CORRELATION RELATIONS FOR RANDOM ELECTRIC CURRENTS AND FIELDS AT LOW TEMPERATURES

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Correlation functions have been obtained between the components of a random electric field with account of spatial dispersion. The correlation radius depends on the frequency: for $\delta(\omega) \gg l$ it is identical with the depth of the skin layer, while for $\delta(\omega) \ll l$ it coincides with the length of the mean free path.

As has been shown by Leontovich and Rytov,[1*] the correlation between the random side currents in a metal is determined by the conductivity tensor σ_{ik}

$$[j_i(\mathbf{r})\, j_k(\mathbf{r}')]_\omega = \frac{\hbar\omega}{2\pi}\sigma_{ik}\coth\frac{\hbar\omega}{2T}\,\delta(\mathbf{r}-\mathbf{r}'), \qquad (1)$$

while for $\hbar\omega \ll T$

$$[j_i(\mathbf{r})\, j_k(\mathbf{r}')]_\omega = \frac{T}{\pi}\sigma_{ik}\delta(\mathbf{r}-\mathbf{r}'). \qquad (2)$$

By means of Eqs. (1) or (2) we can find the correlation relations between the components of random fields (for example, see Ref. 4, § 90).

However, at low temperatures, Ohm's law, $j_i = \sigma_{ik}E_k$, which was used in the derivation of Eq. (1), must be replaced by the integral relation between the current density $\mathbf{j}$ and the direction of the electric field $\mathbf{E}$.[5] This connection is obtained from the solution of the kinetic equation. In the general case, the integral connecting the current and the field has the following form:

$$j_i(\mathbf{r}) = \int K_{ik}(\mathbf{r}, \mathbf{r}')\, E_k(\mathbf{r}')\, dv'. \qquad (3)$$

The space correlation function between the components of the electric current is expressed simply by the components of the kernel $K_{ik}(\mathbf{r}, \mathbf{r}')$. Actually, according to Eq. (90.5) of Ref. 4, we can introduce the generalized coordinates x_a and the generalized forces f_a in the following manner: $x_a \to \mathbf{E}\Delta v/4\pi$ (Δv = element of volume), $f_a \to \mathbf{K}$, where $\mathbf{K}$ [not to be confused with $K_{ik}(\mathbf{r}, \mathbf{r}')$!] is connected with the current density by the relation $\mathbf{j} = -(i\omega/4\pi)\mathbf{K}$. Transforming in Eq. (3) from integration to summation, and replacing $\mathbf{j}$ by $\mathbf{K}$, we find the following connection between the generalized force and the generalized coordinates:

$$K_i(\mathbf{r}) = \frac{16\pi^2 i}{\omega}\sum_{k,\,\mathbf{r}'} K_{ik}(\mathbf{r}, \mathbf{r}')\, E_k(\mathbf{r}')\frac{\Delta v}{4\pi}.$$

From a comparison of this expression with (88.10) of Ref. 4, it is seen that the kinetic coefficients α_{ab}^{-1} in the given case are equal to $(16\pi^2 i/\omega) K_{ik}(\mathbf{r}, \mathbf{r}')$, whence, in accord with Eq. (88.11) of Ref. 4, we obtain the following relation for the generalized forces $K_i(\mathbf{r})$:

$$[K_i(\mathbf{r})\, K_k(\mathbf{r}')]_\omega = -\frac{8\pi\hbar}{\omega}\operatorname{Re} K_{ik}(\mathbf{r}, \mathbf{r}')\coth\frac{\hbar\omega}{2T},$$

or, turning again to the currents:

$$[j_i(\mathbf{r})\, j_k(\mathbf{r}')]_\omega = \frac{\hbar\omega}{2\pi}\operatorname{Re} K_{ik}(\mathbf{r}, \mathbf{r}')\coth\frac{\hbar\omega}{2T}. \qquad (4)$$

Thus, to find the correlation relations, it is only necessary to write out the concrete form of the connection between the current density $\mathbf{j}$ and the direction of the electric field $\mathbf{E}$.

For this purpose, we make use of the linearized kinetic equation (for simplicity, the dispersion law is taken to be quadratic and we introduce the relaxation time τ):

$$\frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial \mathbf{r}}\mathbf{v} + \frac{f_1}{\tau} = -\frac{\partial f_0}{\partial \varepsilon} e\mathbf{E}\cdot\mathbf{v}. \qquad (5)$$

Here $\partial f_0/\partial\epsilon = -\delta(\epsilon - \epsilon_0)$, and ϵ_0 is the Fermi energy; the current density is determined by the following:

$$\mathbf{j} = \frac{2e}{(2\pi\hbar)^3}\int f_1 \mathbf{v}\, d\tau_p. \qquad (6)$$

In the case of an unbounded metal, replacing $\mathbf{E}$ and f_1 in the Fourier integral by (5) and (6), we find

$$K_{ik}(\mathbf{r}, \mathbf{r}') = \frac{\sigma_0}{(2\pi)^3}\frac{3}{4\pi}\iint \frac{n_i n_k do\, e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}}{1 + i(\mathbf{k}\mathbf{n}l - \omega\tau)}\, d\tau_{\mathbf{k}}. \qquad (7)$$

*See also Refs. 2 and 4. The most satisfactory account of the problem of correlations for our purposes is contained in Electrodynamics of Continuous Media by L. D. Landau and E. M. Lifshitz.[4]

Here σ_0 is the static conductivity, $\mathbf{n} = \mathbf{v}/v$, do = element of solid angle in momentum space, $l = v/\tau$ = length of the mean free path (v is the constant value of the velocity on the Fermi surface).

It is easy to see that in these cases, when we can neglect spatial dispersion ($l = 0$), we get Eq. (1), wherein in this case,

$$\sigma_{ik} = \sigma_0\delta_{ik}/(1+\omega^2\tau^2).$$

Making use of Eq. (7), we compute the dependence of the correlation function on $\boldsymbol{\rho} = \mathbf{r} - \mathbf{r}'$:

$$[j_i(\mathbf{r})\, j_k(\mathbf{r}')]_\omega = \frac{\hbar\omega}{4\pi}\coth\frac{\hbar\omega}{2T}\cdot\frac{3\sigma_0}{4\pi}\Big\{\frac{\delta_{ik}}{\rho l^2}\times\int_0^1\Big(\frac{1}{u^2}-1\Big)e^{-\rho/ul}\Big(\cos\frac{\rho\omega\tau}{ul}+\omega\tau\sin\frac{\rho\omega\tau}{ul}\Big)du+\frac{1}{1+\omega^2\tau^2}\frac{\partial^2}{\partial x_i\partial x_k}\times\Big[\frac{1}{\rho}\int_0^1(3u^2-1)\,e^{-\rho/ul}\Big(\cos\frac{\rho\omega\tau}{ul}-\omega\tau\sin\frac{\rho\omega\tau}{ul}\Big)du\Big]\Big\}. \quad (8)$$

Clearly, there is a broad range of frequencies (up to the infrared) in which spatial dispersion is important, while time dispersion can be neglected ($\omega\tau \ll 1$). In this region, Eq. (8) can be materially simplified:

$$[j_i(\mathbf{r})\, j_k(\mathbf{r}')]_\omega = \frac{\hbar\omega}{4\pi}\coth\frac{\hbar\omega}{2T}\cdot\frac{3\sigma_0}{4\pi}\Big\{\frac{\delta_{ik}}{\rho l^2}\int_0^1\Big(\frac{1}{u^2}-1\Big)e^{-\rho/ul}\,du + \frac{\partial^2}{\partial x_i\partial x_k}\Big[\frac{1}{\rho}\int_0^1(3u^2-1)\,e^{-\rho/ul}\,du\Big]\Big\} \quad (9)$$

or

$$[j_i(\mathbf{r})\, j_k(\mathbf{r}')]_\omega = \frac{\hbar\omega}{4\pi}\coth\frac{\hbar\omega}{2T}\cdot\frac{3\sigma_0}{4\pi}\Big\{\delta_{ik}\Big[\frac{e^{-\rho/l}}{\rho^2 l}-\frac{1}{\rho l^2}E_2\Big(\frac{\rho}{l}\Big)\Big] + \frac{\partial^2}{\partial x_i\partial x_k}\Big[\frac{1}{\rho}\Big(3E_4\Big(\frac{\rho}{l}\Big)-E_2\Big(\frac{\rho}{l}\Big)\Big)\Big]\Big\}, \quad (10)$$

where

$$E_n(x) = \int_1^\infty e^{-zx}z^{-n}\,dz.$$

In two limiting cases, it follows from Eq. (10) that

$$[j_i(\mathbf{r})\, j_k(\mathbf{r}')]_\omega \approx \frac{\hbar\omega}{4\pi}\coth\frac{\hbar\omega}{2T}\cdot\frac{3\sigma_0}{4\pi}\frac{\delta_{ik}}{\rho^2 l} \quad (\rho/l \ll 1);$$
$$[j_i(\mathbf{r})\, j_k(\mathbf{r}')]_\omega \approx \frac{\hbar\omega}{4\pi}\coth\frac{\hbar\omega}{2T}\cdot\frac{3\sigma_0}{2\pi}\frac{\nu_i\nu_k}{\rho^2 l}e^{-\rho/l};$$
$$\nu_i = \frac{\rho_i}{\rho} \quad \Big(\frac{\rho}{l}\gg 1\Big). \quad (11)$$

Therefore the random currents, as was to be expected, are correlated at distances of the order of the mean free path length l.

Using the correlation relations for the components of the random currents, we can, by means of Maxwell's equations, obtain correlation relations between the components of the electric and magnetic fields.[1]

However, it is better to proceed otherwise; from Maxwell's equations, we have

$$(\mathrm{curl}\,\mathbf{H})_i = \frac{4\pi}{c}\int K_{ik}(\mathbf{r},\mathbf{r}')\,E_k(\mathbf{r}')\,dv' + \frac{4\pi}{c}j_i^{\mathrm{for}}, \quad (12)$$
$$\mathrm{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}.$$

Eliminating $\mathbf{H}$ from the system (12), and solving for $\mathbf{E}$, we get the following expression for the electric field:

$$E_i = \frac{4\pi i}{\omega}\int\Lambda_{ik}(\mathbf{r},\mathbf{r}')\,j_k^{\mathrm{for}}(\mathbf{r}')\,dv'. \quad (13)$$

Introducing the generalized coordinates and generalized forces $x_a \to \mathbf{E}$, $f_a \to \mathbf{K}\Delta v/4\pi$, and also transforming (13) from an integration to a summation, just as was done in deriving Eq. (4), we find

$$[E_i(\mathbf{r})\,E_k(\mathbf{r}')]_\omega = 2\hbar\coth\frac{\hbar\omega}{2T}\cdot\mathrm{Im}\,\Lambda_{ik}(\mathbf{r},\mathbf{r}'). \quad (14)$$

It is simplest to find the kernel $\Lambda_{ik}(\mathbf{r},\mathbf{r}')$ in the case under consideration by going over to Fourier components in (12). Then, solving the resultant system relative to the Fourier components of the electric field intensity, and employing the inverse Fourier transformation, we get the following formulas:

$$\Lambda_{ik}(\boldsymbol{\rho}) = \frac{\omega^2}{4\pi^2c^2}\Big\{\frac{\delta_{ik}}{i\rho}\int_{-\infty}^{\infty}\frac{e^{ik\rho}}{L_1}k\,dk + \frac{\delta^2}{6}\frac{\partial^2}{\partial x_i\partial x_k}\frac{1}{\rho}\int_{-\infty}^{\infty}\frac{L_2}{L_1}\frac{(kl)^2e^{ik\rho}k\,dk}{1-\arctan kl/kl}\Big\}. \quad (15)$$

Here

$$L_1 = k^2 - \frac{3i}{\delta^2}\Big[\Big(1+\frac{1}{(kl)^2}\Big)\frac{\arctan kl}{kl}-\frac{1}{(kl)^2}\Big], \quad (16)$$
$$L_2 = -\Big\{1+\frac{3i}{(k\delta)^2}\Big[\frac{3}{(kl)^2}-\Big(1+\frac{3}{(kl)^2}\Big)\frac{\arctan kl}{kl}\Big]\Big\},$$

$\delta = c/\sqrt{2\pi\sigma_0\omega}$ is the penetration depth of the low frequency field in metals. As the mean free path approaches zero, Eq. (14) takes the form

$$[E_i(\mathbf{r})\,E_k(\mathbf{r}')]_\omega = \frac{\hbar}{2\pi}\coth\frac{\hbar\omega}{2T}\Big\{\frac{\omega^2}{c^2\rho}e^{-\rho/\delta}\sin\frac{\rho}{\delta}\,\delta_{ik} - \frac{\omega}{4\pi\sigma_0}\frac{\partial^2}{\partial x_i\partial x_k}\frac{e^{-\rho/\delta}}{\rho}\cos\frac{\rho}{\delta}\Big\}. \quad (17)$$

This equation can be obtained directly from Eqs. (90.23) of Ref. 4, if we assume the dielectric constant of the metal to be equal to $\epsilon = 4\pi\sigma_0 i/\omega$.

We consider the opposite limiting case (large mean free path, $l \gg \delta$). The general formulas (for any ratio between ρ and l) are very rough We limit ourselves to asymptotic expressions in the two cases:

1. $\rho \ll \delta \ll l$:

$$[E_i(\mathbf{r})E_k(\mathbf{r}')]_\omega = 2\hbar\coth\frac{\hbar\omega}{2T}\left\{A\frac{\omega^2}{c^2(\delta^2 l)^{1/3}}\delta_{ik} - \frac{\omega^2}{c^2}\frac{\delta^2 l^2}{6}\frac{\partial^2}{\partial x_i \partial x_k}\delta(\rho)\right\}, \qquad (18)$$

where

$$A = 1729/2^{1/3}\pi^{2/3}3^{1/6} \approx 570.$$

Here the spectral density of the energy per unit volume of the electric field in the metal, W_ω, is equal to

$$W_\omega = \frac{1}{8\pi}\{E^2\}_\omega = \frac{3A\hbar}{4\pi}\coth\frac{\hbar\omega}{2T}\cdot\frac{\omega^2}{c^2(\delta^2 l)^{1/3}}. \qquad (19)$$

In averaging over an infinitely small volume, the second component in Eq. (18) drops out.

2. $\delta \ll l \ll \rho$. For calculation of the integrals entering into Eq. (15), it is appropriate in this case to extend the path of integration to infinity[5] (we consider k a complex variable, $\mathrm{Im}\,k \to \infty$), taking into account all the singularities of the integral. The singularities of the integrand are the zeros of the denominator and $k = i$ (branch point). After transformation, the integrals reduce to a sum of residues and an integral from i to $i\infty$.

It is easy to see that for $\delta \ll l$ the sum of the residues is appreciably less than the integral, the asymptotic value of which leads to a correlation function of the following sort:

$$[E_i(\mathbf{r})E_k(\mathbf{r}')]_\omega = \frac{\hbar\omega}{6\pi\sigma_0}\coth\frac{\hbar\omega}{2T}\left[\frac{\delta_{ik}}{\rho} - \frac{\nu_i\nu_k}{l\ln^2(\rho/l)}\right]\frac{e^{-\rho/l}}{\rho^2}. \qquad (20)$$

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QUANTUM OSCILLATIONS OF THE HIGH-FREQUENCY SURFACE IMPEDANCE

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On the basis of general formulas obtained earlier by the author, a quantum-mechanical formula is found for the total surface impedance of metals at high frequencies, where the skin depth is small in comparison with the Larmor radius and with the electronic mean free path. The analysis is carried out for an arbitrary law of dispersion for the conduction electrons. Cases involving constant magnetic fields both parallel to the surface of the metal and inclined with respect to it are studied. The influence of the thickness of the metallic film on the quantum oscillations is ascertained. It is shown that an experimental study of the surface impedance in a strong magnetic field makes it possible in principle to reconstruct the shape of the Fermi surface and to determine the velocities of the electrons on it.

INTRODUCTION

In Ref. 1 we found the first non-vanishing quantum correction to the current density Δ_j^{qu} at high frequencies in a film whose width D satisfies the inequality

$$D > d = \left|\frac{c}{eH}\int_{t_0'}^{t_0''} v_\zeta\, dt_2\right|_{p_z = p_z^{ext}}$$

$$v_\zeta(t_0') = v_\zeta(t_0'') = 0, \quad dv_\zeta/dt_0' > 0, \quad dv_\zeta/dt_0'' < 0, \qquad \mathbf{(1.1)}$$