STABILITY OF THE STATIONARY CONVECTIVE FLOW OF AN ELECTRICALLY CONDUCTING LIQUID BETWEEN PARALLEL VERTICAL PLATES IN A MAGNETIC FIELD

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Submitted to JETP editor September 19, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 675-683 (March, 1958)

The effect of a constant magnetic field on the stability of the stationary convective flow of an electrically conducting liquid in the space between two parallel vertical plates is investigated. The equations for the amplitudes of the perturbations are solved by approximations, using the method of Galerkin. The study shows that a magnetic field greatly increases the stability of the stationary flow. In the case of a longitudinal field, the instability is always in the form of a "standing" perturbation. The critical Grasshof number and the critical wave number for standing and running perturbations have been determined as functions of the field strength.

In the preceding paper[1] we considered the stationary convective flow of an electrically conducting liquid between parallel plates, heated to different temperatures, in the presence of an external magnetic field. In this paper we consider the hydrodynamical stability of this flow. (The corresponding problem for the case where the magnetic field is absent has already been solved.[2]) The effect of the magnetic field is, firstly, to slow down the stationary motion, and secondly, to hinder the growth of perturbations; both these effects should greatly increase the flow stability. Studies of hydrodynamical stability in the presence of external magnetic fields have been made for the case of plane Poiseuille flow[3,4] and for the flow between rotating cylinders.[5] No studies of the stability of stationary convective flows in a magnetic field have been made previously, to our knowledge.

In this paper we shall investigate the stability of convective flow between vertical plates. The generalization to the case of arbitrary orientation of the plates is more complex than for the stationary-flow problem, and can be carried out in a manner analogous to that of Gershuni.[6]

1. PERTURBATION EQUATIONS

Let us denote by $\mathbf{v}_0$, T_0, p_0, and $\mathbf{H}_0$ the velocity, temperature, pressure, and magnetic field strength in the stationary flow, and consider small, non-stationary perturbations $\mathbf{v}$, T, p, and $\mathbf{H}$ of these quantities. In the perturbed motion, the quantities $\mathbf{v}_0 + \mathbf{v}$, $T_0 + T$, $p_0 + p$, and $\mathbf{H}_0 + \mathbf{H}$ must satisfy Eqs. (8) to (11) of Ref. 1. Considering that the stationary solution also satisfies these equations, and neglecting the squares of the small perturbation terms, we obtain the following equations for the perturbations*

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}_0 = -\nabla\left(p + \frac{M^2}{P_m}\mathbf{H}_0\mathbf{H}\right) + \nabla^2 \mathbf{v} + G\gamma T + \frac{M^2}{P_m}(\mathbf{H}_0 \cdot \nabla \mathbf{H} + \mathbf{H} \cdot \nabla \mathbf{H}_0); \quad (1)$$

$$\frac{\partial T}{\partial t} + \mathbf{v}_0 \nabla T + \mathbf{v} \nabla T_0 = \frac{1}{P}\nabla^2 T; \quad (2)$$

$$\frac{\partial \mathbf{H}}{\partial t} + \mathrm{curl}\,(\mathbf{H} \times \mathbf{v}_0) + \mathrm{curl}\,(\mathbf{H}_0 \times \mathbf{v}) = \frac{1}{P_m}\nabla^2 \mathbf{H}; \quad (3)$$

$$\mathrm{div}\,\mathbf{v} = 0, \quad \mathrm{div}\,\mathbf{H} = 0. \quad (4)$$

Let us consider a plane perturbation, in which $v_y = 0$, $H_y = 0$, and all quantities depend only on x, z, and t. Then because of (4), we may introduce a flow function Ψ, connected with the velocity components by the relations

$$v_x = -\partial\Psi/\partial z, \quad v_z = \partial\Psi/\partial x, \quad (5)$$

and also a vector potential field $\mathbf{A}$ given by

$$H_x = -\partial A_y/\partial z, \quad H_z = \partial A_y/\partial x; \quad (A_x = A_z = 0). \quad (6)$$

We assume the dependence of the perturbation on z and t to be of the form

$$\Psi = \psi(x)\,e^{i(\omega t + kz)}, \quad A_y = \varphi(x)\,e^{i(\omega t + kz)}, \quad T = \theta(x)\,e^{i(\omega t + kz)}.$$

Here k is the wave number and ω is the frequency (generally complex) of the perturbation. It is well known that the sign of the imaginary part of the frequency ω determines the behavior of

*For the notation, choice of units, and orientation of coordinates axes, see Ref. 1.

small perturbations; if the imaginary part is positive, the perturbation decays with time, i.e., the stationary motion is stable. If the imaginary part of the frequency is negative, the perturbation grows — the stationary flow is unstable.

Equations for ψ, φ, and θ can be obtained if we eliminate the pressure from (1) by taking the curl of both sides, and expressing the velocity and the field in terms of the flow function and the vector potential in all equations. From this substitution we obtain differential equations for the amplitude of a perturbation. (The primes denote differentiation with respect to x):

$$(\psi^{IV} - 2k^2\psi'' + k^4\psi) - (i\omega + ikv_0)(\psi'' - k^2\psi) + ikv_0''\psi + G\theta' = -\frac{M^2}{P_m}[H_{0x}(\varphi''' - k^2\varphi') + ikH_{0z}(\varphi'' - k^2\varphi) - ikH_{0z}''\varphi]; \quad (7)$$

$$-\frac{1}{P_m}(\varphi'' - k^2\varphi) + (i\omega + ikv_0)\varphi = H_{0x}\psi' + ikH_{0z}\psi; \quad (8)$$

$$-ikT_0'\psi + (i\omega + ikv_0)\theta - \frac{1}{P}(\theta'' - k^2\theta) = 0. \quad (9)$$

Perturbations of the velocity and temperature must vanish at the boundary between the liquid and the plates, so that the boundary conditions for ψ and θ will be

$$\psi(-1) = \psi(1) = \psi'(-1) = \psi'(1) = 0,$$
$$\theta(-1) = \theta(1) = 0. \quad (10)$$

The perturbations of the magnetic field are not, in general, required to vanish at the plates; the boundary conditions for the field are the usual conditions at the junction of two media. Thus, field perturbations may extend into the medium surrounding the liquid. In this case we must investigate the field in the external region also, which greatly complicates the problem. It would be possible to assume, as Fermi has done[5] in solving similar problems, that the surrounding material is an ideal conductor; such an assumption would naturally lead to very simple boundary conditions. In our case, however, it is possible, following Stuart[3] and Lock,[4] to eliminate the function $\varphi(x)$ from Eqs. (7) and (8). In order to be able to do this we must first simplify the equations by making use of the smallness of the parameter P_m.

In what follows, we shall consider two orientations of the constant external field: (1) a constant, uniform, external field perpendicular to the parallel plates, and therefore also perpendicular to the velocity vector of the stationary liquid flow (for brevity, this case will be referred to as the "transverse field" case); and (2) a constant external field in the direction of the velocity, i.e., along the z axis (the "longitudinal field" case). We shall first simplify Eqs. (7) and (8) for the transverse field case. It has been shown in Ref. 1 that the ratio H_{0z}/H_{0x} is proportional to P_m, so that for liquid metals it is extremely small, even for relatively large Grasshof numbers. Therefore in the right-hand sides of Eqs. (7) and (8) we may eliminate the terms containing the induced field H_{0z}. In the left-hand side of (8), obviously, the only important term is the one containing $1/P_m$. Thus we may write (8) in the approximate form

$$\varphi'' - k^2\varphi = -H_{0x}P_m\psi'.$$

When this substitution is made for $\varphi'' - k^2\varphi$, the right-hand side of Eq. (7) becomes $M^2\psi''H_{0x}^2$. It will be recalled that, as a result of our choice of units, H_{0x} is equal to ± 1, the two signs corresponding to the two possible directions of the perpendicular external field. Thus the right-hand side will reduce to $M^2\psi''$. This term obviously represents the effect of the magnetic field on the perturbations; its effect on the stationary flow is expressed in the dependence of the stationary profile v_0 on the magnetic field. Finally, for the transverse field case Eq. (7) takes the form

$$(\psi^{IV} - 2k^2\psi'' - M^2\psi'' + k^4\psi) - (i\omega + ikv_0)(\psi'' - k^2\psi) + ikv_0''\psi + G\theta' = 0. \quad (11)$$

In the longitudinal field case, we have $H_{0x} = 0$, and H_{0z} does not depend on x. Therefore we now have, instead of (8), keeping only the terms on the left-hand side which contain $1/P_m$,

$$\varphi'' - k^2\varphi = -ikH_{0z}P_m\psi.$$

The right-hand side of Eq. (7) is now equal to $-k^2M^2\psi$ (since $H_{0z}^2 = 1$). The equation for ψ in the longitudinal field case can be written

$$(\psi^{IV} - 2k^2\psi'' + k^4\psi + k^2M^2\psi) - (i\omega + ikv_0)(\psi'' - k^2\psi) + ikv_0''\psi + G\theta' = 0. \quad (12)$$

Thus the amplitude of the vector potential of the field perturbations, $\varphi(x)$, can be eliminated from the equations in both the longitudinal and transverse field cases. The problem reduces to the determination of the amplitudes of the flow and temperature functions $\psi(x)$ and $\theta(x)$ from Eqs. (9) and (11) or (12), with the boundary conditions (10). This is obviously an eigenvalue problem; a non-trivial solution for given values of parameters occurring in the equations will exist for only a few values of the complex number ω. The stability problem reduces to the problem of finding these characteristic frequencies ω. A neutral state, separating the regions of stability and instability, will obviously occur when the imaginary part of the complex frequency reduces to zero.

2. STABILITY INVESTIGATION

For an approximate solution of the problem, we shall make use of the method of Galerkin.

The unknown functions $\psi(x)$ and $\theta(x)$, which are to be determined, are approximated by a linear combination of functions which satisfy the boundary conditions. The coefficients are then determined by Galerkin's method (see, for example, Kantorovich and Krylov[7]). In what follows, we shall limit ourselves to the approximation

$$\bar{\psi}(x) = a_1\psi_1 + a_2\psi_2, \quad \bar{\theta}(x) = b_1\theta_1 + b_2\theta_2, \tag{13}$$

where the approximating functions ψ_1, ψ_2, θ_1, and θ_2 must satisfy the boundary conditions (10). From the form of the equations for ψ and θ it follows that these functions are not perfectly even functions, since the unperturbed profile v_0 which enters the equations is an odd function of x. Therefore, in constructing the approximate solutions (13), we shall choose ψ_1 and θ_1 to be even functions, and ψ_2 and θ_2 to be odd. The coefficients in the approximate solutions (13) are determined from the following set of linear homogeneous equations:

$$\int_{-1}^{1} L(\bar{\psi}, \bar{\theta})\psi_i\,dx = 0, \quad \int_{-1}^{1} M(\bar{\psi}, \bar{\theta})\theta_i\,dx = 0, \quad (i = 1, 2). \tag{14}$$

Here L is an operator which corresponds to the left-hand side of Eq. (11) in the transverse field case, and of Eq. (12) in the longitudinal field case. M is the operator corresponding to the left-hand side of Eq. (9).

The condition for the existence of a non-trivial solution of (14) is that the determinant of the set of equations should equal zero. In order to express this condition more concisely, we shall first introduce the following notation:

$$A_{ik} = \int_{-1}^{1} (\psi_i^{IV} - 2k^2\psi_i'' - M^2\psi_i'' + k^4\psi_i)\psi_k dx \quad \text{for } \mathbf{H} \parallel \mathbf{x},$$

$$A_{ik} = \int_{-1}^{1} (\psi_i^{IV} - 2k^2\psi_i'' + k^4\psi_i + k^2M^2\psi_i)\psi_k dx \quad \text{for } \mathbf{H} \parallel \mathbf{z},$$

$$B_{ik} = -\int_{-1}^{1} (\psi_i'' - k^2\psi_i)\psi_k dx, \quad C_{ik} = \int_{-1}^{1} (f_0''\psi_i - f_0\psi_i'' + k^2 f_0\psi_i)\psi_k dx, \quad D_{ik} = \int_{-1}^{1} \theta_i'\psi_k dx,$$

$$a_{ik} = -\frac{1}{P}\int_{-1}^{1} (\theta_i'' - k^2\theta_i)\theta_k dx, \quad b_{ik} = \int_{-1}^{1} \theta_i\theta_k dx,$$
$$c_{ik} = \int_{-1}^{1} f_0\theta_i\theta_k dx, \quad d_{ik} = -\int_{-1}^{1} T_0'\psi_i\theta_k dx; \quad f_0 = \frac{v_0}{G} \tag{15}$$

Note that, by virtue of the symmetry properties of the trial functions ψ_i and θ_i and the stationary profile f_0, the integrals A_{ik}, B_{ik}, a_{ik}, b_{ik}, and d_{ik} are equal to zero for $i \neq k$; and so are the integrals C_{ik}, D_{ik}, and c_{ik} for $i = k$. Using these results and the notation of (15), the vanishing of the determinant of system (14) can be written in the form:

$$\begin{vmatrix} A_{11} + i\omega B_{11} & ikGC_{21} & 0 & GD_{21} \\ ikGC_{12} & A_{22} + i\omega B_{22} & GD_{12} & 0 \\ ikd_{11} & 0 & a_{11} + i\omega b_{11} & ikGc_{21} \\ 0 & ikd_{22} & ikGc_{12} & a_{22} + i\omega b_{22} \end{vmatrix} = 0. \tag{16}$$

This equation determines the characteristic frequencies ω of the perturbation. In order to find the conditions for a neutral perturbation state [i.e., $\mathrm{Im}(\omega) = 0$], the real and imaginary parts of determinant (16), with ω taken to be real, must be equated separately to zero. In this way we obtain two equations which we write provisionally in the form

$$L_{40}k^4G^4 + L_{04}\omega^4 + L_{22}k^2G^2\omega^2 + L_{20}k^2G^2 + L_{02}\omega^2 + L_{00} = 0, \tag{17}$$

$$\omega(N_{20}k^2G^2 + N_{02}\omega^2 + N_{00}) = 0, \tag{18}$$

the coefficients representing terms involving the coefficients in determinant (16).

For a perturbation with a given wave number, and for given values of the parameters M and P, Eqs. (17) and (18) make it possible to find the critical Grasshof number G and the real perturbation frequency ω [for, of course, only the real solutions of the set (17) — (18) have any physical meaning]. If we eliminate the real frequency ω from Eqs. (17) and (18), we define a relation between G and k which specifies the neutral curve in the G-k plane. The minimum in the curve $G = G(k)$ defines the critical wave number k_m for a perturbation and the minimum critical Grasshof number G_m.

In order to carry out all these calculations, it is necessary first to choose the trial functions in Eqs. (13). Let us take ψ_1 and ψ_2 to be polynomials which vanish, together with their first derivatives, at $x = \pm 1$:

$$\psi_1 = (1 - x^2)^2, \quad \psi_2 = x(1 - x^2)^2. \tag{19}$$

In choosing the functions θ_1 and θ_2 we note that the temperature must satisfy the boundary condition

$$\theta''(-1) = \theta''(1) = 0. \quad (20)$$

as can be seen from (9) and the boundary condition (10). Therefore we choose θ_1 and θ_2 to be polynomials which reduce to zero at $x = \pm 1$, together with their second derivatives:

$$\theta_1 = (1 - x^2)(5 - x^2), \quad \theta_2 = x(1 - x^2)(7 - 3x^2). \quad (21)$$

By using expressions (19) and (21) as the trial functions we can calculate from (15) the elements of the determinant (16), and find the coefficients in (17) and (18). In order to calculate the integrals C_{ik} and c_{ik}, which depend on the stationary velocity profile v_0, we must use expression (19) of Ref. 1 for the transverse field case and expression (21) of the same reference for the longitudinal field case, since the longitudinal field does not affect the profile.

We shall consider the results of the stability investigations separately for the transverse and longitudinal field cases.

1. Transverse Field Case

The coefficients of determinant (16) could be found from their elements, but this is a rather cumbersome method of calculation; we shall not go into details of the derivation of these complicated expressions here, since they are of no interest in themselves.

From (17) and (18) it is evident that these

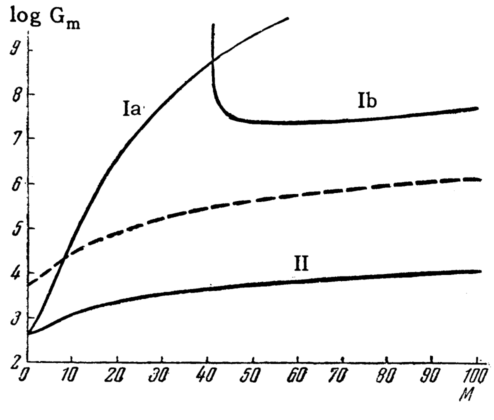

FIG. 1. Dependence of the critical Grasshof number on the Hartmann number (for P = 0.02): Ia – transverse field (standing perturbations). Ib – transverse field (running perturbations). II – longitudinal field. For comparison, the dashed line shows the dependence of the critical Grasshof number upon the field, for the onset of convection in a horizontal layer heated from below, in the presence of a transverse magnetic field.[9]

equations would be satisfied if $\omega = 0$ and if the value of G were a root of the equation

$$L_{40}k^4G^4 + L_{20}k^2G^2 + L_{00} = 0. \quad (22)$$

If there is a real positive root of Eq. (22), it implies that for given parameters M and P and for a given wave number k, the motion is unstable for this value of G, because at the critical point there will be a corresponding perturbation with the real part of the frequency ω equal to zero, and consequently with a phase velocity of zero. (Perturbations of this type may be called "standing," to distinguish them from "running" perturbations, for which $\omega \neq 0$.) The use of the approximation method clearly shows that the possibility of a standing perturbation, which is interesting in itself, is connected with the antisymmetry of the stationary velocity profile v_0; if the profile is completely symmetric, the imaginary part of the determinant (16) does not contain a real frequency ω as a factor, and no solution with $\omega = 0$ exists. By the same token, the existence of a standing perturbation in a flow where the mean velocity is different from zero (for example, of the Poiseuille or Couette type) is hardly possible from the physical point of view, since the existence of the perturbation would destroy the flow pattern.

Equation (22) has a real positive root for values of wave number in the interval $0 < k < \bar{k}$, where $\bar{k}$ is determined by the condition that the coefficient L_{40} should reduce to zero. The values $k = 0$ and $k = \bar{k}$ give the asymptotic neutral curves $G = G(k)$. Between these points the curve has a minimum at $k = k_m$. The critical wave number and the value of the minimum critical Grasshof number G_m have been calculated for $P = 0.02$ and for different values of the parameter M (i.e., as a function of the field strength). In the absence of a field ($M = 0$) the critical Grasshof number is equal to $G_m = 405$. This indicates a relatively low stability of the motion; for comparison, we may note that the critical Grasshof number corresponding to the onset of convection in a plane horizontal layer heated from below[8] is $G_m = 5340$ for $P = 0.02$. The presence of a field leads to a considerable increase in stability; as the parameter M increases the critical number G_m rises rapidly (see the table and Curve Ia in Fig. 1). For example, at $M = 10$ the critical number G_m is more than 100 times higher than its value in the absence of a field.

The critical wave number k_m decreases monotonically with increase in M (Curve Ia in Fig. 2), i.e., the critical wavelength of these (standing) perturbations increases with increasing magnetic field.

In addition to the standing perturbation type of instability, described above, in the presence of a transverse magnetic field the stationary motion also exhibits instability caused by running perturbations. This type of instability takes place when the field is sufficiently high, and corresponds to a solution of (17) and (18) with $\omega \neq 0$. To find the neutral curve $G(k)$ for the running perturbations, ω must be eliminated from (17) and (18) after dividing by ω. If desired, it is also possible to find from (17) and (18) the real frequency, and hence the phase velocity, of the running perturbations. Since only the even powers of ω appear in the equations, the perturbations travel both ways along the z axis. The critical Grasshof number G_m is plotted as a function of M in Fig. 1, Curve Ib. It can be seen that for $M > 42$ a breakdown in the stationary flow can arise as the result of either standing or running perturbations, but that the critical number G_m for running perturbations is very much lower than the corresponding critical number for standing perturbations, i.e., the flow is very much less stable toward running perturbations. Nevertheless, even at very high fields there is always a possibility that the instability may set in as a standing perturbation. To make this happen, some means would have to be taken to prevent the formation of running perturbations; for instance, running perturbations obviously cannot exist in a channel with very small vertical dimensions.

Figure 2 (Curve Ib) shows the dependence of the critical wave number k_m on the field. It is evident that for $M > 42$ the critical wave number for running perturbations is greater than the critical wave number for standing perturbations. This is physically reasonable, since for large transverse fields the running perturbations lead to breakdown at lower G numbers than the standing ones; at high transverse fields the formation of a short wavelength perturbation, covering a relatively large distance in the direction of the transverse field, is energetically favored, since in this

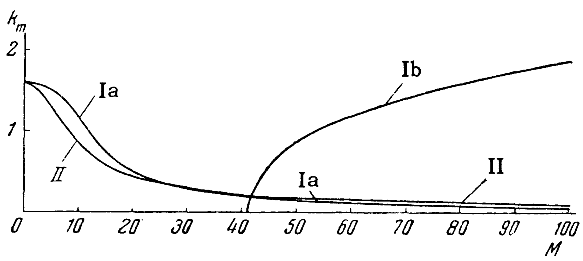

FIG. 2. Dependence of the critical wave number on the Hartmann number (for P = 0.02): Ia – transverse field (standing perturbations). Ib – transverse field (running perturbations). II – longitudinal field.

case the Joule dissipation will be less than for a long wavelength perturbation. The same reasoning explains why the critical wave number k_m for running perturbations increases with an increase in the applied field.

2. Longitudinal Field Case

In the case of a longitudinal field, the only possible type of instability is a standing perturbation with $\omega = 0$. If we assume that $\omega \neq 0$, then Eqs. (17) and (18) give complex roots G when ω is eliminated, indicating that with respect to running waves, the system is always stable. The critical Grasshof number G_m for standing perturbations is shown as a function of the Hartmann number M in Fig. 1, Curve II. It will be seen that the critical number G_m increases much more slowly with the field than it does in the transverse field case. At large fields, the critical number G_m increases in direct proportion to the field, in accordance with the asymptotic formula

$$G_m = 107\,M. \tag{23}$$

A longitudinal field is much less stabilizing than a transverse field. This can be explained by the fact that a longitudinal field merely hinders the development of perturbations, while the transverse field, in addition, greatly slows down the stationary flow. The effect of this second factor on the stability is much greater — so much so that Lock,[4] in his

$\omega = 0$			$\omega \neq 0$		
M	k_m	G_m	M	k_m	G_m
0	1.6	405	41.3	0	∞
5	1.5	$3.85\cdot10^3$	45	0.62	$3.26\cdot10^7$
10	1.16	$4.64\cdot10^4$	50	0.87	$2.38\cdot10^7$
20	0.52	$3.40\cdot10^6$	60	1.2	$2.31\cdot10^7$
30	0.31	$5.70\cdot10^7$	70	1.4	$2.63\cdot10^7$
40	0.21	$4.20\cdot10^8$	80	1.6	$3.18\cdot10^7$
50	0.16	$1.80\cdot10^9$	90	1.75	$3.80\cdot10^7$
100	0.06	$2.0\cdot10^{11}$	100	1.9	$4.6\cdot10^7$

study of the stability of Poiseuille flow in a transverse field, did not even consider the effect which the field would have on the perturbations.

In the longitudinal field case the critical wave number k_m decreases monotonically as the field increases (Curve II of Fig. 2); this is a natural result, since in distinction to the transverse field case, it is the long wavelength perturbations which are energetically favored in a longitudinal field.

In conclusion, we note that the quantitative results which we have obtained could be made more accurate by using better approximations. This could be done in two way: either by an increase in the number of trial functions in (13), or by choosing the trial functions in a different way. It seems to us that the second method is more comprehensive, from the following considerations. It is known that, at high fields in stationary flows, a sort of boundary layer is formed; hence it may be expected that some such layer would also accompany a perturbation which is formed in a high field. However, the polynomial trial functions (19) and (21) which we have chosen make no provision for this kind of structural singularity in the perturbations (if such a singularity should exist). It should be noted that both of the methods for improving the accuracy of the results would greatly complicate the numerical computations.

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Translated by D. C. West
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