# THE INTERACTION OF ELECTRONS WITH LATTICE VIBRATIONS

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This work investigates the stability of the lattice is investigated with account of the interaction of the electrons with the phonon field. A criterion for stability is established without the aid of perturbation theory.

In his well known work, Wentzel[1] investigated the limits of applicability of perturbation theory to the interaction of electrons with the phonon field. By calculating the second-order correction to the phonon self-energy in perturbation theory, he found that the velocity of sound of the phonons is renormalized by writing

$$s = s_0(1-\rho), \quad \rho = (g^2/2\pi^2)\, k_F^2/\varepsilon'(k_F), \tag{1}$$

where $s_0$ is the velocity of sound before renormalization, $s$ is its renormalized value, $k_F$ is the wave vector at the Fermi boundary, $\epsilon'(k_F)$ is the derivative of the electron self-energy with respect to the wave vector, and $g$ is the coupling constant. Wentzel concludes that a necessary condition for the applicability of perturbation theory is that

$$\rho \ll 1. \tag{2}$$

As to what will occur if (2) is not satisfied, Wentzel asserts, on the basis of an analysis of the one-dimensional problem, that if in this case the coupling constant is sufficiently great, the phonon self-energy will become imaginary rather than negative. This, in turn, leads to breakdown of the lattice. We shall show below how an exact criterion for the stability of a crystal lattice can be established without the aid of perturbation theory.

We start with the Hamiltonian describing the interaction of electrons with lattice vibrations in the form proposed by Fröhlich. This is

$$H = H_0 + H_{\rm int},$$

where

$$H_0 = \sum_{k,\sigma} \varepsilon(k)\, a_{k\sigma}^+ a_{k\sigma} + \sum_q \hbar\omega(q)\, b_q^+ b_q, \tag{3}$$

$$H_{\rm int} = \frac{g}{\sqrt{2V}} \sum_{k,k',\sigma} \sqrt{\hbar\omega(k'-k)}\,(a_{k'\sigma}^+ a_{k\sigma} b_{k'-k} + a_{k\sigma}^+ a_{k'\sigma} b_{k'-k}^+),$$

in which $a_{k\sigma}^+$ and $a_{k\sigma}$ are electron creation and annihilation operators, $b_k^+$ and $b_k$ are phonon creation and annihilation operators, and $V$ is the volume of the region of periodicity.

In what follows we shall be interested in phonons whose energies are so low that

$$\hbar\omega \ll \Delta\varepsilon,$$

where $\Delta\epsilon$ is the mean energy difference in electron transitions. A rather good idea of the situation which arises can be obtained with the aid of the so-called adiabatic approximation in the form given by Bogoliubov and Tiablikov.[2,3] In agreement with the basic concept of this approximation, we shall formally introduce in (3) a small parameter into the phonon frequency $\hbar\omega$ and project our Hamiltonian with the necessary accuracy onto the subspace of states each of which is an electron Fermi vacuum.* We obtain

$$(E-E_0)\Phi = \sum_q (\hbar\omega(q)\, b_q^+ b_q - g^2\hbar\omega(q)\, A(q) \times (b_q b_{-q} + b_q^+ b_q + b_q b_q^+ + b_q^+ b_{-q}^+))\,\Phi, \tag{4}$$

where $\Phi$ is the wave function in the above-mentioned subspace,

$$E_0 = 2\sum_{|k|<k_F} \varepsilon(k)$$

is the energy of the Fermi vacuum, and

$$A(q) = \frac{1}{V} \sum_{\substack{|k|<k_F \\ |k+q|>k_F}} \frac{1}{\varepsilon(k+q)-\varepsilon(k)}. \tag{5}$$

Equation (4) is easily solved, since the Hamiltonian entering into it is a quadratic form in the Bose operators. We set up the secular equations (for more details see Bogoliubov and Tiablikov[3] and the monograph mentioned in the last footnote)

*A detailed description of the technique of projection can be found in N. N. Bogoliubov's monograph Лекції з квантової статистики (Lectures on Quantum Statistics) (in Ukranian).

$$(\hbar\omega(q) - 2g^2\hbar\omega(q)A(q) - E)C_q - 2g^2\hbar\omega(q)A(q)C^+_{-q} = 0,$$
$$(\hbar\omega(q) - 2g^2\hbar\omega(q)A(q) + E)C^+_{-q} - 2g^2\hbar\omega(q)A(q)C_q = 0, \quad (6)$$

where $C_q$ and $C_q^+$ are treated as c-numbers, and E is the energy of an elementary excitation (the energy difference between an excited state and the ground state). The condition that (6) be a soluble set of equations is that its determinant vanish.

Solving (6), we obtain

$$E(q) = \hbar\omega(q)\sqrt{1 - 4g^2A(q)} \quad (7)$$

(it follows from the general theory that the negative root of (6) may be ignored). Equation (7) shows immediately that if $g^2$ is sufficiently large, E(q) becomes imaginary so that the state under consideration becomes unstable due to the breakdown of the crystal lattice. Thus the criterion for stability is

$$4g^2A(q) < 1, \quad (8)$$

for all q. Noting that $\epsilon(q) = \hbar^2q^2/2m$, we can calculate the integral in (5). Calculation gives

$$4g^2A(q) = \rho f(x), \quad (9)$$

where

$$f(x) = 1 + \frac{4 - x^2}{4x}\ln\left|\frac{x+2}{x-2}\right|; \quad x = \frac{q}{k_F}. \quad (10)$$

The function f(x) takes on its maximum value f = 2 at x = 0. If this maximum value is used in the stability condition (8), we obtain finally

$$\rho \leqslant 1/2. \quad (11)$$

Since it is the phonons with low momenta, and therefore also with low energies, which are responsible for the breakdown of the crystal lattice, we have an a posteriori verification of the consistency of the approximation being used.

It is interesting to note that if, in the spirit of perturbation theory, we had expanded the root of (7) in powers of $\rho$, we would have obtained just Wentzel's Eq. (1). We note also that the stability criterion of (1) remains valid if we project the original Hamiltonian on a subspace of states close to the Fermi vacuum such as, in particular, the superconducting state.

We have treated the case in which the Hamiltonian does not involve the Coulomb interaction of electrons. When the Coulomb interaction is included, the stability criterion (8) will differ because of the different variation of A(q) for small q. One may suppose, however, that the lattice will remain unstable for sufficiently high coupling constants.

We have established the stability criterion in the adiabatic approximation. We will now show that the same expression is easily obtained by using the principle of compensation of "dangerous" diagrams.[4]

Let us perform the canonical transformation

$$b_q + b^+_{-q} = \sqrt{\frac{\omega(q)}{\Omega(q)}}(B_q + B^+_{-q});$$
$$b_q - b^+_{-q} = \sqrt{\frac{\Omega(q)}{\omega(q)}}(B_q - B^+_{-q}), \quad (12)$$

on the Bose operators $b_k$, where $\Omega(q)$ is the renormalized phonon energy, which will be found below. Equation (3) then becomes

$$H = E_0 + H_0 + H_1$$
$$E_0 = \frac{1}{2}\sum_q(\hbar\Omega(q) - \hbar\omega(q));$$
$$H_0 = \sum_{k,\sigma}\varepsilon(k)a^+_{k\sigma}a_{k\sigma} + \sum_q\hbar\Omega(q)B^+_qB_q;$$
$$H_1 = \frac{g}{\sqrt{2V}}\sum_{k,\sigma,q}\sqrt{\hbar\Omega(q)}\frac{\omega(q)}{\Omega(q)}(a^+_{k+q,\sigma}a_{k\sigma}B_q + a^+_{k-q,\sigma}a_{k\sigma}B^+_q) + \sum_q\hbar\Omega(q)\frac{\omega^2(q) - \Omega^2(q)}{4\Omega^2(q)}$$
$$\times(B_{-q}B_q + B^+_qB^+_{-q} + B^+_qB_q + B_{-q}B^+_{-q}). \quad (13)$$

We shall treat the term $H_1$ as a perturbation. It should be noted that the second term in $H_1$ is in fact of order $g^2$.

Let us now determine $\Omega$ from the requirement that in the second approximation in g all diagrams with two phonon lines at the output must compensate.[4] This leads to an equation for the heretofore unknown function $\Omega(q)$. As a result we obtain

$$\Omega^2(q) = \omega^2(q)\left\{1 - \frac{4g^2}{V}\sum_{\substack{|k|<k_F \\ |k-q|>k_F}}\frac{1}{\varepsilon(k-q) - \varepsilon(k) + \hbar\Omega(q)}\right\}, \quad (14)$$

and we may make the approximation

$$\Omega(q) = \omega(q)\left\{1 - \frac{4g^2}{V}\sum_{|k|<k_F,\ |k-q|>k_F}\frac{1}{\varepsilon(k-q) - \varepsilon(q) + \hbar\omega(q)}\right\}^{1/2}. \quad (15)$$

It is clear that the lattice will be stable if the expression in curly brackets in (14) or (15) is positive. It is easily seen that this condition

$$\frac{4g^2}{V}\sum_{|k|<k_F,\ |k-q|>k_F}\frac{1}{\varepsilon(k-q) - \varepsilon(q) + \hbar\omega(q)} > 1$$

becomes identical with (8) if the term $h\omega(q)$ is neglected in the denominator.

It should be noted that obtaining $\Omega$ from the requirement that the "dangerous" diagrams compensate in the second order in g is equivalent to obtaining it by minimizing the ground state energy to the same approximation in g. One may suppose that this equivalence will be true also in higher orders in g.

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[1] G. Wentzel, Phys. Rev. **83**, 168 (1952).

[2] N. N. Bogoliubov and S. V. Tiablikov, Вестник МГУ (Bulletin, Moscow State Univ.) **3**, 35, (1949); J. Exptl. Theoret. Phys. (U.S.S.R.) **19**, 151 (1949).

[3] N. N. Bogoliubov and S. V. Tiablikov, J. Exptl. Theoret. Phys. (U.S.S.R.) **19**, 257 (1949).

[4] N. N. Bogoliubov, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 58 (1958), Soviet Phys. JETP **7**, 41 (1958).

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# *RADIATIVE CORRECTIONS TO COMPTON SCATTERING TAKING INTO ACCOUNT POLARIZATION OF THE SURROUNDING MEDIUM*

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A general method for taking into account polarization of the medium in the calculation of radiative corrections in phenomenological quantum mechanics is developed. The effect of a nonconducting medium on radiative corrections to Compton scattering is taken into account for an arbitrary dependence of the dielectric constant of the medium on frequency. It is shown that in some cases, account of the medium substantially changes the cross section in the region of small scattering angles.

## 1. INTRODUCTION

THE influence of the medium in the calculation of higher approximations in perturbation theory must, in general, be taken into account, because the integrations over the 4-momenta of virtual photons include a region of long-wave photons for which it is impossible to ignore the presence of neighboring atoms of the medium. This situation was first indicated by Landau and Pomeranchuk,[1] who noted that multiple scattering by the atoms of the medium should lead to a change in radiative corrections in those cases in which infrared catastrophes occur, i.e., where the region of soft quanta is essential. Ter-Mikaelian[2] noted that the difference of the dielectric constant of the medium from unity for soft quanta should strongly influence the radiative corrections.

A method of taking into account the multiple scattering by atoms of the medium was developed by Migdal.[3] In the following, we consider the influence of the medium on radiative corrections, connected with the difference of the dielectric constant and magnetic permeability of the medium, $\epsilon$ and $\mu$, from unity in the region of soft quanta; we shall not take account of multiple scattering.

In order to develop a general method for taking into account the polarization of the medium in higher orders of perturbation theory, it is convenient to use a generalization by the author[4] of the Feynman-Dyson covariant perturbation theory to the case of phenomenological quantum electrodynamics in media. The general method obtained in this way will be applied to the Compton scattering, in order to obtain the cross section of sixth power in e, with account of the polarization of the