APPLICATION OF DISPERSION RELATIONS TO π–N SCATTERING AT LOW ENERGIES

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The consequences of the dispersion relations for $\pi - N$ scattering at low energies are studied without resort to the use of the Low equations. Relations between the scattering lengths in various states are obtained from the values of the derivatives of the dispersion relations at $k^2 = 0$.

1. In the application of the dispersion relations to the low-energy region it is common practice to go from these relations to the Chew-Low equations and then study the consequences of these equations.[1] In the present paper the consequences of the dispersion relations for $\pi - N$ scattering at low energies are studied without resorting to the use of the Low equations.

Let us consider the dispersion relations for $\pi - N$ scattering,[2] written in the form

$$D_\pm(k) - \frac{1}{2}\left(1 + \frac{\omega}{\mu}\right) D_\pm(0) - \frac{1}{2}\left(1 - \frac{\omega}{\mu}\right) D_\mp(0) = k^2 J_\pm(\omega) \pm \frac{2f^2}{\mu^2} \frac{k^2}{\omega \mp \mu^2/2M}, \quad (1)$$

where we have used the notation $J_\pm(\omega)$ to denote the dispersion integral

$$J_\pm(\omega) = \frac{1}{4\pi^2} \mathrm{P} \int_\mu^\infty \frac{d\omega'}{k'} \left[\frac{\sigma_\pm(\omega')}{\omega' - \omega} + \frac{\sigma_\mp(\omega')}{\omega' + \omega}\right], \quad (2)$$

ω is the total energy of the meson in the laboratory system, and the other notations are obvious. We shall assume that in Eq. (1) the meson energy is low, $\omega - \mu \ll \mu$. For $\eta^2 = k^2/\mu^2 \to 0$ both sides of Eq. (1) go to zero. Therefore, wishing to obtain the consequences of the dispersion relations for $\eta^2 \to 0$, we turn to the values of the derivatives of Eq. (1) with respect to η^2 at $\eta^2 \to 0$. In calculating the derivatives we use the form of the energy dependence of the phase shifts as given by the "effective range theory"[3]

$$\eta_b^{2l+1} \cot \delta_l = 1/a_{2l+1} + P_l \eta_b^2 + Q_l \eta_b^4, \quad (3)$$

where $a_{2\ell+1}$ is the scattering length in the ℓ-state, and $k_b = \eta_b \mu$ is the momentum of the meson in the center-of-mass system, so that

$$k/k_b = \eta/\eta_b = (1 + 2\omega/M + \mu^2/M^2)^{1/2}. \quad (4)$$

The derivatives of the second and third terms in the left member of Eq. (1) will have the form

$$\tfrac{1}{2}(\omega/\mu)^{(n)}[D_\mp(0) - D_\pm(0)] = \pm\tfrac{1}{2}(\omega/\mu)^{(n)}[D_-(0) - D_+(0)].$$

Since

$$\left(\frac{\omega}{\mu}\right)' = \frac{1}{2}\left(\frac{\mu}{\omega}\right); \quad \left(\frac{\omega}{\mu}\right)'' = -\frac{1}{4}\left(\frac{\mu}{\omega}\right)^3; \quad \left(\frac{\omega}{\mu}\right)''' = \frac{3}{8}\left(\frac{\mu}{\omega}\right)^5, \ldots$$

(the prime denotes differentiation with respect to η^2), we have for $\eta^2 \to 0$

$$\pm\frac{1}{4}[D_-(0) - D_+(0)] = \pm\frac{\lambda_c}{6}\left(1 + \frac{\mu}{M}\right)(a_1 - a_3); \quad \left(\lambda_c = \frac{\hbar}{\mu c}\right) \quad (5)$$

for the first derivative and

$$\mp\frac{1}{8}[D_-(0) - D_+(0)] = \mp\frac{1}{12}\lambda_c\left(1 + \frac{\mu}{M}\right)(a_1 - a_3) \quad (6)$$

for the second derivative. Here, as usual, a_3 and a_1 mean the scattering lengths in the s states with isotopic spins $T = 3/2$ and $T = 1/2$, respectively.

The unobserved region gives for the first derivative

$$\pm 2f^2\lambda_c\left\{\frac{1}{\omega/\mu \mp \mu/2M} - \frac{\eta^2}{(\omega/\mu \mp \mu/2M)^2}\frac{1}{2}\frac{\mu}{\omega}\right\}_0 = \pm\frac{2f^2\lambda_c}{(1 \mp \mu/2M)} \quad (7)$$

and for the second derivative

$$\mp 2f^2\lambda_c/(1 \pm \mu/2M)^2 \quad (8)$$

The index 0 means throughout that the value of the expression is taken for $\eta^2 = 0$. If we confine ourselves to the values of only the first two derivatives at $\eta^2 = 0$, it suffices to represent $D_+(k)$, for example, in the form

$$2k_b^2 D_+(k) = k\{\sin 2\alpha_3 + \sin 2\alpha_{31} + 2(\sin 2\alpha_{33} + \sin 2\delta_{33}) + 3\sin 2\delta_{35}\}, \quad (9)$$

where δ_{33} and δ_{35} denote the phase shifts for the states $d_{3/2}$ and $d_{5/2}$ with $T = 3/2$ and the usual notations are used for the phase shifts of the s and p states. Using Eq. (3), we write the expression (9) in the form

$$k_b D_+(k) = \lambda_c k \left\{ \frac{A_3}{A_3^2 + x} + \frac{xA_{31}}{A_{31}^2 + x^3} + 2\left[\frac{xA_{33}}{A_{33}^2 + x^3} + \frac{x^2 B_{33}}{B_{33}^2 + x^5}\right] + \frac{3x^2 B_{35}}{B_{35}^2 + x^5} \right\}, \qquad (10)$$

where A_3, A_{31}, A_{33}, B_{33}, and B_{35} denote the right members of Eq. (3) for the states $s_{1/2}$, $p_{1/2}$, $p_{3/2}$, $d_{3/2}$, and $d_{5/2}$ with $T = 3/2$, and $x = \eta_b^2$.

For the value of the first derivative of the expression (10) at $x = 0$ we get

$$D'_{+0}(k) = \frac{\lambda_c}{(1 + \mu/M)} \left\{2a_{33} + a_{31} + \frac{\mu}{2M} a_3 - P_3 a_3^2 - a_3^3\right\}. \quad (11)$$

Combining Eqs. (6), (7), and (11), from the value of the first derivative of Eq. (1) for $\eta^2 = 0$ we have the relation

$$\frac{\lambda_c}{(1+\mu/M)}\left\{2a_{33} + a_{31} + \frac{\mu}{2M} a_3 - P_3 a_3^2 - a_3^3\right\} + \frac{\lambda_c}{6}\left(1 + \frac{\mu}{M}\right)(a_1 - a_3) = \{k^2 J_+\}'_0 + \frac{2f^2\lambda_c}{(1 - \mu/2M)}, \qquad (12)$$

which establishes the connection between the scattering lengths for mesons in various states which follows from the dispersion relations. In what follows we shall denote $k^2 J_\pm(\omega)$ by $F_\pm(\omega)$. For the scattering of negative mesons in hydrogen, expressing $D(k)$ in terms of the amplitudes in states with definite values of the isotopic spin,

$$3D_-(k) = 2D_1(k) + D_3(k),$$

we get in a similar way

$$\frac{\lambda_c}{3(1+\mu/M)}\left\{\frac{\mu}{2M}(2a_1 + a_3) - (2a_1^3 + a_3^3) - (2P_1 a_1^2 + P_3 a_3^2) + 4a_{13} + 2(a_{33} + a_{11}) + a_{31} - \frac{1}{2}\left(1 + \frac{\mu}{M}\right)^2 (a_1 - a_3)\right\} = F'_{-0} - \frac{2f^2\lambda_c}{1 + \mu/2M}. \qquad (13)$$

For the half-sum of (11) and (12), representing the scattering of π^0 mesons by nucleons, we have

$$\frac{1}{3}\frac{\lambda_c}{(1+\mu/M)}\left\{\frac{\mu}{2M}(a_1 + 2a_3) - (a_1^3 + 2a_3^3) - (P_1 a_1^2 + P_3 a_3^2) + 2(a_{13} + a_{31}) + a_{11} + 4a_{33}\right\} = \left\{\frac{F_{+0} + F_{-0}}{2}\right\}' + \frac{\mu}{2M}\frac{2f^2\lambda_c}{[1 - (\mu/2M)^2]}. \qquad (14)$$

Substitution of the experimental data on the phase shifts[4]

$\alpha_3 = -(0.105 \pm 0.010)\eta_b$, $P_{33} = 0.6$, $d_{33} = 0.0035$, $\alpha_1 = (0.165 \pm 0.012)\eta_b$, $Q_{33} = -0.8$, $d_{35} = -0.0035$, $\alpha_{33} = 0.235$

(the scattering lengths in the $d_{3/2}$ and $d_{5/2}$ states are denoted by d_{33} and d_{35}), with $2f^2 = 0.16$ (and $2f^2 = 0.19 \pm 0.01$[5]) and zero values of the other coefficients, gives

$$\lambda_c^{-1} D'_{+0} = 0.40, \quad \lambda_c^{-1} D'_{-0} = 0.14$$

and

$$F'_{+0} = 0.28\,(0.25)\,\lambda_c, \quad F'_{-0} = 0.24\,(0.27)\,\lambda_c,$$

where the contribution of the unobserved region [-0.173 (0.206) and $+0.149$ (0.177)] is considerable. For the half-sum, however,

$$\{^1/_2 (F_+ + F_-)'\}_0 = 0.26\,\lambda_c$$

the contribution of the term containing f^2 amounts to only $-0.01\lambda_c$. The calculation of $F'_{\pm 0}$ from the data on the total cross-sections is discussed in the following section.

For the second derivative of Eq. (10) we have

$$D''_{+0}(k) = \lambda_c\left(1 + \frac{\mu}{M}\right)[2a_3^3(a_3 + 2P_3)(a_3 + P_3) - 2a_3^2(Q_3 + a_3 P_3^2) - 3(a_{31}^2 P_{31} + 2a_{33}^2 P_{33}) + 2(2d_{33} + 3d_{35})] + \frac{\mu}{M}\frac{\lambda_c}{(1+\mu/M)^3}[2a_{33} + a_{31} - P_3 a_3^2 - a_3^3] - \frac{\mu}{4M}\frac{\lambda_c}{(1+\mu/M)}\left[1 + \frac{\mu}{M}\left(1 + \frac{\mu}{M}\right)^2\right] a_3. \qquad (15)$$

By means of Eqs. (10), (6), and (8) we get from the value of the second derivative of Eq. (1) the second group of relations

$$D''_{+0}(k) - \frac{\lambda_c}{12}\left(1 + \frac{\mu}{M}\right)(a_1 - a_3) = F''_{+0} - \frac{2f^2\lambda_c}{(1 + \mu/2M)^2} \qquad (16)$$

for $\pi^+ p$ scattering;

$$D''_{-0}(k) + \frac{\lambda_c}{12}\left(1 + \frac{\mu}{M}\right)(a_1 - a_3) = F''_{-0} + \frac{2f^2\lambda_c}{(1 - \mu/2M)^2} \qquad (17)$$

for the scattering of negative mesons; and

$$\frac{1}{2}(D''_+ + D''_-)_0 = \frac{1}{2}(F''_+ + F''_-)_0 - \frac{\mu}{M}\frac{2f^2\lambda_c}{[1 - (\mu/2M)^2]}, \qquad (18)$$

where D''_- is constructed in analogy with Eq. (15).

By using the data on the phase shifts given earlier we have

$$\lambda_c^{-1} D''_{+0} = \left(1 + \frac{\mu}{M}\right)[2a_3^5 - 6a_{33}^2 P_{33} + 2(2d_{33} + 3d_{35})] + \frac{\mu/M}{(1+\mu/M)^3}(2a_{33} - a_3^3) - \frac{\mu/4M}{(1+\mu/M)}\left[1 + \frac{\mu}{M}\left(1 + \frac{\mu}{M}\right)^2\right] a_3; \qquad (19)$$

$$\begin{aligned}3\lambda_c^{-1} D_{-0}(k)'' = &\left(1+\frac{\mu}{M}\right)[2(2a_1^5+a_3^5) \\ &- 6a_{33}^2 P_{33} + 4(2d_{13}+d_{33}) \\ &+ 6(2d_{15}+d_{35})] + \frac{\mu}{M}\frac{2a_{33}-(2a_1^3+a_3^3)}{(1+\mu/M)^3} \\ &- \frac{\mu/4M}{(1+\mu/M)}\left[1+\frac{\mu}{M}\left(1+\frac{\mu}{M}\right)^2\right](2a_1+a_3).\end{aligned} \tag{20}$$

Substitution of the experimental data on the phase shifts gives

$$D''_{+0} = -0.180\,\lambda_c; \quad D''_{-0} = -0.052\,\lambda_c;$$
$$\{{}^1/_2(D''_+ + D''_-)\}_0 = -0.116\,\lambda_c;$$
$$F''_{+0} = -0.019\,\lambda_c \ (2f^2 = 0.16);$$
$$F''_{+0} = +0.016\,\lambda_c \ (2f^2 = 0.19);$$
$$F''_{-0} = -0.162\,\lambda_c \ (2f^2 = 0.16);$$
$$F''_{-0} = -0.187\,\lambda_c \ (2f^2 = 0.19);$$
$$\{{}^1/_2(F''_- + F''_+)\}_0 = -0.090\,\lambda_c \quad (2f^2 = 0.16);$$
$$\{{}^1/_2(F''_- + F''_+)\}_0 = -0.085\,\lambda_c, \quad (2f^2 = 0.19),$$

where the contribution of the unobserved region turns out to be very important for the first two values and amounts to about 35 percent for the half-sum. The value of the second derivative of F_+ consists mainly of the difference of the resonance contribution and that from the unobserved region (this latter contribution being $+0.188\lambda_c$ and $0.222\lambda_c$). For the scattering of negative mesons the resonance transition is of less importance, so that the contribution of the unobserved region, amounting to $-0.138\lambda_c$ and $-0.161\lambda_c$, is the main one.

2. In the preceding section the values of the derivatives of $F_\pm$ were obtained from the experimental data on the phase shifts. These same quantities can be calculated directly from Eq. (2) and the experimental data on the total cross-sections. This provides a more searching check on the correspondence between the experimentally determined phase shifts and the dispersion relations.

Having obtained $F_\pm$, one can calculate the values of the derivatives by direct differentiation. But the differentiation of a curve drawn from experimental data introduces large errors; therefore we shall represent the derivative in another form. For this purpose, being interested in the values of the derivatives at $\eta^2 \to 0$, we break up the dispersion integral $J_+(\omega)$ into three parts:

$$4\pi^2 J_+(\omega) = \mathrm{P}\int_\mu^z \frac{d\omega'}{k'}\frac{\sigma_+(\omega')}{\omega'-\omega} + \int_z^\infty \frac{d\omega'}{k'}\frac{\sigma_+(\omega')}{\omega'-\omega} + \int_\mu^\infty \frac{d\omega'}{k'}\frac{\sigma_-(\omega')}{\omega'+\omega}, \tag{21}$$

so that the second integral already contains no singularity $(z > \omega)$. We choose the value of z so that with the limitation to s and p states and use of Eq. (3) the expression for $\sigma_+(\omega)$ can be put in the form

$$\sigma_+(\omega) = 4\pi\left\{a_3^2 + \frac{(a_{31}^2+2a_{33}^2)}{(1+\mu/M)^4}\eta^4\right\}. \tag{22}$$

Substituting Eq. (22) into the first integral in Eq. (21), we get for it

$$\begin{aligned}\pi L(\omega) &= (4\pi)^{-1}\,\mathrm{P}\int_\mu^z \frac{d\omega'}{k'}\frac{\sigma_+(\omega')}{\omega'-\omega} \\ &= a_3^2 J_\alpha(p) + \frac{(a_{31}^2+2a_{33}^2)}{(1+\mu/M)^4} J_\beta(p),\end{aligned} \tag{23}$$

where

$$J_\alpha(p) = -\frac{1}{p}\ln\left[1+\frac{(\omega-\mu)(z+\mu)+p\sqrt{z^2-\mu^2}}{\mu(z-\omega)}\right],$$

$$\begin{aligned}\mu^4 J_\beta(p) = &\frac{1}{3}(z^2-\mu^2)^{3/2} + \frac{1}{2}(z^2-\mu^2)^{1/2}(\omega z + p^2) + p^4 J_\alpha(p) \\ &- \frac{1}{2}\mu(\mu^2-p^2)\ln\left[\frac{\sqrt{z^2-\mu^2}+z}{\mu}\right].\end{aligned} \tag{23'}$$

For small values of p^2

$$\begin{aligned}\mu^2 J_\alpha(p) = &-\sqrt{z^2-\mu^2}\left\{\frac{1}{z'-1} + \frac{p^2}{6}\frac{2-z'}{(z'-1)^2}\right. \\ &\left.+ \frac{p^4}{40}\frac{3z'^2-9z'+8}{(z'-1)^3} + \ldots\right\} \quad \left(z' = \frac{z}{\mu}\right),\end{aligned} \tag{23''}$$

so that (with $z = 1.43\mu$)

$$J_\alpha(0) = -\frac{\sqrt{z^2-\mu^2}}{\mu(z-\mu)} = -2.38\,\lambda_c,$$

$$\begin{aligned}\mu^4 J_\beta(0) = &\frac{(z^2-\mu^2)^{3/2}}{3} + \frac{\mu z (z^2-\mu^2)^{1/2}}{2} \\ &- \frac{\mu^3}{2}\ln\left[\frac{\sqrt{z^2-\mu^2}+z}{\mu}\right] = 0.639\,\mu^3,\end{aligned} \tag{24}$$

$$\begin{aligned}J_+(\mu) = &\frac{a_3^2}{\pi}J_\alpha(0) + \frac{(a_{31}^2+2a_{33}^2)}{\pi(1+\mu/M)^4}J_\beta(0) + \frac{1}{4\pi^2}\int_z^\infty \frac{d\omega'}{k'}\frac{\sigma_+(\omega')}{\omega'-\mu} \\ &+ \frac{1}{4\pi^2}\int_\mu^\infty \frac{d\omega'}{k'}\frac{\sigma_-(\omega')}{\omega'+\mu}.\end{aligned}$$

The expression for $J_-(\mu)$ is obtained from Eq. (24) by the replacements

$$\sigma_+ \rightleftarrows \sigma_-, \quad 3a_3^2 \to a_3^2 + 2a_1^2; \quad 3a_{33}^2 \to a_{33}^2 + 2a_{13}^2, \quad 3a_{31}^2 \to a_{31}^2 + 2a_{11}^2. \tag{25}$$

From Eqs. (21) and (23) there follow expressions for the dispersion relations that are convenient for calculations in the range of energies $\omega - \mu \ll \mu$. Before using them to calculate the values of the de-derivatives for $\eta^2 \to 0$, we make several remarks.

In the second term of Eq. (22) (the contribution of the p waves) the change from the center-of-mass system to the laboratory system has been carried out approximately. Working more exactly, using Eq. (4), instead of $J_\beta(p)/(1+\mu/M)^4$ we get

$$L_\beta(p) = \int_\mu^z \frac{d\omega'}{\omega'-\omega}\frac{k'^3}{(1+2\omega'/M+\mu^2/M^2)^2} = \left(1+\frac{2\omega}{M}+\frac{\mu'}{M^2}\right)^{-2}\int_{\mu-\omega}^{z-\omega}\frac{R^{3/2}dx}{x(1+\gamma x)^2},$$

where

$$\gamma = \frac{2\mu}{M}\left(1+\frac{2\omega}{M}+\frac{\mu^2}{M^2}\right)^{-2}, \quad R = p^2+2\omega x+x^2,$$

or

$$\left(1+\frac{2\omega}{M}+\frac{\mu^2}{M^2}\right)^2 L_\beta(p) = J_\beta(p) - \gamma\left\{\int_{\mu-\omega}^{z-\omega}\frac{R^{3/2}dx}{1+\gamma x}+\int_{\mu-\omega}^{z-\omega}\frac{R^{3/2}dx}{(1+\gamma x)^2}\right\} = J_\beta(p)-\gamma N_\beta(p). \tag{26}$$

We give the expression for $N_\beta(p)$ which is obtained when the denominators of the integrands are expanded in series up to terms in γ^2:

$$N_\beta(p) = \left[\left(\frac{N^2}{2}-\frac{3}{4}\right)zN - \frac{3}{16}\ln(\omega+2z+N)\right] - 3\gamma\left[\frac{N}{5}-z\left(\frac{\omega N^3}{4}-\frac{3}{8}N\right)-\frac{3}{8}\ln(\omega+2z+N)\right] + 4\gamma^2\left[\left(\frac{z-1}{6}-\frac{7\omega}{30}\right)N^5+\frac{(6\omega^2+1)}{6}z\left(\frac{N^3}{4}-\frac{3N}{8}\right)+\frac{(6\omega^2+1)}{16}\ln(\omega+2z+N)\right], \tag{27}$$

where $N=(z^2-1)^{1/2}$ $(\mu=1)$. The last term in Eq. (27) gives less than 6.5 percent of the value of N_β at $p=0$. The value of $\gamma N_\beta(0)$ itself is 0.085, which leads to the change from $J_\beta(0)=0.639$ to $L_\beta(0)=0.554$, i.e., gives a correction of about 15 percent. Use of only the first term in Eq. (27) gives $L_\beta(0)=0.530$, from which it is clear that the remaining terms give less than 5 percent. Since the entire contribution from $J_\beta(0)$ is not large, one can just use Eq. (23′), bringing in the first term of Eq. (27) as a correction.

We shall now obtain the expressions for the derivatives of $F_\pm$. For arbitrary momenta

$$F'_+(\omega) = J_+(\omega)+\eta^2 J'_+(\omega). \tag{28}$$

From Eqs. (21), (23), and (27) it can be seen that

$$\eta^2 J'_{+0}(\omega) = 0. \tag{29}$$

Since Eq. (28) is also correct for $\eta^2 J'_-$, we get

$$F'_{\pm 0}(\omega) = J_\pm(\mu). \tag{30}$$

This last equation enables us to reduce the calculation of the derivative of F to the value of the dispersion integral at the single point $\omega=\mu$. The value of the derivative is calculated from Eqs. (30), (23), and (24). For $z=1.43$ the contribution of the first two terms in Eq. (23) gives $-0.026/\pi$ for the s wave and $+0.0350/\pi$ for the p waves, which is a very small fraction of the value $F'_{+0} = 0.28\,(0.25)\,\lambda_c$ obtained in the preceding section.

Similarly, for the interaction of negative mesons with protons we have instead of Eq. (24)

$$J_-(\mu) = \frac{(a_3^2+2a_1^2)}{3\pi}J_\alpha(0) + \frac{(a_{31}^2+2a_{11}^2+a_{33}^2+2a_{13}^2)}{3\pi(1+\mu/M)^4}L_\beta(0) + \frac{1}{4\pi^2}\int_z^\infty\frac{d\omega'}{k'}\frac{\sigma_-(\omega')}{\omega'-\mu}+\frac{1}{4\pi^2}\int_\mu^\infty\frac{d\omega'}{k'}\frac{\sigma_+(\omega')}{\omega'+\mu}. \tag{31}$$

For the contribution of the first two terms in Eq. (31) we get

$$\pi L_- = -0.051+0.012 = -0.039.$$

Here also the contributions of the s and p waves, which are comparable with each other, turn out at $z=1.43$ to be tiny in comparison with the contribution of the integral terms in Eq. (31), if we judge by the results of the preceding section. For the second derivative of F at arbitrary momenta we find

$$F''(\omega) = 2J'(\omega)+\eta^2 J''(\omega).$$

For $\eta^2\to 0$ the second term vanishes, so that

$$F''_0(\omega) = 2J'(\mu). \tag{32}$$

Breaking $J'_+(\omega)$ up into parts as in Eq. (21), we get from Eq. (23) for the contribution of the s and p waves

$$\pi L'_+(\mu) = -a_3^2\frac{N(2-z)}{6(z-1)^2}+\frac{a_{31}^2+2a_{33}^2}{\pi(1+\mu/M)^4}\times\left\{\frac{N}{2}\left(1+\frac{z}{2}\right)+\frac{1}{2}\ln(N+z)\right\}; \tag{33}$$

$$J'_+(\mu) = L'_+(\mu)+\frac{1}{4\pi^2}\int_z^\infty\frac{d\omega'}{k'}\frac{\sigma_+(\omega')}{(\omega'-\mu)^2} - \frac{1}{4\pi^2}\int_\mu^\infty\frac{d\omega'}{k'}\frac{\sigma_-(\omega')}{(\omega'+\mu)^2}. \tag{34}$$

The contribution of the terms outside the integrals is $+0.002\lambda_c$. The expression for $J'_-(\mu)$ is obtained from Eq. (34) by the replacements (25). For $\pi^- - p$ the contribution of the terms outside the integrals is $-0.002\lambda_c$.

3. The numerical values of the dispersion integrals are

$$J_+(\mu) = 6.96/4\pi^2 = 0.176, \qquad J_-(\mu) = 4.99/4\pi^2 = 0.126. \tag{35}$$

Comparison of Eqs. (35) and (30) with the previously obtained values 0.28 (0.25) and 0.24 (0.27) provides a further confirmation of the positive sign of a_{33}. Moreover, it follows from Eqs. (35) and (12) that a_{31} is a negative quantity, and

$$\text{for } 2f^2 = 0.16: \quad D'_+(0) = 0.258\ \lambda_c, \quad a_{31} = -0.115,$$
$$\text{for } 2f^2 = 0.19: \quad D_+(0) = 0.320\ \lambda_c, \quad a_{31} = -0.080. \tag{36}$$

From Eqs. (35) and (13) we have as a consequence of the dispersion relations that

$$\text{for } 2f^2 = 0.16: \quad D'_-(0) = 0.026\ \lambda_c,$$
$$2a_{13} + a_{11} = -0.139;$$
$$\text{for } 2f^2 = 0.19: \quad D_-(0) = -0.00(4)\ \lambda_c, \tag{37}$$
$$2a_{13} + a_{11} = -0.208.$$

On the hypothesis that $a_{13} = a_{31}$ it follows from Eq. (37) that

$$\text{for } 2f^2 = 0{,}16: \quad a_{11} = 0.09,$$
$$\text{for } 2f^2 = 0{,}19: \quad a_{11} = -0.05. \tag{38}$$

It must be emphasized that it is difficult to determine the errors in the numerical values of a_{31} and a_{11}. Although the idea of a small and negative scattering length a_{31} and a small a_{11} corresponds roughly to the experimental data on $\pi - N$ scattering, the value of $D'_-(0)$ is in noticeable disagreement with the experimental data.

For the values of the derivatives of the dispersion integrals we get by numerical integration

$$2J'_+(\mu) = 0.08\ \lambda_c, \quad 2J'_-(\mu) = 0.04\ \lambda_c. \tag{39}$$

It is at present difficult to get results analogous to Eq. (36) — (38), since for this more precise data are needed.

Thus the analysis that has been given by using information about the scattering phase shifts at low energies only confirms the results of Puppi and Stanghellini.[5] For the scattering of π^+ mesons by protons the dispersion relations enable us to determine from the data on a_1, a_3, and a_{33} a value of a_{31} that agrees with the experimental data, but the consequences of the dispersion relations for $\pi^- - p$ scattering cannot be brought into agreement with experiment. The actual disagreement at low energies is not large, as it is at higher energies. It is not clear just how far it exceeds the limits of the experimental errors in the phase shifts.

The causes of the disagreement remain unclear. All kinds of isotopically noninvariant corrections are small. In connection with the important part played by isotopic invariance in the derivation of the final form of the dispersion relations, the possibility of an additional check on the isotopic invariance in the $\pi - N$ scattering is discussed in the Appendix. The contribution of the πp mesic atom calls for special examination.

In the situation that has arisen, it is very desirable that additional experiments be carried out, and also that the experimental data be processed more precisely. Experiments that could be of particular value are studies of the polarization of recoil nucleons, and also of the interference with the Coulomb scattering.

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APPENDIX

As is well known, the unitary character of the S matrix, together with invariance under time reversal, decidedly reduces the number of independent parameters occurring in the S matrix. Recently it has been shown[6] that these conditions make it possible to reconstruct the scattering amplitude in the case in which only elastic scattering takes place, unaccompanied by inelastic processes.

We shall now consider the unitary conditions for $\pi^- - p$ scattering, in which in parallel with the elastic scattering there occurs an inelastic process — the conversion of the charged meson into a neutral meson — and shall show how these conditions can aid in checking the isotopic invariance.

We introduce the S-matrix elements

$$S_{ik}^{Jl} = \rho_{ik}^{Jl} \exp\left[2i\alpha_{ik}^{Jl}\right]$$

(α_{ik} and ρ are real numbers) for a state characterized by the angular momentum J and by $l = J + 1/2$ (hereafter the indices J, l are omitted). S_{11} corresponds to the elastic scattering of the π^- meson by a proton, $\pi^- + p \to \pi^-$; $S_{12} = S_{21}$ corresponds to the exchange scattering $\pi^- + p \to \pi^0$; and S_{22}, to the elastic scattering $\pi^0 + n \to \pi^0$.

The requirement that the matrix be unitary, written in the form

$$\sum_k S_{ik}^* S_{lk} = \delta_{il}, \tag{A.1}$$

is easily seen to give three independent conditions:

$$|S_{11}|^2 + |S_{12}|^2 = 1; \tag{A.2}$$
$$|S_{22}|^2 + |S_{12}|^2 = 1; \tag{A.3}$$
$$S_{11}^* S_{12} + S_{12}^* S_{22} = 0. \tag{A.4}$$

For the $\pi^+ - p$ scattering Eq. (1) gives instead of

Eqs. (2) — (4) only a single condition, which makes it possible to introduce real phases.

From Eqs. (2) and (3) we get the relations between absolute values

$$|S_{11}|^2 = |S_{22}|^2 = \rho_{11}^2 = \rho_{22}^2 = \rho^2, \quad |S_{12}|^2 = \rho_{12}^2 = 1 - \rho^2, \tag{A.5}$$

and from Eqs. (4) and (5) a connection between the phases

$$4\alpha_{12} = 2(\alpha_{11} + \alpha_{22}) + n\pi. \tag{A.6}$$

Consequently in this case, in virtue of the three relations (2) — (4) the amplitudes of the three processes, including both elastic and inelastic scattering, can be expressed in terms of three real numbers, which must be determined by experiment.

We note that when one goes to a larger number of channels Eq. (6) remains an approximately true relation if the probabilities of transitions between channels are small in comparison with those for transitions within the channels (cf. Ref. 7, for example).

Relations (5) and (6) can be put in another form. The situation here recalls that which is encountered when, in considering the transitions ${}^3P_2 \rightarrow {}^3P_2$, ${}^3F_2 \rightarrow {}^3F_2$, ${}^3P_2 \rightleftarrows {}^3F_2$ in nucleon-nucleon scattering, one introduces real phases and mixing coefficients.[8] In analogy with this we introduce two real phases δ_J^I and δ_J^{II} and a mixing parameter ϵ_J:

$$\begin{aligned} S_{11}^{Jl} &= \exp[2i\delta_J^I]\cos^2\varepsilon_J + \exp[2i\delta_J^{II}]\sin^2\varepsilon_J, \\ 2S_{12}^{Jl} &= \{\exp[2i\delta_J^I] - \exp[2i\delta_J^{II}]\}\sin 2\varepsilon_J, \\ S_{22}^{Jl} &= \exp[2i\delta_J^I]\sin^2\varepsilon_J + \exp[2i\delta_J^{II}]\cos^2\varepsilon_J \end{aligned} \tag{A.7}$$

(unlike the case of N — N scattering, here both $\delta_J^{I,II}$ and also ϵ_J depend on ℓ as well as on J, and the ϵ_J do not drop out of the expressions for the integrated cross-sections). The connection bebetween the quantities ρ, α_{11}, α_{22} and δ_J^I, δ_J^{II}, ϵ_J follows from Eqs. (5), (6), and (7). The elements of the S matrix are expressed in this way in the exact formulation when the existence of isotopic invariance is not assumed. As is well known, under this further condition one has

$$3S_{11}^{Jl} = 2b_1^{Jl} + b_3^{Jl}; \quad 3S_{12}^{Jl} = \sqrt{2}\,(b_3^{Jl} - b_1^{Jl}); \quad 3S_{22}^{Jl} = b_1^{Jl} + 2b_3^{Jl} \tag{A.8}$$

($b_{2T}^{J\ell} = \exp[2i\delta_{2T}^{J\ell}]$ characterizes the scattering in a state with given J and isotopic spin T). From a comparison of Eqs. (8) and (7) it can be seen that isotopic invariance corresponds to the case in which ϵ_J, not depending on J and ℓ (there remain two real parameters) takes the constant value arc tan $2^{1/2} \approx 55°$, if δ_J^I corresponds to δ_3 and δ_J^{II} to δ_1.

By checking these last assertions (lack of dependence of ϵ_J on J and ℓ and definite value of ϵ_J) one has a possibility for confirming the isotopic invariance in the scattering of charged mesons. It might seem that, since the cross-section for elastic scattering of π^0 mesons appears in Heitler's relation

$$d\sigma(\pi^+ + p \to \pi^+) + d\sigma(\pi^- + p \to \pi^-) = 2d\sigma(\pi^0 + p \to \pi^0) + d\sigma(\pi^- + p \to \pi^0)$$

the π — N scattering cannot be used for checking the isotopic invariance. Fermi has pointed out[7] that, at least when one takes into account only the s and p waves, such a check can be carried out with experiments with charged mesons. From the results of this appendix it follows that this can be done with the inclusion of an arbitrary number of states. The check proposed here goes beyond this. It makes it possible to see to what accuracy isotopic invariance is fulfilled in each state of the π — N system.

We note a generalization of the symmetry pointed out by Minami[9,10] to the case in which an inelastic process occurs in parallel with the elastic process. By direct verification one readily finds that the unpolarized cross-sections of the processes in question remain unchanged if one makes the simultaneous replacements

$$\delta_{J,\,J-1/2}^{I,\,II} \rightleftarrows \delta_{J,\,J+1/2}^{I,\,II}; \quad \varepsilon_{J-1/2}^{J} \rightleftarrows \varepsilon_{J+1/2}^{J}.$$

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TRANSITION BETWEEN HYPERFINE STRUCTURE LEVELS IN MU-MESIC HYDROGEN

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It is shown that, through the mechanism of "jumping" of the μ^- meson from one proton to another, which was proposed by Ia. B. Zel'dovich, mesic hydrogen atoms convert completely to the ground state of the hyperfine structure during the lifetime of the μ meson. As a result, there is complete depolarization of μ mesons in hydrogen, and the neutrons which are formed from capture of μ^- mesons by protons via $\mu^- + p = n + \nu$ will be completely polarized along their direction of motion.

THE separation between the upper ($F = 1$, where F is the total spin of the mesic atom) and lower ($F = 0$) levels of the hyperfine structure for a μ meson in the K orbit of a mesic hydrogen atom, is[1]

$$\Delta\varepsilon = \frac{16\pi}{3}\beta_\mu\beta_N g_i |\psi(0)|^2 = 0.25 \text{ ev} \qquad (1)$$

(β_μ is the μ-mesonic and β_N the nuclear Bohr magneton, $g_i = 2 \times 2.79$ is the gyromagnetic ratio of the proton).

Because of the smallness of this separation, the radiative transition to the lower state is extremely improbable ($\tau_{rad} \sim 10^6$ sec). However, because of the neutrality of mesic hydrogen there is a very effective mechanism via which there is a complete transition into the lower hyperfine structure state during the lifetime of the μ meson. This mechanism is the "jump" of the μ meson from one proton to another with simultaneous transition into the lower state of the hyperfine structure.* Since the hyperfine splitting is much greater than the thermal energy in collisions of a proton and a mesic atom, the process is irreversible. In the present paper we give an estimate of the cross section for this transition.

In mesic units ($e = 1$, $\hbar = 1$, $m_\mu = 1$), the Hamiltonian for the interaction of a meson with a pair of protons, including the interaction of the spins of the meson and the protons is

$$\hat{H} = -\frac{1}{2M}\Delta_{\mathbf{R}_1} - \frac{1}{2M}\Delta_{\mathbf{R}_2} - \frac{1}{2}\Delta_{\mathbf{r}} - \frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{R} + \frac{4}{3} g_i\beta_\mu\beta_N\left(\frac{\delta(r_1)}{r_1^2}\mathbf{s}\mathbf{i}_1 + \frac{\delta(r_2)}{r_2^2}\mathbf{s}\mathbf{i}_2\right) \qquad (2)$$

($\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{r}$ are the coordinates and $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{s}$ are the spins of the protons and the meson, $R = |\mathbf{R}_1 - \mathbf{R}_2|$ is the distance between the protons, while $r_1 = |\mathbf{r} - \mathbf{R}_1|$ and $r_2 = |\mathbf{r} - \mathbf{R}_2|$ are the distances of the meson from the two protons). We are neglecting the spin-spin interaction between the protons and between the meson and the second proton when the meson is at the position of the first proton.

At the velocities we are considering, the relative motion of the protons is described by an s wave, so that the total spin is conserved. The spin of the system consisting of two protons and a μ

*This was called to the attention of the author by Ia. B. Zel'dovich.

up the crystals as infinitely long unidimensional or two-dimensional atom complexes, bound together by "small" forces of one nature, whereas in the complex itself the atoms are bound by "big" forces of another nature.

6. The difference between the typical molecular crystals (e.g., the CH_4 or C_6H_6 crystals) and the heteropolar molecular crystals (such as the NaCl, $HgCl_2$ or PbS crystals) lies: (1) in the degree of molecularity β; (2) in the nature of the forces in the molecules; (3) in the nature of intermolecular forces. The quantity β is defined as the ratio of the intramolecular energy $U^a \cong D$ (D is the energy of dissociation of the diatomic molecule into ions) to the intermolecular energy U^e per bond. For the substances for which β is given below, it is possible to take $U^e \approx 2S/l$. Example:

$\beta = 300(CH_4)$, $200(HCl)$, $22(HgCl_2)$, $10(NaCl)$ taking $l = 12$ in all four cases.

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ERRATA

Volume 5

Page	Line	Reads	Should Read
1043	Eq. (4)		$W = y^2 a_{14}^2 \sin 2\phi/2\rho\,(a_{11}a_{44} - \alpha_{14}^2 \sin^2 3\phi)$ The coefficient k_2 equals $0.185 \times 10^{-3}\,cm^{-1}$.
1044	3 from bottom (l.h.)	$\Delta y = 2.87 \times 10^{-3}$ cm	$\Delta y = 3.18 \times 10^{-3}$ cm
	4 from top (r.h.)	$\Delta\vartheta_\Sigma = 7.2 \times 10^{-5}$ radians	$\Delta\vartheta_\Sigma = 5.9 \times 10^{-5}$ radians

Volume 6

Page	Line	Reads	Should Read
1090	4 and 5 from top	2—(d, 3n); and of the I_{53}^{127} cross section, 3—(d, 2n); 4—(d, 3n)	2—(d, 3n) on I_{53}^{127} and 3—(d, 3n); 4—(d, 3n) on Bi_{83}^{209}
1091	6 from bottom expression for determinant C(y)	ρ, γp, h, $1/\rho$	ρy_2, $\gamma p y_2$, $h y_2$, y_2/ρ
1094	7 from bottom	For $\gamma = 5/3$, μ has . . .	Here μ has . . .

Volume 7

Page	Line	Reads	Should Read
55	16 from bottom	Correct submittal date is April 5, 1957	
169	17 from bottom	Delete "Joint Institute for Nuclear Research"	
215	Table	Add: Note. Columns 2—9 give the number of counts per 10^6 monitor counts	
215	Table, column headings	1, 2, 3, 4-7, 8	1, 2, 3, 4, 8-7
312	Eq. (8)	. . . $(1 \pm \mu/2M)^2$	. . . $(1 \mp \mu/2M)^2$
313	2, r.h. col.	$\alpha_{33} = 0.235$	$a_{33} = 0.235$
692	Eq. (5)	$m_B/M_B = \ldots \mp [1 + \ldots]$	$m_B/M_B = \mp [1 + \ldots]$
461	Title	. . . Elastically Conducting	. . . Electrically Conducting