+\frac{1}{2i}(\hat{p}_1\hat{p}_3 - \hat{p}_3\hat{p}_1)\rho_{13}^{\psi\varphi\overline{\psi}}(p_1^2, q^2, p_3^2)\Big\}, \quad (14)$$

$$\langle\varphi_i(x_2)\psi(x_1)\overline{\psi}(x_3)\rangle = \frac{1}{(2\pi)^9}\int d^4p_1 d^4p_3 e^{ip_1(x_2-x_1)+ip_3(x_1-x_3)}\vartheta(p_{10})\vartheta(p_{30})\gamma_5\tau_i$$
$$\times\Big\{\rho_0^{\varphi\psi\overline{\psi}}(p_1^2, q^2, p_3^2) + \hat{p}_1\rho_1^{\varphi\psi\overline{\psi}}(p_1^2, q^2, p_3^2) + \hat{p}_3\rho_3^{\varphi\psi\overline{\psi}}(p_1^2, q^2, p_3^2)$$
$$+\frac{1}{2i}(\hat{p}_1\hat{p}_3 - \hat{p}_3\hat{p}_1)\rho_{13}^{\varphi\psi\overline{\psi}}(p_1^2, q^2, p_3^2)\Big\}. \quad (14a)$$

The causality condition establishes a connection between $\rho_i^{\psi\varphi\overline{\psi}}$ and $\rho_i^{\varphi\psi\overline{\psi}}$. In order to bring out this connection we shall, as before, seek to express $\rho_i^{\psi\varphi\overline{\psi}}$, $\rho_i^{\varphi\psi\overline{\psi}}$, etc., in the forms

$$\vartheta(p_{10})\vartheta(p_{30})\rho_0^{\psi\varphi\overline{\psi}}(p_1^2, q^2, p_3^2)$$
$$= \int d^4l\,\vartheta(k_{10})\vartheta(k_{20})\vartheta(k_{30}) f_0^{\psi\varphi\overline{\psi}}(-k_1^2, -k_2^2, -k_3^2),$$

$$\vartheta(p_{10})\vartheta(p_{30})[\hat{p}_1\rho_1^{\psi\varphi\overline{\psi}} + \hat{p}_3\rho_3^{\psi\varphi\overline{\psi}}]$$

*Nambu[3] took this rule as the basis for his derivation of the spectral representations of the Green's functions, but used it also for l_i^2 not satisfying the condition $l_1 + l_2 + l_3 = 0$. This last fact obviously makes his representations incorrect.

After the present paper was completed, the writer learned of a report by Schwinger at the Seventh Rochester Conference, in which similar representations were considered from a different point of view.

$$=\int d^4l\vartheta(k_{10})\vartheta(k_{20})\vartheta(k_{30})[(\hat{k}_1+\hat{k}_3)f_1^{\psi\varphi\bar{\psi}}(-k_1^2,-k_2^2,-k_3^2)+\hat{k}_2 f_2^{\psi\varphi\bar{\psi}}(-k_1^2,-k_2^2,-k_3^2)],$$

$$\vartheta(p_{10})\vartheta(p_{30})\frac{1}{2i}(\hat{p}_1\hat{p}_3-\hat{p}_3\hat{p}_1)\rho_{13}^{\psi\varphi\bar{\psi}}$$
$$=\int d^4l\vartheta(k_{10})\vartheta(k_{20})\vartheta(k_{30})\frac{1}{2i}(\hat{k}_1\hat{k}_3-\hat{k}_3\hat{k}_1)f_3^{\psi\varphi\bar{\psi}}(-k_1^2,-k_2^2,-k_3^2); \tag{15}$$

$$\vartheta(p_{10})\vartheta(p_{30})\rho^{\varphi\psi\bar{\psi}}(p_1^2,q^2,p_3^2)=\int d^4l\vartheta(k_{10})\vartheta(k_{20})\vartheta(k_{30})f_0^{\varphi\psi\bar{\psi}}(-k_1^2,-k_2^2,-k_3^2),$$
$$\vartheta(p_{10})\vartheta(p_{30})[\hat{p}_1\rho_1^{\varphi\psi\bar{\psi}}+\hat{p}_3\rho_3^{\varphi\psi\bar{\psi}}]$$
$$=\int d^4l\vartheta(k_{10})\vartheta(k_{20})\vartheta(k_{30})[(\hat{k}_2+\hat{k}_3)f_1^{\varphi\psi\bar{\psi}}(-k_1^2,-k_2^2,-k_3^2)$$
$$+\hat{k}_1 f_2^{\varphi\psi\bar{\psi}}(-k_1^2,-k_2^2,-k_3^2)], \tag{15a}$$
$$\vartheta(p_{10})\vartheta(p_{30})\frac{1}{2i}(\hat{p}_1\hat{p}_3-\hat{p}_3\hat{p}_1)\rho_{13}^{\varphi\psi\bar{\psi}}$$
$$=\int d^4l\vartheta(k_{10})\vartheta(k_{20})\vartheta(k_{30})\frac{1}{2i}(\hat{k}_2\hat{k}_3-\hat{k}_3\hat{k}_2)f_3^{\varphi\psi\bar{\psi}}(-k_1^2,-k_2^2,-k_3^2).$$

The convenience of just such a choice of the f_i will be evident in what follows.

Substituting Eq. (15) into the causality condition

$$\langle\psi(x_1)\varphi_i(x_2)\bar{\psi}(x_3)\rangle=\langle\varphi_i(x_2)\psi(x_1)\bar{\psi}(x_3)\rangle,\quad x_{12}^2>0,$$

we get (here $\partial=\hat{\gamma}_\mu\partial/\partial x_\mu$)

$$\int d\varkappa_1^2 d\varkappa_2^2 d\varkappa_3^2\gamma_5\tau_i\Delta^+(x_{12},\varkappa_3)\Delta^+(x_{13},\varkappa_2)\Delta^+(x_{23},\varkappa_1)$$
$$\times\{f_0^{\psi\varphi\bar{\psi}}(\varkappa_1^2,\varkappa_2^2,\varkappa_3^2)-f_0^{\varphi\psi\bar{\psi}}(\varkappa_2^2,\varkappa_1^2,\varkappa_3^2)\}=0,$$
$$\int d\varkappa_1^2 d\varkappa_2^2 d\varkappa_3^2\gamma_5\tau_i\{[\Delta^+(x_{12},\varkappa_3)\Delta^+(x_{13},\varkappa_2)\partial\Delta^+(x_{23},\varkappa_1)$$
$$+\partial\Delta^+(x_{12},\varkappa_3)\Delta^+(x_{13},\varkappa_2)\Delta^+(x_{23},\varkappa_1)]$$
$$\times[f_1^{\psi\varphi\bar{\psi}}(\varkappa_1^2,\varkappa_2^2,\varkappa_3^2)-f_1^{\varphi\psi\bar{\psi}}(\varkappa_2^2,\varkappa_1^2,\varkappa_3^2)]+\Delta^+(x_{12},\varkappa_3)\partial\Delta^+(x_{13},\varkappa_2)\Delta^+(x_{23},\varkappa_1)$$
$$\times[f_2^{\psi\varphi\bar{\psi}}(\varkappa_1^2,\varkappa_2^2,\varkappa_3^2)-f_2^{\varphi\psi\bar{\psi}}(\varkappa_2^2,\varkappa_1^2,\varkappa_3^2)]\}=0, \tag{16}$$
$$\int d\varkappa_1^2 d\varkappa_2^2 d\varkappa_3^2\gamma_5\tau_i\frac{1}{2i}[\partial\Delta^+(x_{12},\varkappa_3)\Delta^+(x_{13},\varkappa_2)+\partial\Delta^+(x_{13},\varkappa_2)\Delta^+(x_{12},\varkappa_3)]\Delta^+(x_{23},\varkappa_1)$$
$$\times[f_3^{\psi\varphi\bar{\psi}}(\varkappa_1^2,\varkappa_2^2,\varkappa_3^2)-f_3^{\varphi\psi\bar{\psi}}(\varkappa_2^2,\varkappa_1^2,\varkappa_3^2)]=0,$$

from which it follows that

$$f_k^{\psi\varphi\bar{\psi}}(\varkappa_1^2,\varkappa_2^2,\varkappa_3^2)=f_k^{\varphi\psi\bar{\psi}}(\varkappa_2^2,\varkappa_1^2,\varkappa_3^2). \tag{17}$$

Analogous relations can be obtained for $f_k^{\psi\varphi\bar{\psi}}$, $f_k^{\psi\bar{\psi}\varphi}$, etc. Using them and the definition of the functions $\Delta_F(x,\kappa)$, we can proceed as in the previous case to go over to the spectral representation of $\langle T\psi(x_1)\varphi_i(x_2)\bar{\psi}(x_3)\rangle$. We get here:

$$\langle T\psi(x_1)\varphi_i(x_2)\bar{\psi}(x_3)\rangle=\int d\varkappa_1^2 d\varkappa_2^2 d\varkappa_3^2\gamma_5\tau_i\Big\{f_0^{\psi\varphi\bar{\psi}}(\varkappa_1^2,\varkappa_2^2,\varkappa_3^2)$$
$$+f_1^{\psi\varphi\bar{\psi}}(\varkappa_1^2,\varkappa_2^2,\varkappa_3^2)\,i(\partial_1+\partial_3)+f_2^{\psi\varphi\bar{\psi}}(\varkappa_1^2,\varkappa_2^2,\varkappa_3^2)\,i\partial_2$$
$$+\frac{1}{2i}(\partial_3\partial_1-\partial_1\partial_3)f_3^{\psi\varphi\bar{\psi}}(\varkappa_1^2,\varkappa_2^2,\varkappa_3^2)\Big\}\Delta_F(x_{12},\varkappa_3)\Delta_F(x_{13},\varkappa_2)\Delta_F(x_{23},\varkappa_1) \tag{18}$$

and correspondingly in the momentum representation

$$\tau(l_1,l_2,l_3)=\frac{\pi^2}{2}\delta(l_1+l_2+l_3)\gamma_5\tau_i\int d\varkappa_1^2 d\varkappa_2^2 d\varkappa_3^2\int_0^1 d\alpha\int_0^1 d\beta\int_0^1 d\gamma\delta(\alpha+\beta+\gamma-1)$$
$$\times\left\{\frac{f_0^{\psi\varphi\bar{\psi}}(\varkappa_1^2,\varkappa_2^2,\varkappa_3^2)+(\gamma\hat{l}_1-\alpha\hat{l}_3+\beta\hat{l}_2)f_1^{\psi\varphi\bar{\psi}}(\varkappa_1^2,\varkappa_2^2,\varkappa_3^2)+(\alpha\hat{l}_1-\gamma\hat{l}_3)f_2^{\psi\varphi\bar{\psi}}(\varkappa_1^2,\varkappa_2^2,\varkappa_3^2)}{l_1^2\beta\gamma+l_2^2\alpha\gamma+l_3^2\alpha\beta+\alpha\varkappa_1^2+\beta\varkappa_2^2+\gamma\varkappa_3^2-i\varepsilon}\right.,$$
$$\left.+\frac{\frac{\beta}{2i}(\hat{l}_1\hat{l}_3-\hat{l}_3\hat{l}_1)f_3^{\psi\varphi\bar{\psi}}(\varkappa_1^2,\varkappa_2^2,\varkappa_3^2)}{l_1^2\beta\gamma+l_2^2\alpha\gamma+l_3^2\alpha\beta+\alpha\varkappa_1^2+\beta\varkappa_2^2+\gamma\varkappa_3^2-i\varepsilon}\right\}. \tag{19}$$

Using the invariance of the theory with respect to charge conjugation, we get:

$$\langle T\psi(x_1)\varphi_i(x_2)\overline{\psi}(x_3)\rangle = -(C^{-1}\langle T\psi(x_3)\varphi_i(x_2)\overline{\psi}(x_1)\rangle C)^T;$$

$$\overline{\psi}' = C^{-1}\psi;\quad \psi' = C\overline{\psi};\quad \varphi_i' = -\varphi_i;\quad C^{-1}\gamma_\mu C = -\gamma_\mu^T;\ C^{-1}\tau_j C = -\tau_j^T. \tag{20}$$

From this we find, by considerations analogous to those used in Sec. 2, that the $f_k^{\psi\varphi\overline{\psi}}(\kappa_1^2, \kappa_2^2, \kappa_3^2)$ are real functions. $f_0^{\psi\varphi\overline{\psi}}(\kappa_1^2, \kappa_2^2, \kappa_3^2)$ is an antisymmetric function and $f_1^{\psi\varphi\overline{\psi}}$, $f_2^{\psi\varphi\overline{\psi}}$, and $f_3^{\psi\varphi\overline{\psi}}$ are symmetric functions, with respect to interchange of κ_1^2 and κ_3^2.

We obtain further information about the $f_k^{\psi\varphi\overline{\psi}}(\kappa_1^2, \kappa_2^2, \kappa_3^2)$ in just the same way as in the previous case. From the condition that $\rho_k^{\psi\varphi\overline{\psi}}(p_1^2, q^2, p_3^2) = 0$ if $-p_1^2 \le m^2$ or $-p_3^2 \le m^2$ and from Eq. (15) it follows, just as in Sec. 2, that $f_k^{\psi\varphi\overline{\psi}}(\kappa_1^2, \kappa_2^2, \kappa_3^2) = 0$ if $\kappa_1 + \kappa_2 \le m$ or $\kappa_2 + \kappa_3 \le m$ (m is the mass of the nucleon). From the condition $\rho_k^{\varphi\psi\overline{\psi}}(p_1^2, q^2, p_3^2) = 0$ if $-p_1^2 \le \mu^2$ or $-p_3^2 \le m^2$ and from Eq. (15a) it follows that $f_k^{\varphi\psi\overline{\psi}}(\kappa_1^2, \kappa_2^2, \kappa_3^2) = 0$ if $\kappa_1 + \kappa_2 \le m$ or $\kappa_2 + \kappa_3 \le \mu$ (μ is the mass of the meson).

Combining these conditions with the condition (17), we get

$$f_k^{\psi\varphi\overline{\psi}}(\varkappa_1^2, \varkappa_2^2, \varkappa_3^2) \neq 0 \text{ for } \varkappa_1 + \varkappa_2 \geqslant m,$$
$$\varkappa_2 + \varkappa_3 \geqslant m,\ \varkappa_1 + \varkappa_3 \geqslant \mu. \tag{21}$$

The last result means that if we disregard the imaginary quantities occurring in $\hat{l}_i$ and γ_5, τ_i,

$$\text{Im}''\tau(l_1, l_2, l_3) = 0, \tag{22}$$
$$\text{if} - l_1^2 < m_1^2,\ -l_3^2 < m^2,\ -l_2^2 < \mu^2.$$

We obtain a more complete representation for $\tau(l_1, l_2, l_3)$ if we note that it can be written in the form

$$\tau(l_1, l_2, l_3) = -\frac{1}{l_2^2+\mu^2}\frac{1}{i\hat{l}_1+m}\tau'(l_1, l_2, l_3)\frac{1}{i\hat{l}_3-m}. \tag{23}$$

We get an expression for $\tau'(l_1, l_2, l_3)$ if we go back to the coordinate representation

$$\tau'(x_1, x_2, x_3)$$
$$= (\Box_2 - \mu^2)(\hat{\partial}_1 + m)(\hat{\partial}_3^T - m)\langle T\psi(x_1)\varphi_i(x_2)\psi(x_3)\rangle$$
$$= \langle Tu(x_1)j_i(x_2)\overline{u}(x_3)\rangle$$
$$+ \gamma_4\delta(t_1 - t_2)\langle T[\psi(x_1)\ j_i(x_2)]\,u\overline{(x_3)}\rangle$$
$$+ \delta(t_2 - t_3)\langle Tu(x_1)[\overline{\psi}(x_3)\,j(x_2)]\rangle\gamma_4$$
$$+ \delta(t_1 - t_2)\delta(t_2 - t_3)\gamma_4\langle\{\overline{\psi}(x_3),\ [\psi(x_1)\,j_i(x_2)]\}\rangle\gamma_4;$$

$$j_i(x) = (\Box - \mu^2)\varphi_i(x);\quad u(x) = (i\hat{\partial} + m)\psi(x);$$
$$\overline{u}(x) = (i\hat{\partial}^T - m)\overline{\psi}(x). \tag{24}$$

Setting $\langle Tu(x_1)j_i(x_2)\overline{u}(x_3)\rangle = \tau_c(x_1, x_2, x_3)$, we see that $\tau_c(x_1, x_2, x_3)$ has precisely the same structure as $\tau(x_1, x_2, x_3)$, i.e., it is a T-product of Heisenberg operators. Therefore we can repeat all the arguments of this chapter with respect to $\tau_c(x_1, x_2, x_3)$. By so doing we arrive at formulas which coincide with Eqs. (18) and (19) in the coordinate and momentum representations, respectively. The corresponding functions $f_k^{uj\overline{u}}$ will satisfy the same conditions of symmetry and reality. The condition (22) is changed, however, since the matrix elements of the operators u, $\overline{u}$, and j_i between the vacuum and one-particle states are equal to zero.

Instead of the previous conditions we get

$$\rho_k^{uj\overline{u}}(p_1^2, q^2, p_3^2) = 0,$$
$$\text{if } -p_1^2 < (m+\mu)^2 \text{ or } -p_3^2 < (m+\mu)^2,$$
$$\rho_k^{ju\overline{u}}(p_1^2, q^2, p_3^2) = 0,$$
$$\text{if } -p_1^2 < 9\mu^2 \quad \text{or } -p_3^2 < (m+\mu)^2,$$

from which it follows that

$$f_k^{uj\overline{u}}(\varkappa_1^2, \varkappa_2^2, \varkappa_3^2) \neq 0,\quad \text{if } \varkappa_1 + \varkappa_2 \geqslant m + \mu; \tag{25}$$
$$\varkappa_2 + \varkappa_3 \geqslant m + \mu,\ \varkappa_1 + \varkappa_3 \geqslant 3\mu.$$

From this it follows in turn that

$$\text{„Im"}\tau_c(l_1, l_2, l_3) = 0,\quad \text{if } -l_1^2 < (m+\mu)^2, \tag{26}$$
$$-l_3^2 < (m+\mu)^2,\ -l_2^2 < 9\mu^2.$$

If $j_i(x) = ig\overline{\psi}\gamma_5\tau_i\psi + \lambda(\varphi_i\varphi_k\varphi_k - \delta\mu^2\varphi)$, then the terms

$$\delta(t_1 - t_2)\gamma_4\langle T[\psi(x_1)j_i(x_2)]\overline{u}(x_3)\rangle$$
$$+ \delta(t_2 - t_3)\langle Tu(x_1)[\psi(x_3)j(x_2)]\rangle\gamma_4$$
$$+ \delta(t_1 - t_2)\delta(t_2 - t_3)\gamma_4\langle\{[\psi(x_1)j_i(x_2)]\psi(x_3)\}\rangle\gamma_4$$
$$= ig\gamma_5\tau_i\langle T\psi(x_2)\overline{u}(x_3)\rangle\delta(x_1 - x_2)$$
$$- ig\delta(x_2 - x_3)\langle Tu(x_1)\overline{\psi}(x_2)\rangle\gamma_5\tau_i$$
$$+ ig\gamma_5\tau_i\delta(x_1 - x_2)\delta(x_2 - x_3) \tag{27}$$

reduce to the vacuum expectation values of T-products of two operators, for which the spectral representations are well known.

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APPENDIX

Calculation of the integral (10)

$$I = \int \vartheta(p_1 - k)\,\vartheta(p_3 - k)\,\vartheta(k)\,\delta(k^2 + \varkappa_2^2) \times \delta((p_1 - k)^2 + \varkappa_3^2)\,\delta((p_3 - k)^2 + \varkappa_1^2)\,d^4k.$$

The most important point is to find out the conditions under which $I \neq 0$. In order that the products

$$\vartheta(k)\,\vartheta(p_1 - k)\,\delta(k^2 + \varkappa_2^2)\,\delta((p_1 - k)^2 + \varkappa_3^2), \quad \vartheta(k)\,\vartheta(p_3 - k)\,\delta(k^2 + \varkappa_2^2)\,\delta((p_3 - k)^2 + \varkappa_1^2) \tag{A1}$$

be not equal to zero, it is necessary that

$$-p_1^2 \geqslant (\varkappa_2 + \varkappa_3)^2, \qquad -p_3^2 \geqslant (\varkappa_1 + \varkappa_2)^2 \tag{A2}$$

These conditions are not, however, sufficient to secure that $I \neq 0$. What is needed is the existence of common values of k for which these products are nonvanishing. To obtain the sufficient conditions we go over to a definite reference system with

$$\mathbf{p}_1 = -\mathbf{p}_3 = \mathbf{p}.$$

In this system we have

$$p_1^2 + \varkappa_3^2 - \varkappa_2^2 + 2p_{10}k_0 - 2pkx = 0, \quad p_3^2 + \varkappa_1^2 - \varkappa_2^2 + 2p_{30}k_0 + 2pkx = 0, \quad x = \cos(\mathbf{pk}). \tag{A3}$$

Solving these equations and substituting the resulting solutions into the condition

$$k_0^2 - \varkappa_2^2 \geqslant k^2x^2,$$

we get

$$\frac{1}{4(p_{10} + p_{30})^2}(2\varkappa_2^2 - \varkappa_1^2 - \varkappa_3^2 - p_1^2 - p_3^2)^2 - \varkappa_2^2 \geqslant \frac{1}{16p^2}\Big[\varkappa_1^2 - \varkappa_3^2 - p_1^2 + p_3^2 + \frac{p_{30} - p_{10}}{p_{30} + p_{10}}(2\varkappa_2^2 - \varkappa_1^2 - \varkappa_3^2 - p_1^2 - p_3^2)\Big]^2. \tag{A4}$$

The conditions (A2) and (A4) are sufficient conditions for $I \neq 0$. When the conditions (A2) and (A4) are satisfied,

$$I = \pi / 4p(p_{10} + p_{30}). \tag{A5}$$

It can easily be shown that

$$p_{10} + p_{30} = [q^2 - 2p_1^2 - 2p_3^2]^{1/2},$$

$$p^2 = \frac{[q^2 + (\sqrt{-p_1^2} + \sqrt{-p_3^2})^2]\,[q^2 + (\sqrt{-p_1^2} - \sqrt{-p_3^2})^2]}{4(q^2 - 2p_1^2 - 2p_3^2)}. \tag{A6}$$

By means of these formulas we can write I in invariant form. Substituting (A6) into Eqs. (A5) and (A4), we obtain the result given in the text.

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