consider the case of "specular" reflection, i.e., where the integral in (19) becomes identically zero, since this has no physical significance. In this case one would have to consider higher terms in the expansion of ζ'_{surf} in $v/\omega\delta$.

It is interesting to compare the magnitudes of ζ'_{surf} and ζ'_{el}. It stands to reason that they can be estimated only roughly; in any case, no more accurately than the nearest order of magnitude, since at the present time the functions which enter into the formulas are not known. This is particularly true of ζ'_{el}, for which the expression is of an extremely complicated type. The estimates give

$$\zeta'_{surf} \sim v/c \sim 10^{-2}; \quad \zeta'_{el} \sim m^{3/2}\sigma\omega^2/\hbar e n^{1/2} \sim 10^{-30}\omega^2.$$

It can be seen that even up to a frequency of $\omega \sim 10^{14}$, ζ'_{el} is, generally speaking, less than ζ'_{surf}.

As for the absorption of light accompanied by the emission of phonons, Holstein[5] has shown, in the case $\hbar\omega \gg k\Theta$ (Θ being the Debye temperature) that the corresponding component of ζ', like ζ'_{surf}, is independent of ω and is of the same order of magnitude as ζ'_{surf}, and is correspondingly smaller at lower frequencies. It would be very difficult to obtain an exact formula for this, since it depends chiefly on the short wavelength phonons with $\hbar\omega \sim k\Theta$, i.e., with wavelengths of the order of the lattice spacing.

In conclusion I would like to thank Academician L. D. Landau for advice and discussions.

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SCATTERING OF PARTICLES OF ARBITRARY SPIN

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We study the geometrical consequences for elastic scattering of the fact that nuclear particles possess a spin. The scattering matrix for particles of arbitrary spin is constructed, and those quantities which are experimentally measurable (cross section, polarization, and polarization correlation) are expressed in terms of its matrix elements. We consider the question of the completeness of a polarization experiment. We show that to reconstruct the scattering matrix it is necessary to measure either the cross section for scattering of a polarized beam by a polarized target, or the polarization correlation after scattering (with an initially unpolarized state), or finally measure the change of polarization of the incident particles after scattering (repeated scattering). The last experiments will be sufficient only if the spin of the particles in the beam is not less than the target spin.

THE analysis of angular distributions and polarization in nuclear reactions is done by two methods. The first method, that of phase analysis, has been investigated in detail and generalized to the case of arbitrary spins.[1] The second method, that of Dalitz, Wolfenstein and Ashkin,[2] which constructs the scattering amplitude as a function of the initial and final wave vectors and spin operators, has been investigated for reactions involving particles of spin 0, ½ and 1.[3-6] The present paper gives the extension of this method to the elastic scattering of particles of arbitrary spin.

1. CONSTRUCTION OF THE SCATTERING MATRIX

The general method for constructing the scattering matrix is to form all possible scalars from the spin operators and the initial and final wave vectors $\mathbf{k}_i$ and $\mathbf{k}_f$. As spin operators, we shall use the irreducible tensor operators T^q_κ, normalized by the condition

$$\mathrm{Sp}\,\{T^q_\varkappa (T^{q'}_{\varkappa'})^+\} = \delta_{qq'}\delta_{\varkappa\varkappa'}.$$

If the spin of the incident particles is s_1 and the target spin s_2, $(s_1, s_2 \neq 0)$ we can use the irreducible combinations of products of tensor operators:

$$T^q_\varkappa(q_1, q_2) = \sum_{\varkappa_1\varkappa_2} (q_1 q_2 \varkappa_1 \varkappa_2 \mid q\varkappa)\, T^{q_1}_{\varkappa_1} \times T^{q_2}_{\varkappa_2}. \tag{1}$$

The following functions, which transform according to an irreducible representation of the rotation group, can be formed from the two unit vectors $\mathbf{k}_i$ and $\mathbf{k}_f$

$$\Psi^\varkappa_{ql_1l_2}(\mathbf{k}_i, \mathbf{k}_f) = \sum_{m_1m_2} (l_1 l_2 m_1 m_2 \mid q\varkappa)\, Y^{m_1}_{l_1}(\mathbf{k}_i)\, Y^{m_2}_{l_2}(\mathbf{k}_f).$$

The spherical harmonics are normalized by the condition $\sum_m |Y^m_l|^2 = 1$. Since the scattering matrix must be even under space inversion, it can contain only those functions $\Psi^\kappa_{ql_1l_2}$ for which $l_1 + l_2$ is even. If $l_1 + l_2 > q + 1$, the function $\Psi^\kappa_{ql_1l_2}$ can be written as a linear combination with scalar coefficients of the same functions, but with $l_1 + l_2 = q$ if q is even, or with $l_1 + l_2 = q + 1$ if q is odd. Thus in constructing the scattering matrix we can restrict ourselves to functions of $\mathbf{k}_i$ and $\mathbf{k}_f$ of the form

$$\Psi^\varkappa_{q\lambda}(\mathbf{k}_i, \mathbf{k}_f) = \sum_{m_1m_2} (r+\lambda \;\; r-\lambda \;\; m_1 m_2 \mid q\varkappa)\, Y^{m_1}_{r+\lambda}(\mathbf{k}_i)\, Y^{m_2}_{r-\lambda}(\mathbf{k}_f), \tag{2}$$

where $r = q/2$ if q is even, and $r = (q+1)/2$ if q is odd. The scattering matrix can thus be written as

$$M(\mathbf{k}_i, \mathbf{k}_f) = \sum_{q\varkappa q_1 q_2} [T^q_\varkappa(q_1, q_2)]^+ \sum_{\lambda=-r}^{r} a^q_\lambda(q_1, q_2)\, \Psi^\varkappa_{q\lambda}(\mathbf{k}_i, \mathbf{k}_f). \tag{3}$$

If the spin of one of the particles is zero, the matrix becomes

$$M(\mathbf{k}_i, \mathbf{k}_f) = \sum_{q\varkappa} (T^q_\varkappa)^+ \sum_{\lambda=-r}^{r} a^q_\lambda \Psi^\varkappa_{q\lambda}(\mathbf{k}_i, \mathbf{k}_f). \tag{3a}$$

The number of independent scalar functions of angle and energy, $a^q_\lambda(q_1, q_2)$, is reduced if we impose the condition of time reversibility of the scattering process. If the time reversal operator is written as UK, where K is the complex conjugation (cf. Ref. 7), the reversibility condition is expressed as:

$$UM^*(\mathbf{k}_i, \mathbf{k}_f)\, U^+ = M^+(\mathbf{k}_f, \mathbf{k}_i), \tag{4}$$

where * denotes complex conjugation and + means Hermitian conjugation. We then get

$$a^q_{-\lambda}(q_1, q_2) = (-1)^{q_1+q_2+q} a^q_\lambda(q_1, q_2); \tag{5}$$

the operator U has the property

$$UT^q_\varkappa(q_1, q_2)\, U^+ = (-1)^{q+\varkappa} T^q_{-\varkappa}(q_1, q_2).$$

If one of the particles has spin zero, the reversibility condition gives

$$a^q_\lambda = a^q_{-\lambda}. \tag{5a}$$

If the particles are identical, a symmetry condition is imposed on the scattering matrix, giving the relation

$$a^q_\lambda(q_1, q_2) = a^q_\lambda(q_2, q_1). \tag{6}$$

2. DENSITY MATRIX. CROSS SECTION. POLARIZATION.

The density matrix for the spin state of the two particles, (the incident and target particles) is also conveniently expressed in terms of the tensor operators $T^q_\kappa(q_1, q_2)$:

$$\rho = \sum_{q_1q_2q\varkappa} \rho_{q\varkappa q_1 q_2} T^q_\varkappa(q_1, q_2).$$

The expansion coefficients are identical with the statistical tensors introduced by Fano, and have the physical significance of being polarization tensors. The final density matrix ρ is related to the initial density matrix $\rho^{(0)}$ by

$$\rho = M\rho^{(0)}M^+/\sigma, \qquad \sigma = \mathrm{Sp}\,\{M\rho^{(0)}M^+\}. \tag{7}$$

If we go over to the polarization tensors, we get

$$\rho_{q_2\varkappa_2q_{21}q_{22}} = \frac{1}{\sigma} \sum K^{q_1\varkappa_1q_{11}q_{12}}_{q_2\varkappa_2q_{21}q_{22}}(\mathbf{k}_i, \mathbf{k}_f)\, \rho^{(0)}_{q_1\varkappa_1q_{11}\varkappa_{12}}, \tag{8}$$

where

$$K^{q_1\varkappa_1q_{11}q_{12}}_{q_2\varkappa_2q_{21}q_{22}}(\mathbf{k}_i, \mathbf{k}_f) = \mathrm{Sp}\,\{M(\mathbf{k}_i, \mathbf{k}_f)\, T^q_\varkappa(q_1, q_2)\, M^+(\mathbf{k}_i, \mathbf{k}_f)\, [T^q_\varkappa(q_1, q_2)]^+\}. \tag{9}$$

Taking the trace gives the following expression:

$$K^{q_1\varkappa_1q_{11}q_{12}}_{q_2\varkappa_2q_{21}q_{22}}(\mathbf{k}_i, \mathbf{k}_f) = (-1)^{q_2+\varkappa_2} \sum_z (q_1 q_2 \varkappa_1 - \varkappa_2) \mid z\zeta)$$
$$\times \sum_{p_i\lambda_i} \varphi^{z\zeta}_{p_1\lambda_1p_2\lambda_2}(\mathbf{k}_i, \mathbf{k}_f) \sum_{p_i k} a^{p_1}_\lambda(p_{11}, p_{12})\, [a^{p_2}_{\lambda_2}(p_{21}, p_{22})]^*$$

$$\times \sum_{z_i} \begin{pmatrix} s_1 & s_1 & q_{11} \\ s_1 & s_1 & q_{21} \\ p_{11} & p_{21} & z_1 \end{pmatrix} \begin{pmatrix} s_2 & s_2 & s_{12} \\ s_2 & s_2 & q_{22} \\ p_{12} & p_{22} & z_2 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} & p_1 \\ p_{21} & p_{22} & p_2 \\ z_1 & z_2 & z \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} & q_1 \\ q_{21} & q_{22} & q_2 \\ z_1 & z_2 & z \end{pmatrix}, \quad (10)$$

where

$$\varphi^{z\zeta}_{p_1\lambda_1 p_2\lambda_2}(\mathbf{k}_i,\ \mathbf{k}_f) = \sum_{ll'} (r_1 + \lambda_1\ r_2 + \lambda_2 \,\|\, l)(r_1 - \lambda_1\ r_2 - \lambda_2 \,\|\, l')$$
$$\times \begin{pmatrix} r_1 + \lambda_1 & r_1 - \lambda_1 & p_1 \\ r_2 + \lambda_2 & r_2 - \lambda_2 & p_2 \\ l & l' & z \end{pmatrix} \Psi^{\zeta}_{zll'}(\mathbf{k}_i,\ \mathbf{k}_f),$$
$$(l_1 l_2 \,\|\, L) \equiv (l_1 l_2 00 \,|\, L0). \quad (11)$$

If one of the spins, for example the target spin s_2, is zero, the expression simplifies somewhat:

$$K^{q_1\varkappa_1}_{q_2\varkappa_2}(\mathbf{k}_i,\ \mathbf{k}_f) = (-1)^{q_2+\varkappa_2} \sum_z (q_1 q_2 \varkappa_1 - \varkappa_2 \,|\, z\zeta)$$
$$\times \sum_{p_i\lambda_i} \varphi^{z\zeta}_{p_1\lambda_1 p_2\lambda_2}(\mathbf{k}_i,\ \mathbf{k}_f)\, a^{p_1}_{\lambda_1} [a^{p_2}_{\lambda_2}]^* \begin{pmatrix} s_1 & s_1 & q_1 \\ s_1 & s_1 & q_2 \\ p_1 & p_2 & z \end{pmatrix} \quad (11a)$$

The meaning of all the quantities is clear from the way they were introduced. The three-by-three tableaus in parentheses are the coefficients of the unitary transformation between different schemes for coupling of four vectors.

The following important equality follows from time reversibility:

$$K^{q_1\varkappa_1 q_{11} q_{12}}_{q_2\varkappa_2 q_{12} q_{22}}(\mathbf{k}_i,\ \mathbf{k}_f) = (-1)^{q_1+\varkappa_1+q_2+\varkappa_2} K^{q_2 - \varkappa_2 q_{21} q_{22}}_{q_1 - \varkappa_1 q_{11} q_{12}}(\mathbf{k}_f,\ \mathbf{k}_i). \quad (12)$$

This is not the only condition which the coefficients $K^{\cdots}_{\cdots}$ satisfy. From the conservation of parity, these coefficients are even functions with respect to simultaneous change of the signs of the vectors $\mathbf{k}_i$ and $\mathbf{k}_f$. Therefore, for example, in the coordinate system in which the z axis is along $[\mathbf{k}_i\mathbf{k}_f]$, the sum $\kappa_1 + \kappa_2$ can take on only even values.

It is not difficult to express all measurable quantities — cross section, polarization, and polarization correlation, in terms of the coefficients $K^{\cdots}_{\cdots}$. We now give these expressions.

A. The cross section for scattering of an unpolarized beam by an unpolarized target is

$$\sigma_0 = K^{0000}_{0000}(\mathbf{k}_i, \mathbf{k}_f). \quad (13)$$

B. The polarization of the scattered beam under these same initial conditions is

$$\rho_{q\varkappa}(\mathbf{k}_i,\ \mathbf{k}_f) = K^{0000}_{q\varkappa q0}(\mathbf{k}_i,\ \mathbf{k}_f) / \sqrt{2s_1 + 1}\, \sigma_0. \quad (14)$$

As a consequence of the parity condition mentioned above, in the coordinate system in which the z axis is parallel to $[\mathbf{k}_i\mathbf{k}_f]$, κ takes on only even values. In particular, this means that the polarization is always along $[\mathbf{k}_i\mathbf{k}_f]$. In the system in which the z axis is along $\mathbf{k}_i$ or $\mathbf{k}_f$ and the y axis is along $[\mathbf{k}_i\mathbf{k}_f]$, the same condition is expressed differently:

$$\rho_{q-\varkappa}(\mathbf{k}_i,\ \mathbf{k}_f) = (-1)^{q+\varkappa} \rho_{q\varkappa}(\mathbf{k}_i,\ \mathbf{k}_f). \quad (14a)$$

Thus single scattering gives rise to a state of polarization of a special type. In order to obtain a general state of polarization, double scattering is necessary (if we disregard other means of polarization of particles, such as the use of a magnetic field).

C. The cross section for scattering of a polarized beam by an unpolarized target is

$$\sigma(\mathbf{k}_i,\ \mathbf{k}_f) = \sqrt{2s_1 + 1} \sum_{q\varkappa} K^{q\varkappa q0}_{0000}(\mathbf{k}_i,\ \mathbf{k}_f)\, \rho^{(0)}_{q\varkappa}. \quad (15)$$

If follows immediately from (12) that measurement of the polarization gives the same information as measurement of the scattering cross section of a polarized beam. Considerations analogous to those of case B show that the angular distribution is affected by only those $\rho^{(0)}_{q\kappa}$ for which κ is even (in the system with the z axis along $[\mathbf{k}_i\mathbf{k}_f]$). In order to determine the remaining part of the polarization tensor, one must use double scattering (once again assuming that we disregard other methods for analyzing the polarization). If we make use of (12) and (14), the expression (15) can be rewritten as

$$\sigma(\mathbf{k}_i,\ \mathbf{k}_f) = (2s_1 + 1)\,\sigma_0 \sum_{q\varkappa} (-1)^{q+\varkappa} \rho_{q-\varkappa}(\mathbf{k}_f, \mathbf{k}_i)\, \rho^{(0)}_{q\varkappa}. \quad (15a)$$

It is also not difficult to obtain the cross section for double scattering of an initially unpolarized beam by an unpolarized target. From (15a) we get immediately

$$\sigma_d = (1/\sigma^{(1)}_0) \sum_{q\varkappa} K^{(1)0000}_{q\varkappa q0}(\mathbf{k}_1,\ \mathbf{k}_2)\, K^{(2)q\varkappa q0}_{0000}(\mathbf{k}_2,\ \mathbf{k}_3), \quad (16)$$

or

$$\sigma_d = (2s_1 + 1)\,\sigma^{(2)}_0 \sum_{q\varkappa} (-1)^{q+\varkappa} \rho^{(1)}_{q\varkappa}(\mathbf{k}_1, \mathbf{k}_2)\, \rho^{(2)}_{q-\varkappa}(\mathbf{k}_3, \mathbf{k}_2) \quad (16a)$$

($\mathbf{k}_1$ is the direction of incidence of the beam, $\mathbf{k}_2$ its direction after the first scattering, and $\mathbf{k}_3$ its direction after the second scattering).

In a coordinate system with the z axis along $\mathbf{k}_2$, the double scattering cross section is given by

$$\sigma_d = (2s_1 + 1)\,\sigma^{(2)}_0 \sum_{\varkappa=0}^{2s_1} A_\varkappa \cos \varkappa\varphi;$$
$$A_\varkappa = 2 \sum_{q=\varkappa}^{2s_1} \rho^{(1)}_{q\varkappa}(\vartheta_1)\, \rho^{(2)}_{q\varkappa}(\vartheta_2) \quad (\varkappa \neq 0);$$
$$A_0 = \sum_{q=0}^{2s_1} \rho^{(1)}_{q0}(\vartheta_1)\, \rho^{(2)}_{q0}(\vartheta_2). \quad (16b)$$

Here $\rho^{(1)}_{q\kappa}(\vartheta_1)$ and $\rho^{(2)}_{q\kappa}(\vartheta_2)$ denote the values of the tensors $\rho^{(1)}_{q\kappa}(\mathbf{k}_1, \mathbf{k}_2)$ and $\rho^{(2)}_{q\kappa}(\mathbf{k}_3, \mathbf{k}_2)$ in the coordinate systems with the y axis perpendicular to the plane of the first and second scattering, respectively (in both cases the z axis is along $\mathbf{k}_2$); φ is the angle between the planes of scattering.

D. The polarization of the scattered beam when a polarized beam is scattered by an unpolarized target is

$$\rho_{q\varkappa}(\mathbf{k}_i, \mathbf{k}_f) = \frac{1}{\sigma(\mathbf{k}_i, \mathbf{k}_f)} \sum_{q'\varkappa'} K_{q\varkappa q0}^{q'\varkappa' q'0}(\mathbf{k}_i, \mathbf{k}_f)\, \rho_{q'\varkappa'}^{(0)}. \quad (17)$$

E. The cross section for scattering of a polarized beam by a polarized target is

$$\sigma = \sqrt{(2s_1+1)(2s_2+1)} \sum_{q_1 q_2 q\varkappa} K_{0000}^{q\varkappa q_1 q_2}(\mathbf{k}_i, \mathbf{k}_f)\, \rho_{q\varkappa q_1 q_2}^{(0)}. \quad (18)$$

F. The polarization correlation in scattering of an unpolarized beam by an unpolarized target is

$$\rho_{q\varkappa q_1 q_2}(\mathbf{k}_i, \mathbf{k}_f) = K_{q\varkappa q_1 q_2}^{0000}(\mathbf{k}_i, \mathbf{k}_f) / \sigma_0 \sqrt{(2s_1+1)(2s_2+1)}. \quad (19)$$

From (12) it follows immediately that experiments E and F are equivalent from the point of view of the information which they give.

3. COMPLETENESS OF POLARIZATION EXPERIMENTS

The number of independent scalar functions of energy and angle, $a_\lambda^q(q_1, q_2)$, which appear in the scattering matrix (3), is equal (if we disregard time reversibility) to

$$N_0(s_1, s_2) = {}^1/_2[(2s_1+1)^2(2s_2+1)^2+1], \quad (20)$$

if both spins are integral, and to

$$N_0(s_1, s_2) = {}^1/_2(2s_1+1)^2(2s_2+1)^2. \quad (20a)$$

if one or both of the spins are half-integral. If we impose condition (5) on the functions $a_\lambda^q(q_1, q_2)$, their number decrease and becomes

$$N(s_1, s_2) = {}^1/_2 N_0(s_1, s_2) + {}^1/_2(2s_1+1)(2s_2+1). \quad (21)$$

For identical particles the number of independent functions $a_\lambda^q(q_1, q_2)$ appearing in $M(\mathbf{k}_i, \mathbf{k}_f)$ is decreased still further and becomes ($s_1 = s_2 = s$)

$$N(s) = {}^1/_2 N(s, s) + {}^1/_3(2s+1)(2s^2+2s+{}^3/_2). \quad (22)$$

The functions $a_\lambda^q(q_1, q_2)$ are complex, so that all the numbers we have given should be doubled. However, the scattering matrix satisfies the unitarity condition[8]

$$M(\mathbf{k}_i, \mathbf{k}_f) - M^+(\mathbf{k}_f, \mathbf{k}_i) = \frac{ik}{2\pi}\int M^+(\mathbf{k}_f, \mathbf{n})\, M(\mathbf{k}_i, \mathbf{n})\, d\mathbf{n}, \quad (23)$$

which imposes as many relations on the complex functions $a_\lambda^q(q_1, q_2)$ as the number of such functions appearing in the scattering matrix.

In experiments E and F we find the quantities $K_{0000}^{q\kappa q_1 q_2}(\mathbf{k}_i, \mathbf{k}_f)$ (or $K_{q\kappa q_1 q_2}^{0000}(\mathbf{k}_i, \mathbf{k}_f)$ which is the same thing). Their number is $N_0(s_1, s_2)$. Thus each of the groups of experiments E and F gives a sufficient number of equations for reconstructing the scattering matrix.

The experiments B and the equivalent experiments C enable us to study only the quantities $K_{q\kappa q0}^{0000}$, whose number is in general less than the number of elements in the scattering matrix (though this is not the case for scattering by a spin zero target). Double scattering may thus be insufficient for determining $M(\mathbf{k}_i, \mathbf{k}_f)$, which is the situation we meet when we study nucleon-nucleon scattering.

The number of coefficients $K_{q_2\kappa_2 q_2 0}^{q_1\kappa_1 q_1 0}$ which appear in the quantities of group D is $N(s_1, s_1)$, and consequently repeated scattering in principle enables us to reconstruct the scattering matrix, but only when $s_1 \geq s_2$.

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