EFFECT OF COLLECTIVE INTERACTION OF ELECTRONS IN CYCLIC ACCELERATORS

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A mathematical technique is developed which permits one to take into account collective interaction of particles in cyclic accelerators. The method is then applied to the problem of electron capture in the betatron acceleration regime. The results obtained are compared with experiment.

It has been shown in a number of experimental papers[1-4] that in the case of certain processes in cyclic accelerators, particularly during injection, the collective interaction of particles plays a significant and even a dominant role.

Among the various kinds of interaction the following are the most essential ones:

1. Coulomb interaction which in the first approximation leads to a change in the frequency of betatron oscillations associated with a decrease in the effective restoring force, and also to the appearance of azimuthal inhomogeneities. As will be shown below, Coulomb repulsion plays an essential role in the process of the capture of electrons into the acceleration regime.

2. Magnetic interaction which fundamentally reduces to two effects. Firstly, attraction of currents which counteracts Coulomb repulsion; this has to be taken into account only at relativistic energies. Secondly, the effect of beam inductance which was pointed out by Kerst.[5] According to estimates that have been made, such an effect can be significant only in the case of a very rapid change of current in the chamber which never occurs in practice (see, for example, Kerst[5] and Rusanov[2]).

In addition to the above effects one could in principle take into account effects of two-body collisions in the beam which, however, produce an appreciable effect only at such high densities that in practice they can be always neglected.

A typical problem in which the collective interaction plays an essential role is the calculation of the current captured after injection into the betatron acceleration regime.

Recently, a number of papers (primarily by Soviet authors), among which one should particularly note Refs. 1 — 4, have been devoted to the experimental investigation of betatron capture.

The main experimental fact requiring theoretical explanation is the large value of the captured current which does not fit within the framework of the original concepts of Kerst and Serber.[6] Indeed, the slow increase of the magnetic field can produce a displacement of the instantaneous orbit and a change in the amplitude of oscillations only by an amount of the order of 10^{-3} to 10^{-2} cm/turn, which gives a very low value for the captured current since the majority of the particles must be lost on the back side of the injector. Moreover, it has been shown experimentally[1-3] that the value of the captured current within wide limits depends only very little on the distance between the filament of the injector and its edge (the so-called "gap"), which also does not agree with the Kerst and Serber theory. Moreover, recently it has been confirmed experimentally that capture takes place into a field constant in time, with the qualitative picture of the phenomenon being essentially undisturbed.[4]

As a result of the above investigations, it has been definitely established that the capture mechanism is unquestionably a collective mechanism, and that Coulomb interaction plays a fundamental role in it.

Since we are unable to give here even a brief review of the basic experimental facts and hypotheses relating to betatron capture (as was done, for example, by Rusanov[2]) we must merely note that the published papers do not contain any mathematical theory of the above phenomenon which would at all satisfactorily explain the available experimental facts. There exists only a series of unrelated attempts and hypotheses the majority of which is in qualitative disagreement with experiment.

One must, however, note that the above situation exists only for large injection currents when the collective interaction is very pronounced. For small currents, the theory agrees with experiment satisfactorily.

In dealing with such problems it is most natural to adopt a purely statistical approach making use of the method of the kinetic equation into which the self-consistent interaction has been introduced. This method is used below for the investigation of the injection of non-relativistic electrons into a betatron. The generality of the method employed allows us to extend it to a number of other problems which will be done in subsequent papers. In particular, we shall investigate collective interactions in accelerators with strong focussing, which is of interest from the point of view of obtaining high intensity beams in them.

1. FORMULATION OF THE PROBLEM

We shall describe the state of the system of electrons in the betatron chamber by means of a distribution function f(p, q, t) defined in coordinate and momentum space. We shall take for our starting point the equation

$$\frac{\partial f}{\partial t} + [\mathcal{H} f] = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} - \frac{f}{\tau} + F_s \ , \qquad (1.1)$$

which differs from the usual form of the kinetic equation by the terms $-f/\tau$ and F_S which take into account, respectively, the loss of particles on the chamber walls and on the injector, and the appearance of new particles due to the operation of the injector. In this equation, $\mathcal{H}$ is the Hamiltonian of the particle taking the self-consistent field into account. Generally speaking, in order to take the loss of particles into account rigorously one should have included in the Hamiltonian the interaction between the particles and the walls and the injector, but this encounters considerable difficulties. Therefore we take the effect of the walls and the injector into account purely phenomenologically, viz., we characterize the particles by some mean lifetime in the chamber τ which is a function of p, q and t and, generally speaking, a function of f(p, q, t). In other words, we shall consider that the change in the distribution function due to the loss of particles is proportional to $-f/\tau$.

In view of the above remarks we can neglect in the present problem the term $(\partial f/\partial t)_{\text{coll}}$ which takes into account two-body collisions.

In order to demonstrate more explicitly the method being employed, we shall discuss in the present article somewhat idealized conditions of betatron capture which, nevertheless, allow us to obtain the basic qualitative characteristics of the phenomenon and some quantitative estimates. We shall treat the one-dimensional case, i.e., we shall take into account only the radial motion of the electrons whose energy coincides with the equilibrium energy. Then, as is well known, the Hamiltonian may be written in the form:

$$\mathcal{H} = (p^2 + \alpha^4 q^2)/2m, \qquad (1.2)$$

where m is the electron mass, p is the radial component of its momentum, q is the deviation from the equilibrium orbit and α^2 (up to a constant factor) is the frequency of betatron oscillations

$$\alpha = (eH/c)^{1/2}(1-x)^{1/4}(1-n)^{1/4}, \qquad (1.3)$$

where x is the average charge density in the chamber which is to be determined expressed in units of the limiting density.* The self-consistent nature of the problem is already clear from the above since the density x is determined by the function f(p, q, t) which in turn is a solution of Eq. (1.1).

The equation is considerably simplified if we go over to new canonical variables P and Q by means of the generating function

$$V(q, Q, t) = \frac{\alpha^2 q^2}{2} \tan Q + \frac{\alpha^2 q^2}{2} \varepsilon, \qquad (1.4)$$

where we take the quantity $\epsilon = m\dot{\alpha}/\alpha^3$ to be small, which means that the adiabatic invariant is applicable to the betatron oscillations. It may then be easily shown that, up to terms of the second order in ϵ, the new Hamiltonian has the form

$$\mathcal{H}' = \alpha^2(t) P/m \qquad (1.5)$$

and Eq. (1.1) takes on the form

$$\frac{\partial \psi}{\partial t} + \frac{\alpha^2(t)}{m}\frac{\partial \psi}{\partial Q} = -\frac{\psi}{\tau} + F_s\ (P, Q, t), \qquad (1.6)$$

where $\psi(P, Q, t)$ is the distribution function in terms of the variables P and Q.

The form of the function τ may be chosen on the basis of the following considerations. Since it is physically obvious that particles, which have an oscillation amplitude a greater than the half width of the chamber a_2, are lost during a time shorter than the period of revolution, it is natural to assume that

$$\tau \equiv 0 \quad \text{for} \quad a > a_2. \qquad (1.7a)$$

On the other hand if the amplitude is smaller than the distance a_0 from the centre of the chamber to

*The limiting density is that density at which the force of Coulomb repulsion becomes equal to the magnetic focussing forces so that the effective restoring force becomes equal to zero [cf. (1.3)].

the edge of the injector then the particles have an infinite lifetime, i.e.

$$\tau = \infty \quad \text{for} \quad a < a_0. \tag{1.7b}$$

Finally, within the interval (a_0, a_2) for the variable a the lifetime τ is a function of the amplitude which for the sake of simplifying the calculations we shall assume to be equal to some constant τ.

It may be easily seen from formula (1.4) that the amplitude a is expressed in terms of the canonical momentum P in the following manner:

$$a = \sqrt{2P} / \alpha(t). \tag{1.8}$$

From this it may be easily seen that the quantity $\alpha(t)$ and consequently also $\tau(a)$ are unknown functions of the time determined by the distribution function ψ.

After integration of Eq. (1.6) over the cyclic variable Q and denoting the average value $\overline{\psi(P, Q, t)}$ simply by $\psi(P, t)$ we obtain the equation

$$\frac{\partial\psi}{\partial t} + \frac{\psi}{\tau} = F_s\ (P, t). \tag{1.9}$$

In future we shall assume that the injector cathode has vanishingly small radial dimensions and emits electrons with a certain angular spread so that the source function may be written in the form:

$$F_s\ (p, q, t) = i(t)\frac{\delta(q - a_1)}{\pi}\frac{p_0}{p_0^2 + p^2}, \tag{1.10}$$

where $i(t)$ is the emission current. Substituting into the above in accordance with (1.4) in place of q and p their values in terms of Q and P, and integrating over Q we obtain after neglecting small terms:

$$F_s\ (P, t) = \frac{i(t)}{\ }\frac{2p_0}{p_0^2 + \alpha^4(2P/\alpha^2 - a_1^2)}\operatorname{Re}\left(\frac{2P}{\alpha^2} - a_1^2\right)^{1/2}. \tag{1.11}$$

We shall assume the density of the beam to be equal to the total number of particles referred to a certain effective beam dimension a_{eff} which we shall fix later*

$$x = (1/a_{eff}\rho_{lim})\int_0^\infty \psi(P, t)\, dP, \tag{1.12}$$

where ρ_{lim} is the limiting density equal to Q_{lim}/a_2 and Q_{lim} is the limiting charge.

Thus relations (1.8), (1.7a, b), (1.11), (1.12) and the initial condition

$$\psi(P, 0) \equiv 0 \tag{1.13}$$

completely determine the problem of finding the current or the total number of particles in the chamber at an arbitrary time t.

2. SINGLE ELECTRON CAPTURE

Before proceeding to collective capture we shall first demonstrate how the fundamental features of single electron capture, i.e., capture at low currents, follow from the present treatment.

When the current circulating in the chamber is small the collective interaction is weak and we can take $x \ll 1$. Then assuming that the magnetic field increases linearly with time, we obtain

$$\alpha^2(t) = (eH_i/c)(1 + \gamma t)(1 - n)^{1/2}, \tag{2.1}$$

where $\gamma = H_i^{-1}\partial H/\partial t$ and H_i is the field at injection. Since α and τ now do not depend on ψ we can employ the Green's function method for the solution of Eq. (1.6). As is well known the Green's function G coincides up to a normalization factor with the solution of Eq. (1.9) if in expression (1.11) we set

$$i(t) = \delta(t - t'). \tag{2.2}$$

Further, we assume for the sake of definiteness that the mean exit angle for particles from the injector is much larger than the chamber aperture, i.e., that

$$p_0^2 \gg \alpha^4(0)\,[a_2^2 - a_1^2].$$

Then after some elementary calculations we obtain

$$G(P, t, t') = \begin{cases} \dfrac{\sqrt{2}}{\pi p_0}\alpha(t')\exp\left[-\displaystyle\int_{t'}^{t}\frac{dx}{\tau(\sqrt{2P}/\alpha(x))}\right] \times \operatorname{Re}\left(P - \dfrac{a_1^2\alpha^2(t')}{2}\right)^{1/2} & \text{for } t > t', \\ 0 & \text{for } t < t' \end{cases} \tag{2.3}$$

We are interested in the captured current, i.e., in the solution for $t \to \infty$. From expressions (1.7a, b) and (2.1) it follows that the Green's function for the captured current is given by

$$G_\infty(t') = \frac{1}{T}\int_0^\infty \lim_{t\to\infty} G(P, t, t')\, dP$$

$$= \frac{\sqrt{2}}{\pi p_0 T}\alpha(t')\int_0^{\alpha^2(t')a_2^2/2}\exp\frac{1}{\tau}\left[t' - \frac{1}{\gamma}\left(\frac{2Pc}{a_0^2 eH_i} - 1\right)\right] \times \operatorname{Re}\left(P - \frac{a_1^2\alpha^2(t')}{2}\right)^{1/2} dP. \tag{2.4}$$

On carrying out the integration we obtain the Green's function $G_\infty(t')$ and then we immediately

*It will be shown below that the results depend only very little on the choice of the parameter a_{eff}.

find the captured current J_c:

$$J_c = \int_{-\infty}^{+\infty} i(t')\, G_\infty(t')\, dt' = \frac{a_0}{p_0 T}\left(\frac{eH_i \gamma\tau}{\pi c}\right)^{1/2} (1-n)^{1/4} \times \int_0^{t_i} i(t')\, \alpha(t') \exp\left\{-\frac{1+\gamma t'}{\gamma\tau}\left(\frac{a_1^2}{a_0^2}-1\right)\right\} \times \Phi\left(\left[\frac{a_2^2-a_1^2}{a_0^2\gamma\tau}(1+\gamma t')\right]^{1/2}\right) dt', \tag{2.5}$$

where Φ is Kramp's function, T is the period of revolution at the end of injection. Since usually $\gamma\tau \ll 1$, $\delta \ll 1$ (where $\delta = 1 - a_0/a_1$), formula (2.5) can be simplified:

$$J_c = \frac{k_i a_0}{a_2 T}\left(\frac{\gamma\tau}{\pi}\right)^{1/2} e^{-2\delta/\gamma\tau} \int_0^{t_i} i(t')(1+\gamma t')^{1/2} e^{-2t'\delta/\tau} dt', \tag{2.6}$$

where t_i is the time when injection ends, and k_i is the ratio of the chamber aperture to the mean angle of spread of the beam leaving the injector: $k_i = (eH_i a_2/cp_0)(1-n)^{1/2}$.

We have thus obtained an expression for the captured current for an arbitrary shape of the injector current pulse [we note that if the rate of increase of the magnetic field H is sufficiently great, i.e., $\gamma\tau > 1$, it is necessary to use the more general formula (2.5)].

We have assumed above that the angle of spread is sufficiently large, i.e., that the coefficient k_i is sufficiently small. We shall quote without giving its derivation the formula for the captured current obtained in the case when the injector gives no angular spread at all and all the particles are injected strictly azimuthally, with the current pulse being of rectangular shape:

$$J_c = \frac{3}{4}\frac{i\tau}{T\delta}\left[1 - e^{-4\delta t_i/3\tau}\right] e^{-4\delta/3\gamma\tau}. \tag{2.7}$$

From formulas (2.6) and (2.7) we obtain the following fundamental characteristics of capture in the case of low current: (1) a low value of the captured current, since the value of the quantity τ/T varies within the limits[4] 5 to 8; (2) exponential dependence of the captured current on the "gap" δ, since in practice always $\gamma\tau \ll 1$; (3) a linear dependence of the captured current on the emission current.

3. COLLECTIVE CAPTURE

In discussing collective capture, we shall neglect the explicit dependence of α on the time, i.e., the variation of field with time, which makes it equivalent to the problem of injection into a constant field. It can be shown that this assumption is quite justified for sufficiently large injector currents. The criterion which determines that collective rather than single-electron capture plays the dominant role will be given below.

Since for $t > t_i$, $F_s = 0$, i.e., Eq. (1.9) reduces to a homogeneous one, the range of variation for the variable t in Eq. (1.9) can be conveniently broken up into two parts: one where $t \le t_i$, and the other where $t \ge t_i$.

We discuss the solution in the first region. On introducing the new variable

$$\varphi = (1-x)^{1/2} \tag{3.1}$$

and taking into account (1.7a, b), (1.9) and (1.11), we obtain the equations which determine the circulating current:

$$\varphi^2 = 1 - \frac{2}{\pi T J_{\lim}} \int_0^t e^{-(t-t')/\tau} i(t') \tan^{-1}\frac{k_i}{\eta}\varphi^{1/2}(t') \times [\eta^2\varphi(t) - \varphi(t')]^{1/2} dt', \quad t \leqslant t_1 \tag{3.2}$$

$$\varphi^2 = 1 - \frac{2}{\pi T J_{\lim}} \int_{\tilde{t}(t)}^t e^{-(t-t')/\tau} i(t') \tan^{-1}\frac{k_i}{\eta}\varphi^{1/2}(t') \times [\eta^2\varphi(t) - \varphi(t')]^{1/2} dt', \quad t \geqslant t_1 \tag{3.3}$$

where $\eta = a_2/a_1$, $J_{\lim} = Q_{\lim}/T$ is the limiting current, while t_1 and $\tilde{t}(t)$ are determined by the relations

$$\varphi(t_1) = \eta^{-2}, \tag{3.4}$$

$$\varphi(\tilde{t}) = \eta^2\varphi(t). \tag{3.5}$$

In the general case, the solution of these equations is quite a complicated problem, and therefore we shall discuss here the case most important in practice when the injector current reaches a stationary value i after a time $\lesssim \tau$.

In this case the final value of the circulating current will be determined by the equation

$$\varphi_{st}^2 - 1 + \frac{2i\tau}{\pi T J_{\lim}} \tan^{-1}\left[\frac{k_i}{\eta}(\eta^2-1)^{1/2}\varphi_{st}\right] = 0. \tag{3.6}$$

We note that if $t_i \gg \tau$, which always holds in practice, then the captured current does not depend on the way the current behaves during injection, but is determined only by its stationary value $1 - \varphi_{2st}$, or more accurately by the value of x at the time when the injector stops operating: $x_i = x(t_i)$.

We now proceed to determine the captured current. From Eq. (1.9) we obtain

$$x = \frac{a_2}{a_{\mathrm{eff}}} \int_0^\infty \exp\left\{-\int_{t_i}^t \frac{d\xi}{\tau(\sqrt{2P}/\alpha(\xi))}\right\} U(P, t_i)\, dP, \tag{3.7}$$

where

$$U(P,\ t_i) = \frac{2p_0}{\pi T}\int_0 \exp\left\{-\frac{t'-t_i}{\tau}\right\}\frac{i(t')}{J_{\lim}}[p_0^2 + 2\alpha^2(t')P$$

$$-\alpha^4(t')a_1^2]^{-1}\ \mathrm{Re}\left[\frac{2P}{\alpha^2(t')} - a_1^2\right]^{-1/2} dt', \qquad (3.8)$$

Generally speaking, a_{eff} is some monotonic function of the time.

On taking into account the explicit dependence of τ on P equation (3.8) can now be brought into the form:

$$x = x_i e^{-(t-t_i)/\tau},\ t \leqslant t_2, \qquad (3.9)$$

$$x = \frac{a_2}{a_{\mathrm{eff}}}\int_{P_1}^{a_0^2\alpha^2(t)/2} U(P)\exp\left\{-\frac{t''(P)-t_i}{\tau}\right\}dP$$

$$+\frac{a_2}{a_{\mathrm{eff}}}\exp\left\{-\frac{(t-t_i)}{\tau}\right\}\int_{a_0^2\alpha^2(t)/2}^{\eta^2 P_1} U(P)\,dP,\ t \geqslant t_2, \qquad (3.10)$$

where $P_1 = a_i^2\alpha^2(t_i)/2$, $t_2 = t''(P_i)$ and $t''(P)$ is determined by the relation

$$\alpha(t'') = \sqrt{2P}/a_0.$$

We note that if the function U(P) is known Eq. (3.10) allows us to obtain the captured current also in the case that the magnetic field H depends explicitly on the time.

As is shown by calculations made in the two limiting cases $a_{\mathrm{eff}} = a_2$ and $a_{\mathrm{eff}} = a_0$ the value of the captured current for usual values of the accelerator parameters depends very little on the magnitude of a_{eff} (Fig. 1). Therefore in order to simplify the calculations we shall assume that $a_{\mathrm{eff}} = \mathrm{const.}$*

In this case the solution of Eq. (3.10) is obtained from

$$\int_{x(t_2)}^{x}\frac{dx'}{\Psi(x')} = -\frac{a_2}{a_{\mathrm{eff}}}\exp\left\{-\frac{t_2-t_i}{\tau}\right\}\left[1-\exp\left\{-\frac{t-t_2}{\tau}\right\}\right], \qquad (3.11)$$

where

$$\Psi(x') = \int_{a_0^2\alpha^2(x)/2}^{\eta^2 P_1} U(P)\,dP,$$

while the value of the captured current is obtained from the equation

$$\int_{x(t_2)}^{x_c}\frac{dx'}{\Psi(x')} = -\frac{a_2}{a_{\mathrm{eff}}}\exp\left\{-\frac{t_2-t_i}{\tau}\right\}. \qquad (3.12)$$

In the general case of arbitrary p_0 in the expression for the source function in (1.11), the integral on the left hand side of (3.11) is not expressible in terms of elementary functions. Therefore, in order to obtain the final expression in analytic form we shall consider the case most frequently encountered in practice when the mean square exit angle from the injector is much greater than the chamber aperture. Calculations carried out in another limiting case of a sharply focussed beam show that the qualitative picture of the phenomenon is preserved.

*It should be noted that similar but more complicated calculations can be carried out in the case that $a_{\mathrm{eff}} = a_{\mathrm{eff}}(t)$.

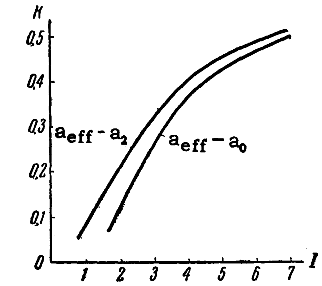

FIG. 1. Dependence of the collective capture coefficient on the emission current.

In the case under discussion, (3.11) is transformed into

$$F\left(\frac{a_0\varphi_3}{a_1\varphi_i}\right) = \frac{a_2}{a_{\mathrm{eff}}}\frac{1-(a_1/a_0)^4\varphi_i^2}{2(a_1/a_0)^4\varphi_i^2}\left[1-\exp\left\{-\frac{t-t_i}{\tau}\right\}\right], \qquad (3.13)$$

where

$$F(z) = -2\beta\left[\frac{1}{3}(z-1)^{3/2} + \frac{\beta}{2}(z-1) + (\beta^2+1)(z-1)^{1/2}\right.$$

$$\left.+\beta(\beta^2+1)\ln\frac{\beta-(z-1)^{1/2}}{\beta}\right],\quad \beta = (\eta^2-1)^{1/2}. \qquad (3.14)$$

At the beginning of this section we noted that the dependence of the magnetic field on the time can be neglected if the injector current exceeds a certain value equal to i_{cr}. In the case under discussion

$$i_{\mathrm{cr}} = \frac{1-(1-\delta)^4}{(1-\delta)^2}\frac{\eta\pi T}{2(\eta^2-1)^{1/2}k_i\tau}J_{\lim}. \qquad (3.15)$$

Thus the condition $i > i_{\mathrm{cr}}$ corresponds to the region of collective capture, $i \lesssim i_{\mathrm{cr}}$ corresponds to the region of mixed capture, and $i \ll i_{\mathrm{cr}}$ to the region of single-electron capture. For example, in the case of the 30 Mev synchroton at the Physics Institute of the Academy of Sciences, we have the value $i_{\mathrm{cr}} \approx 10$ ma. The experimental value of the current at which the mechanism of collective capture begins to play a role is approximately equal to 3 ma. This value is less than i_{cr} just as it should be.

In conclusion we indicate the condition of validity of the above formulas which follows from the possibility of writing the Hamiltonian in the form (1.5), i.e., the condition for the existence of an adiabatic invariant for betatron oscillations. It

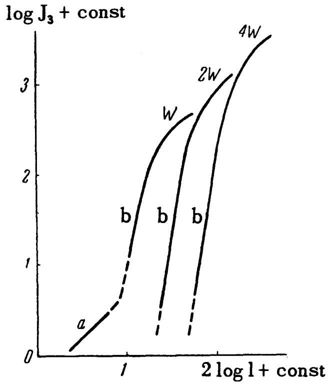

FIG. 2. Dependence of the captured current on the emission current for different injection energies W; a – single electron capture, b – collective capture.

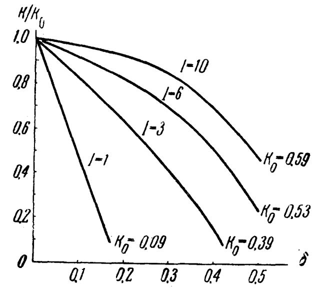

FIG. 3. Dependence of the capture coefficient K on the relative "gap" size:

$$I = \frac{2k_i}{\pi\eta}(\eta^2-1)^{1/2}\frac{\tau}{T}\frac{i}{J_{\text{lim}}}$$

reduces to the requirement $\epsilon < 1$, or taking expression (3.9) into account, to

$$i < \left(\frac{8\pi\tau\sqrt{1-n}}{T}\right)^{1/3}\frac{\eta\pi T}{2(\eta^2-1)^{1/2}k_i\tau}J_{\text{lim}}. \qquad (3.16)$$

4. DISCUSSION OF RESULTS

1. The collective capture coefficient, i.e., the ratio of the captured current to the maximum current circulating in the chamber at first grows rapidly as the injector current increases, and then shows a tendency towards saturation, reaching a value of 0.3 — 0.5 at the boundary of the region of applicability of the adiabatic approximation [see (3.20)]. The characteristic dependence of the capture coefficient on the emission current is given in Fig. 1 for two values of the parameter a_{eff}.

2. Since the maximum current circulating in the chamber is close to the limiting current and depends only slightly on the emission current the dependence of the captured current on the emission current also shows saturation. Figure 2 shows appropriate curves for single electron and for collective capture for different values of the injection energy. These curves show good qualitative agreement with experimental data (see, for example, Refs. 1 — 3). The region of mixed capture for which the theoretical curve has not been calculated is indicated by a dotted line. We note that the ranges of variation of the variables in the graphs shown above all satisfy the condition for the adiabatic approximation (3.16).

3. Figure 3 gives (in relative units) the dependence of the captured current on the "gap", which also agrees well with experiment.[3] As was shown in section 2 for small emission currents the dependence on the "gap" is of exponential nature.

In conclusion the authors express their gratitude to A. A. Kolomenskii, M. S. Rabinovich and P. A. Riasin for fruitful discussions.

Note added in proof (March 18, 1958). Calculations which we have made show that taking into account the deviation of the electron energy from its equilibrium value somewhat reduces the capture coefficient and leads to a lowering of the curves of Fig. 2 for large values of i. In all other respects the description of the phenomenon is preserved.

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