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*A GROUP-THEORETICAL CONSIDERATION OF THE BASIS OF RELATIVISTIC QUANTUM MECHANICS. II. CLASSIFICATION OF THE IRREDUCIBLE REPRESENTATIONS OF THE INHOMOGENEOUS LORENTZ GROUP**

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A classification is obtained for the states of a relativistic quantum system. The irreducible representations of the inhomogeneous Lorentz group are divided into four fundamental classes: P_m, P_{Π}, P_0, O_0. All the representations of classes P_m and P_{Π}, both unitary and non-unitary, are found explicitly.

1. CLASSIFICATION OF THE STATES OF A RELATIVISTIC QUANTUM SYSTEM

WE have previously[1] found all the possible invariants of the inhomogeneous Lorentz group, and have noted that the classification of the irreducible representations of the group reduces to finding the eigenvalue spectra of these invariants. However, we as yet do not know the independent variables contained in the wave functions, which transform according to a particular irreducible representation. In order to find these variables and their domain of variation, we must select from among the operators of the group a complete set, i.e., a complete system of operators which commute with one another (but not with all the operators of the group). The choice of such a system of operators is, of course, not unique. This non-

*Notations used without explanation are the same as in Ref. 1. References like (I.8) are to the corresponding formula in Ref. 1.

uniqueness corresponds to the possibility of performing an equivalence transformation of type (I.8) on the particular representation. As such a system of operators, we might select the invariants of the homogeneous Lorentz group, $M^2_{\mu\nu}$ and $\epsilon_{\mu\nu\lambda\sigma}M_{\mu\nu}M_{\lambda\sigma}$, and in addition the square of the three-dimensional angular momentum M^2_i and one of its projections, M_3. However, this classification is not convenient since it is not translationally invariant. The motion of the system as a whole is not separated out, so that the wave functions of individual states do not belong to a single value of energy and momentum. The most natural classification is one which is translationally invariant, in which the complete set selected consists of the four operators p_μ and one of the projections of Γ_σ, for example Γ_3. In a given irreducible representation, only three of the four operators p_μ are independent, since the eigenvalue of the invariant p^2_μ is the same for all the functions of the irreducible representation. The set of eigenvalue spectra of the operators of the complete set (for example, p_1, p_2, p_3, Γ_3) also give us the complete system of independent variables and their domain of variation for the particular irreducible representation.

Let us summarize our results:

1. In order to find all the irreducible representations of the inhomogeneous Lorentz group, i.e., all the wave functions admissible in quantum mechanics, we must find the eigenvalues and simultaneous eigenfunctions of the operators p_1, p_2, p_3, p_0, Γ^2_σ, Γ_3, satisfying relations (I.41).

2. In a given irreducible representation, only those eigenfunctions can appear which belong to the same eigenvalues of the group invariants p^2_λ and Γ^2_σ. As mentioned in Sec. 12 of Ref. 1, additional invariants may exist for certain classes of representations.

2. THE FOUR FUNDAMENTAL CLASSES OF REPRESENTATIONS OF THE GROUP G

The group of four-dimensional translations characterized by the operator p_μ is a commutative subgroup of the inhomogeneous Lorentz group. Its irreducible representations are one-dimensional and unitary.* Each representation is defined by the set of four numbers p_1, p_2, p_3, $p_0 = p_4/i$, which are eigenvalues of the operators $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$, $\hat{p}_0$. The eigenvalues p_μ can be any real numbers. Only those functions can belong to the same irreducible representation of G which have the same value of p^2_μ. The irreducible representations of the whole group differ qualitatively from one another according as p^2_μ is a negative number (timelike p_μ), positive (spacelike p_μ), or zero. In the last case, representations in which $p_\mu \neq 0$ (p_μ lies on the light cone) and in which $p_\mu = 0$ are qualitatively different. Accordingly, we get four classes of representations of the group G, which we shall investigate in turn:

I. Class P_m: p_μ is a timelike vector.

II. Class P_0: p_μ is a vector on the light cone.

III. Class P_Π: p_μ is a spacelike vector.

IV. Class O_0: $p_\mu = 0$.

For the unitary representations of the inhomogeneous Lorentz group, the division of the representations into these four classes was first done by Wigner by a different method. He also obtained the detailed classification of the unitary representations of classes P_m and P_0.[2] The complete system of irreducible representations of class O_0 coincides with the complete system of representations of the homogeneous Lorentz group which was found by Gel'fand and Naimark.[3]

3. CLASS P_m

For the class P_m, the sign of the energy, $S_H = p_0/|p_0|$, is an invariant of the group, so that for each set of eigenvalues of p^2_μ and Γ^2_σ there will be not one, but two irreducible representations, one for each sign of the energy. Instead of the energy, it is convenient to use the mass m defined as

$$m = (p_0/|p_0|)\sqrt{-p^2_\mu}, \tag{1}$$

*The group of translations also has non-unitary representations, corresponding to complex values of the components of the four-momentum. The representations of the inhomogeneous Lorentz group obtained from them belong to complex values of the invariants m and Π of the classes P_m and P_Π. Representations of this sort occur, for example, when we add an imaginary term to the mass in calculations with Green's functions.

and can be any real number except zero (since zero mass corresponds to classes P_0 and O_0). In each irreducible representation, only those functions can appear which belong to the same value of m. Therefore, of the four variables p_μ in the wave functions of the irreducible representation, only three (for example, p_1, p_2, p_3) will be independent, while the fourth, p_0, will be equal to

$$p_0 = \frac{m}{|m|} E_p \equiv \frac{m}{|m|} \sqrt{\mathbf{p}^2 + m^2}. \quad (2)$$

One can also proceed differently, by forming the 4-velocity vector

$$u_\mu = p_\mu / m, \; u_\mu^2 = -1, \; u_0 = u_4 / i = \sqrt{\mathbf{u}^2 + 1}. \quad (3)$$

In this case, for each value of m the functions of the irreducible representation will depend on the 4-velocity u_μ, which has three independent components.

To find the eigenvalues of Γ_σ^2, it is convenient to go over (for fixed p_μ, which is permissible since p_μ commutes with Γ_σ) to the rest frame in which

$$\mathbf{p} = 0, \; p_4 = im. \quad (4)$$

We then find from (I.39), (I.41) that

$$\Gamma_4 = 0, \; \Gamma_\sigma^2 = \Gamma_i^2, \; [\Gamma_i, \Gamma_j]_- = im\varepsilon_{ijk}\Gamma_k. \quad (5)$$

Defining S_i by

$$\Gamma_i = mS_i, \quad (6)$$

we have

$$[S_i, S_j]_- = i\varepsilon_{ijk} S_k. \quad (7)$$

The commutators (7) define a three-dimensional Euclidean group of rotations, which is natural since $\Gamma_\sigma^2 = m^2 S_i^2$ represents the intrinsic angular momentum of the system. The irreducible representations of the three-dimensional rotation group are well known. All of them are unitary. Each is characterized by a positive integer or half-integer s, where

$$\mathbf{S}^2 \Omega_s = (\Gamma_\sigma^2 / m^2)\, \Omega_s = s(s+1)\, \Omega_s, \quad (8)$$

and the operators S_i are $(2s + 1)$-row matrices. For example, for $s = \frac{1}{2}$, $\mathbf{S} = \boldsymbol{\sigma}/2$, where $\boldsymbol{\sigma}$ is the Pauli matrix vector. The explicit form of the vector Γ_σ in an arbitrary coordinate system is obtained from (6) by Lorentz transformation:

$$\boldsymbol{\Gamma} = m\mathbf{S} + \mathbf{p}(\mathbf{pS}) / (|p_0| + m) = m\{\mathbf{S} + \mathbf{u}(\mathbf{uS}) / (u_0 + 1)\}, \quad \Gamma_0 = \Gamma_4 / i = \mathbf{pS} = m\mathbf{uS}. \quad (9)$$

The commutation relations between the components Γ_σ, as defined by (7) and (9), coincide with (I.41).

For a complete description of the representation, we have only to find the explicit form of the operator g_μ. One can verify directly that the operator $g_\mu = (\mathbf{g}, ig_0)$, where

$$g_i = -(ip_ip_j\partial / \partial p_j) - (im^2\partial / \partial p_i) - 3ip_i + \varepsilon_{ijk}p_jS_k, \quad g_0 = (-ip_0p_i\partial / \partial p_i) - 3ip_0 \quad (10)$$

satisfies the commutation relations (I.41).

By using (I.40), we can find from (9) and (10) the explicit form of the operator $M_{\mu\nu}$ for the 4-angular momentum:

$$\mathbf{M} = -i\,[\mathbf{p}\,\partial / \partial \mathbf{p}] + \mathbf{S} = -i\,[\mathbf{u}\,\partial / \partial \mathbf{u}] + \mathbf{S}, \; \mathbf{N} = ip_0 \frac{\partial}{\partial \mathbf{p}} - \frac{[\mathbf{Sp}]}{p_0 + m} = iu_0 \frac{\partial}{\partial \mathbf{u}} - \frac{[\mathbf{Su}]}{u_0 + 1}, \quad (11)$$

where $M_1 = M_{23}$ etc., $N_i = M_{i4}/i$. The square brackets denote the vector product. In this representation, the 4-momentum operator $\hat{p}_\mu$ has the form

$$\hat{p}_i = p_i, \; \hat{p}_0 = (m/|m|)\sqrt{\mathbf{p}^2 + m^2}, \text{ or } \hat{p}_\mu = mu_\mu. \quad (12)$$

Relations (11) were first used in the theory of elementary particles in Ref. 4.

Thus the irreducible representations of type P_m with timelike 4-momentum are characterized by two numbers: a real number m which is non-zero, and s which is integral or half-integral. The number m determines the rest mass of the system, and s its intrinsic angular momentum (i.e., the spin, in the case of an elementary particle). The wave functions $\Omega_{ms}(p_i)$ [or $\Omega_{ms}(u_i)$] corresponding to a particular representation are matrices of degree $2s+1$, depending on three independent variables p_i defined over the whole real axis.

The probability density $\Omega^*_{ms}\Omega_{ms}$ is a scalar. In calculating mean values (or norms), the integration is carried out with respect to the invariant volume element

$$d_0p = d^3p / |p_0| \text{ or } d_0u = d^3u / u_0,$$

$$\langle \Omega^*(p)\,\Omega(p)\rangle = \int d_0p\,\Omega^*(p)\,\Omega(p), \tag{13}$$

$$\langle \Omega^*(u)\,\Omega(u)\rangle = \int d_0u\,\Omega^*(u)\,\Omega(u). \tag{14}$$

We shall assign representations corresponding to positive and negative masses to different subclasses, and denote them by P_{+m} and P_{-m} respectively. Single-valued representations, corresponding to integer s, will be denoted by P^s_{+m} (P^s_{-m}), and two-valued representations with half-integer s by P'^s_{+m} (P'^s_{-m}). The results found for the classification of P_m coincide with Wigner's[2] results, and can be summarized in the following table:

Table of Representations of Class P_m

Representation	Unitarity Dimensionality in the spin variable	Fundamental invariants $m^2=-p_\mu^2>0$, $\Gamma_\sigma^2/m^2=s(s+1)$, $s=$	Additional invariants, S_H
P^s_{+m}	Unitary, finite-dimensional	0, 1, 2,...	1
P'^s_{+m}	"	1/2, 3/2,...	1
P^s_{-m}	"	0, 1, 2,...	—1
P'^s_{-m}	"	1/2, 3/2...	—1

4. CLASS P_Π

For the class P_Π, the square of the 4-momentum

$$p_\mu^2 = \Pi^2 \tag{15}$$

is positive, i.e., the vector p_μ is spacelike. The search for simultaneous eigenfunctions of the operators p_μ, Γ_σ^2, and one of the projections of Γ_σ, for example $\Gamma_0 = \Gamma_4/i$, is conveniently done in the coordinate system in which

$$p_\mu = (0,\ 0,\ \Pi,\ 0). \tag{16}$$

From (I.39) and (16) we find that in this coordinate system

$$\Gamma_3 = 0. \tag{17}$$

With the notation

$$\Gamma_1 + i\Gamma_2 = \Pi T^+,\ \Gamma_1 - i\Gamma_2 = \Pi T^-,\ \Gamma_0 = \Pi T_0 \tag{18}$$

the commutation relations (I.41) between the components Γ_σ, and the invariant Γ_σ^2 become

$$[T^+,\ T_0]_- = -T^+,\ [T^-,\ T_0]_- = T^-, \tag{19}$$

$$[T^+,\ T^-]_- = -2T_0, \tag{20}$$

$$\Gamma_\sigma^2 / \Pi^2 \equiv T^2 = T^-T^+ - T_0^2 - T_0. \tag{21}$$

The commutators (19), (20) define the group of rotations in three-dimensional pseudo-Euclidean space. All the unitary representations of this group were found by Bargmann.[5] T^2 in formula (21) is, of course, an invariant of this group. The wave functions satisfying (19), (20), are of the form $\Omega_{\alpha\beta}$, where α, β are the eigenvalues of T^2 and T_0 respectively:

$$T^2\Omega_{\alpha\beta} = \alpha\Omega_{\alpha\beta},\ T_0\Omega_{\alpha\beta} = \beta\Omega_{\alpha\beta}. \tag{22}$$

Only functions with the same value of α can appear in an irreducible representation. To determine the spectra of eigenvalues α, β, we operate on $\Omega_{\alpha\beta}$ with the first relation in (19):

$$[T,^{+}T_0]_{-}\Omega_{\alpha\beta} = -(T^{+}\Omega_{\alpha\beta}).$$

Using (22), we then get

$$T_0(T^{+}\Omega_{\alpha\beta}) = (\beta+1)(T^{+}\Omega_{\alpha\beta}). \quad (23)$$

Similarly, from the second relation in (19) we find

$$T_0(T^{-}\Omega_{\alpha\beta}) = (\beta-1)(T^{-}\Omega_{\alpha\beta}). \quad (24)$$

Thus the functions $(T^{+}\Omega_{\alpha\beta})$, $(T^{-}\Omega_{\alpha\beta})$ belong respectively to the eigenvalues $\beta \pm 1$. From this it follows that the spectrum of eigenvalues of T_0 is

$$\beta = \beta_0 + n, \quad (25)$$

where $1 > \beta_0 \geq 0$, $n = \ldots, -2, -1, 0, 1, 2, \ldots$, while the matrix T_0 is infinite-dimensional and has the form

$$(T_0)_{mn} = \beta_0\delta_{mn} + n\delta_{mn}. \quad (26)$$

We do not exclude the possibility that for certain values of β, $T^{+}\Omega_{\alpha\beta}$ (or $T^{-}\Omega_{\alpha\beta}$) may vanish. However, this case need not be treated specially, but will be obtained automatically from the general investigation.

The operator T_0 is the operator for an ordinary three-dimensional rotation in the hyperplane perpendicular to p_μ. The full rotation through 2π must either leave the function unchanged or (for a two-valued representation) multiply it by (-1). Thus β_0 can only be zero for single-valued representations, and $\frac{1}{2}$ for two-valued representations,

$$\beta_0 = 0, \ {}^1/_2, \quad (27)$$

since the matrix for a finite rotation through angle φ has the form

$$(\exp\{iT_0\varphi\})_{mn} = \delta_{mn}\exp\{i\beta_0\varphi + in\varphi\}. \quad (28)$$

To find the eigenvalues of the invariant T^2, we must determine the form of the matrices T^{+}, T^{-}. From (19) and (26) we get

$$T^{+}_{mn}(n-m+1) = 0, \quad (29)$$

$$T^{-}_{mn}(n-m-1) = 0, \quad (30)$$

so that

$$T^{+}_{mn} = a_n\delta_{m,\,n+1}, \quad (31)$$

$$T^{-}_{mn} = b_m\delta_{m+1,\,n}. \quad (32)$$

Substituting (31), (32), (22), and (26) in (21), we find

$$(T^2)_{mn} = \alpha\delta_{mn} = b_m a_n \delta_{m+1,\,l}\delta_{l,\,n+1} - \delta_{mn}(n+\beta_0)(n+\beta_0+1) = \delta_{mn}\{a_n b_n - (n+\beta_0)(n+\beta_0+1)\},$$

i.e.,

$$a_n b_n = \alpha + (n+\beta_0)(n+\beta_0+1), \quad (33)$$

where, according to (27), β_0 is equal to 0 or $\frac{1}{2}$. We should mention that in (29)–(33) there is no summation over the repeated indices m, n.

Formula (33), expressing the coefficients a_m, b_n in terms of the eigenvalue α of the invariant, together with (26) essentially determines all the irreducible representations of the rotation group in three-dimensional pseudo-Euclidean space and all the irreducible representations of class P_{II} for the group G. Only the product $a_n b_n$ of the matrix elements is given uniquely by (33). The elements a_n, b_n themselves are not determined uniquely, which corresponds to the possibility of subjecting the system of operators T^{+}, T^{-}, T_0 to an equivalence transformation using an arbitrary non-degenerate diagonal matrix V, which leaves the operator T_0 unchanged. Formulas (26), (31), (32), and (33) can be rewritten as

$$(T_0)_{m'n'} = \delta_{m'n'}n', \; (T^+)_{m'n'} = a_{n'}\delta_{m', n'+1}, \; (T^-)_{m'n'} = b_{m'}\delta_{m'+1, n'}, \tag{34}$$

$$a_{n'}b_{n'} = \alpha + n'(n'+1). \tag{35}$$

Here the indices m′, n′ are assumed to run through all integral (half-integral) values for single-valued (double-valued) representations.

Now we must first see for which values of α a representation of the whole group corresponds to the infinitesimal representation in the neighborhood of the identity. Secondly, we must check whether all the representations we have found are irreducible, since an additional splitting of the representations is possible, for example, with respect to a sign invariant. Thirdly, we must separate the representations into unitary, real non-unitary and complex representations. Finally, we must construct the operators Γ_σ and g_μ, or the operators $M_{\mu\nu}$.

In order to solve the problem of the continuity of the representations (34) and (35), we shall attempt to construct the operator for an arbitrary finite rotation in the three-dimensional pseudo-Euclidean space. The third axis of our space is the time axis, and a rotation about it is a space rotation. The first and second axes are space axes, while rotations about them are Lorentz transformations. As in ordinary Euclidean three-space, any rotation can in our case be represented as a product of rotations through three Euler angles: a space rotation about the third axis, a Lorentz rotation about the first axis, and a space rotation about the rotated third axis. The matrix for the rotation about the third axis was given in (28). It exists for any α. To get the operator for a rotation around the first axis, it is convenient to go over to the continuous spectrum, i.e., to take as the wave function not an infinite-dimensional matrix, but a continuous function $\Omega(\Phi)$ of a variable Φ which ranges from zero to 2π. Then the operators T^+, T^-, T_0 can be taken in the form:[5]

$$T_0 = (1/i)\partial/\partial\Phi, \tag{36}$$

$$T^+ = e^{i\Phi}\left(\frac{1}{i}\frac{\partial}{\partial\Phi} - l\right), \; T^- = \left(\frac{1}{i}\frac{\partial}{\partial\Phi} + l + 1\right)e^{-i\Phi}, \tag{37}$$

where

$$\alpha = -l(l+1). \tag{38}$$

A direct check will show that the operators T_0, T^+, T^- defined by (36) and (37) satisfy (19) – (22). The transition from (36) and (37) to (34) and vice versa is accomplished by a Fourier transformation. In accordance with (35) and (38), we find for $a_{n'}$, $b_{n'}$, from (37),

$$a_{n'} = (n' - l), \; b_{n'} = (n' + l + 1). \tag{39}$$

Rotation through the angle φ around the third axis is accomplished in the new representation by using the operator $U(\varphi)$:

$$U(\varphi) = \exp(\varphi\,\partial/\partial\Phi), \; U(\varphi)\,\Omega(\Phi) = \Omega(\Phi + \varphi). \tag{40}$$

The operator T_1 for the infinitesimal rotation around the first axis is

$$T_1 = (T^+ + T^-)/2 = -il\sin\Phi - i\cos\Phi\cdot\partial/\partial\Phi. \tag{41}$$

We find the eigenfunctions and eigenvalues of the operator T_1:

$$T_1\psi_{\Phi k} = k\psi_{\Phi k}. \tag{42}$$

After a simple integration we find that for any real k,

$$\psi_{\Phi k} = (4\pi)^{-1/2}(\cos\Phi)^l\left\{\tan\left(\frac{\Phi}{2} + \frac{\pi}{4}\right)\right\}^{ik}. \tag{43}$$

An arbitrary function $\Omega(\Phi)$ can be expanded in terms of the $\psi_{\Phi k}$:

$$\Omega(\Phi) = \int_{-\infty}^{\infty} dk\,\psi_{\Phi k}\Omega_k, \tag{44}$$

so that we may consider $\psi_{\Phi k}$ as the kernel of the operator of linear transformation from Φ to k, i.e., to the basis in which the operator T_1 is diagonal. The kernel of the operator reciprocal to $\psi_{\Phi k}$ is

$$\psi_{k\Phi}^{-1} = (4\pi)^{-1/2}(\cos\Phi)^{-l-1}\left\{\tan\left(\frac{\Phi}{2}+\frac{\pi}{4}\right)\right\}^{-ik}. \tag{45}$$

In fact, we can verify directly that

$$\int_0^{2\pi} d\Phi\, \psi_{k\Phi}^{-1}\psi_{\Phi k'} = \delta(k-k'), \tag{46}$$

$$\int_{-\infty}^{\infty} dk\, \psi_{\Phi k}\psi_{k\Phi'}^{-1} = \delta(\Phi-\Phi'). \tag{47}$$

For the wave function $\Omega(k)$ in the new representation, the operation of rotation through the Lorentz angle χ around the first axis is trivial, and reduces to simply multiplying $\Omega(k)$ by $e^{ik\chi}$:

$$U_k(\chi)\,\Omega(k) = e^{ik\chi}\Omega(k). \tag{48}$$

By using (43), (45), and (I.9), we can transform the operator $U_k(\chi)$ to the Φ-representation,

$$U_{\Phi\Phi'}(\chi) = \int_{-\infty}^{\infty} dk\,\psi_{\Phi k}e^{i\chi k}\psi_{k\Phi'}^{-1} = \delta(\ln\tan\theta - \ln\tan\theta' + \chi)\,\frac{(\cos\Phi)^l}{(\cos\Phi')^{l+1}} = \frac{\delta(\ln\tan\theta - \ln\tan\theta' + \chi)}{\sqrt{\cos\Phi\cos\Phi'}}\left(\frac{\cos\Phi}{\cos\Phi'}\right)^{\sqrt{1/4-\alpha}}, \tag{49}$$

where $\theta = \Phi/2 + \pi/4$, $\theta' = \Phi'/2 + \pi/4$. The rotation operation itself takes the form

$$\{U(\chi)\,\Omega\}_\Phi = \int_0^{2\pi} d\Phi' U_{\Phi\Phi'}(\chi)\,\Omega(\Phi') \tag{50}$$

or, in somewhat different form,

$$\{U(\chi)\,\Omega\}_\Phi = e^{l\chi}\left(\frac{1+\tan^2\theta}{1+e^{2\chi}\tan^2\theta}\right)^{-l}\Omega(\Phi'), \tag{51}$$

where

$$\tan\left(\frac{\Phi}{2}+\frac{\pi}{4}\right) = e^{\chi}\tan\left(\frac{\Phi'}{2}+\frac{\pi}{4}\right).$$

From (51) we see that the transformation for rotation around the first axis actually does exist for any finite values of α, χ. As we said earlier, we can construct the operator for any rotation by using (40) and (51), and this operator will exist for arbitrary α. The theorem we have just demonstrated is not trivial since, for example, for the group of three-dimensional Euclidean rotations there exist representations in the neighborhood of the identity which cannot be extended over the whole group.

We shall now select the real and unitary representations of class P_{II}. According to Sec. 13 of Ref. 1, the representations will be complex for complex α and real for real α. For unitary representations, the operators T^+ and T^- must be Hermitean adjoint to one another. In this case, we find from (31) and (35) that

$$a_{n'} = b_{n'}^*,\ |a_{n'}|^2 = \alpha + n'(n'+1). \tag{52}$$

From (52) it follows that a representation can be unitary only if the quantity $\alpha + n'(n'+1)$ is not negative for any integral (half-integral) values of n'. For integer n', the requirement that (52) be positive is satisfied for $\alpha = a > 0$. (The case of $\alpha = 0$ will be treated later and assigned to another subclass). The corresponding single-valued unitary representations will be denoted by P_{II}^a. For half-integral n', (52) is positive for $\alpha = a > 1/4$ (the case of $\alpha = 1/4$ also will be assigned to another subclass). These two-valued representations will be denoted by $P_{\text{II}}'^a$. However, the subclasses P_{II}^a and $P_{\text{II}}'^a$ do not exhaust the unitary irreducible representations of class P_{II}. The point is that for $\alpha = -s(s+1)$, where s is an integer (half-integer) for single-valued (double-valued) representations, the representation (34) ceases to be irreducible. For $n' = s$ and $n' = -s-1$, the coefficients $a_{n'}$, $b_{n'}$ in (34) and (35) vanish, and the matrices T_0, T^+, T^- simultaneously assume the "block" form.

In this case the representation (34) breaks up into three irreducible representations, two of which are infinite-dimensional and unitary, while the third is finite-dimensional and non-unitary. The matrix elements of the infinitesimal operators are given by formulas (34), (35) for all three cases. In one of the representations, the indices m′, n′ run through values from $-\infty$ to $-s-1$; in the second they go from $s+1$ to ∞, and in the third from $-s$ to s. These representations are respectively designated by $P_\Pi^{-\ell}$, $P_\Pi^{+\ell}$, P_Π^{s} for integral ℓ, s, and by $P_\Pi^{\prime -\ell}$, $P_\Pi^{\prime +\ell}$, $P_\Pi^{\prime s}$ for half-integral ℓ, s. The representations $P_\Pi^{+\ell}$, $P_\Pi^{-\ell}$, $P_\Pi^{\prime +\ell}$, $P_\Pi^{\prime -\ell}$ (and P_Π^{s}, for $s = 0$) are unitary. For them the sign of T_0 is an invariant. The representations P_Π^{s} ($s \neq 0$) and $P_\Pi^{\prime s}$ are non-unitary. For them, the sign of T_0 is not an invariant. The indices ℓ, s can take on the values:

$$\begin{aligned}
&\text{for representations } P_\Pi^{+\ell}: \ \ell = 0,\ 1,\ 2,\ \ldots,\\
&\text{for representations } P_\Pi^{\prime\pm\ell}: \ \ell = -1/2,\ 1/2,\ 3/2,\ 5/2,\ \ldots,\\
&\text{for representations } P_\Pi^{s}: \ s = 0,\ 1,\ 2,\ \ldots,\\
&\text{for representations } P_\Pi^{\prime s}: \ s = 1/2,\ 3/2,\ 5/2,\ \ldots,
\end{aligned} \tag{53}$$

We may mention that the unitary representations P_Π^{+0}, P_Π^{-0}, and P_Π^{0} are all different. The first two are infinite-dimensional and correspond to the case of $\Gamma_\sigma^2 = 0$, $\Gamma_\sigma \neq 0$. The representation P_Π^{0} corresponds to the case of $\Gamma_\sigma = 0$.

A peculiarity of the subclass $P_\Pi^{\prime\pm\ell}$ is the presence of the representation with $\ell = -\frac{1}{2}$, corresponding to the case when (34) splits into two, instead of three, irreducible representations. The finite-dimensional (in the spin variable) non-unitary representations of the group of 3-rotations in pseudo-Euclidean space corresponding to the subclasses P_Π^{s} and $P_\Pi^{\prime s}$ can be gotten by Weyl's[6] unitary trick from the irreducible representations of the ordinary group of 3-rotations.

The real, infinite-dimensional, non-unitary representations, which are not included in the subclasses enumerated above, will be denoted by P_Π^{b} (for integer n′), and by $P_\Pi^{\prime b}$ (for half-integer n′); the complex representations will be assigned to P_Π^{α} (for integral n′), and $P_\Pi^{\prime\alpha}$ (for half-integral n′).

To complete our treatment of class P_Π there remains only the construction of the components of the operator $M_{\mu\nu}$. This rather involved problem is probably solved most simply as follows. In treating the class P_m we had occasion to transform to the rest system, which explicitly singled out the time axis. Accordingly, formula (11) is symmetric with respect to the three space axes but not symmetric with respect to the time axis. In the present section, we explicitly distinguished the third space axis whereas the first two space axes and the time axis were essentially not distinguished. We may therefore expect that we will get formulas which are relatively simple and similar to (11) if we introduce three-dimensional vectors and tensors defined in the pseudo-Euclidean space x_1, x_2, x_4. We shall mark such vectors with a superior tilde. One of the components of each such vector is imaginary. For example, in this space the three-dimensional momentum will have the form

$$\tilde{p}_i = (p_1,\ p_2,\ ip_0), \tag{54}$$

while the component p_3 will be the three-dimensional scalar

$$p_3 = \pm\sqrt{\Pi^2 - \tilde{p}_i^2}. \tag{55}$$

From the operators T^+, T^-, T_0, we can form the vector

$$\tilde{T}_i = (iT_1,\ iT_2,\ T_0),\ T_1 = (T^+ + T^-)/2,\ T_2 = (T^+ - T^-)/2i. \tag{56}$$

Using (19), we can easily verify that the $\tilde{T}_i$ satisfy the covariant commutation relations

$$[\tilde{T}_i\tilde{T}_j]_- = i\tilde{\varepsilon}_{ijk}\tilde{T}_k. \tag{57}$$

We now note that the relations (11) can be rewritten as

$$M_i = {}^1/_2\varepsilon_{ijk}M_{jk} = -(i\varepsilon_{ijk}p_j\partial/\partial p_k) + S_i, \qquad M_{i4} = (ip_4\partial/\partial p_i) + \varepsilon_{ijk}S_jp_k/(p_4 + \sqrt{\mathbf{p}^2 + p_4^2}). \tag{58}$$

where $p^2 + p_4^2 = -m^2 = \text{inv}$, and the three-dimensional vectors are "ordinary" vectors. The tensor $M_{\mu\nu}$, defined by (58), satisfies the commutation relations (I.31); in deriving this, we use only the relations (7) and

$$\partial p_\mu / \partial p_\nu = \delta_{\mu\nu}. \tag{59}$$

Since relations (7) are completely analogous to (57), it is obvious that the tensor $\widetilde{M}_{\mu\nu}$, expressed in terms of the vector $\widetilde{M}_{i3}$ and the pseudo-vector $\widetilde{M}_i$ in the pseudo-Euclidean 3-space (x_1, x_2, x_4)

$$\widetilde{M}_i = {}^1/_2 \widetilde{\varepsilon}_{ijk} \widetilde{M}_{jk} = -i\widetilde{\varepsilon}_{ijk} \widetilde{p}_j \frac{\partial}{\partial \widetilde{p}_k} + \widetilde{T}_i, \qquad \widetilde{M}_{i3} = ip_3 \frac{\partial}{\partial \widetilde{p}_i} - i\widetilde{p}_i \frac{\partial}{\partial p_3} + \widetilde{\varepsilon}_{ijk} \frac{\widetilde{T}_j \widetilde{p}_k}{p_3 + \Pi} \tag{60}$$

will satisfy (I.31). In Eq. (60),

$$i, j, k = 1, 2, 4, \; p_\mu^2 = \Pi^2.$$

Writing (60) in terms of "ordinary" components, we get

$$M_1 = M_{23} = -i(p_2 \partial/\partial p_3 - p_3 \partial/\partial p_2) + (T_1 E_p + T_0 p_1)/(p_3 + \Pi),$$

$$M_2 = M_{31} = -i(p_3 \partial/\partial p_1 - p_1 \partial/\partial p_3) + (T_2 E_p + T_0 p_2)/(p_3 + \Pi), \; M_3 = M_{12} = -i(p_1 \partial/\partial p_2 - p_2 \partial/\partial p_1) + T_0, \tag{61}$$

$$N_1 = M_{14}/i = (iE_p \partial/\partial p_1) - T_1, \; N_2 = M_{24}/i = (iE_p \partial/\partial p_2) + T_2, \; N_3 = M_{34}/i = (iE_p \partial/\partial p_3) + (T_2 p_1 - T_1 p_2)/(p_3 + \Pi).$$

In (61), $E_p = -ip_4 = \pm\sqrt{p^2 - \Pi^2}$. The independent continuous variables are p_1, p_2, p_3. Their domain of variation is limited by the condition $\infty > |p| \geq \Pi$. The energy is not an independent variable, but for given values of p_1, p_2, and p_3, its sign can be arbitrary within the representation. Thus the sign of the energy is a discrete independent variable which takes on two values, and relations (61) will give the correct commutation relations over the whole domain of definition of the independent variables only if this point is taken into account. The simplest way to take account of the two signs of the energy is to express the components of the 4-vector p_μ in terms of the four-dimensional polar angles φ, ϑ, χ:

$$p_1 = \Pi \cosh\chi \sin\vartheta \cos\varphi, \; p_2 = \Pi \cosh\chi \sin\vartheta \sin\varphi, \; p_3 = \Pi \cosh\chi \cos\vartheta, \; p_0 = \Pi \sinh\chi,$$

$$2\pi \geqslant \varphi \geqslant 0, \; \pi \geqslant \vartheta \geqslant 0, \; \infty > \chi > -\infty. \tag{62}$$

The two-valuedness of the energy is then obtained automatically. The components of $M_{\mu\nu}$, expressed in terms of the angle variables, become

$$M_1 = i \sin\varphi \frac{\partial}{\partial\vartheta} + i \cot\vartheta \cos\varphi \frac{\partial}{\partial\varphi} + \frac{T_1 \sinh\chi + T_0 \cosh\chi \sin\vartheta \cos\varphi}{1 + \cosh \cos\vartheta},$$

$$M_2 = -i\cos\varphi \frac{\partial}{\partial\vartheta} + i\cot\vartheta \sin\varphi \frac{\partial}{\partial\varphi} + \frac{T_2 \sinh\chi + T_0 \cosh\chi \sin\vartheta \sin\varphi}{1 + \cosh\chi \cos\vartheta}; \; M_3 = -i\partial/\partial\varphi + T_0,$$

$$N_1 = i \sin\vartheta \cos\varphi \frac{\partial}{\partial\chi} + i \tanh\chi \cos\vartheta \cos\varphi \frac{\partial}{\partial\vartheta} - i \tanh\chi \frac{\sin\varphi}{\sin\vartheta} \frac{\partial}{\partial\varphi} - T_1, \tag{63}$$

$$N_2 = i \sin\vartheta \sin\varphi \frac{\partial}{\partial\chi} + i \tanh\chi \cos\vartheta \sin\varphi \frac{\partial}{\partial\vartheta} + i \tanh\chi \frac{\cos\varphi}{\sin\vartheta} \frac{\partial}{\partial\varphi} + T_2,$$

$$N_3 = i\cos\vartheta \frac{\partial}{\partial\chi} - i \tanh\chi \sin\vartheta \frac{\partial}{\partial\vartheta} + \frac{\cosh\chi \sin\vartheta (T_2 \cos\varphi - T_1 \sin\varphi)}{1 + \cosh\chi \cos\vartheta}$$

Formulas (62), (63), together with (34), (35) give the complete solution of the problem of finding all the irreducible representations of the class P_Π.

One may try to construct the tensor $M_{\mu\nu}$ by a procedure different from ours, by simply replacing the vector S_i in (11) [or, what amounts to the same thing, in (58)] by the vector T_i with components (iT_1, iT_2, T_0). This vector is analogous to the vector $\widetilde{T}_i$ of (56), but, unlike it, is defined in a Euclidean rather than a pseudo-Euclidean 3-space. The components of the angular momentum tensor formed in this way,

$$M_i = {}^1/_2 \varepsilon_{ijk} M_{jk} = -i(\varepsilon_{ijk} p_j \partial/\partial p_k) + T_i, \qquad M_{i4} = (ip_4 \partial/\partial p_i) + \varepsilon_{ijk} T_j p_k / (p_4 + \sqrt{p^2 + p_4^2}) \tag{64}$$

satisfy commutation relations (I.31), and one might get the impression that we have constructed representations which are equivalent to (63). As a matter of fact, however, (64) is equivalent to (63) only for representations which are finite-dimensional in the spin variable. For infinite-dimensional matrices T_i, formulas (64) do not in general define a representation at all, since in this case it is impossible to construct a finite three-dimensional rotation about the x or y axes. This is apparent from the fact that the components of the vector T_i are infinite-dimensional and, at the same time, irreducible with respect to three-dimensional rotations; i.e., they constitute an infinite-dimensional irreducible representation of the group of Euclidean 3-rotations in the neighborhood of the identity. Such a representation cannot be built up over the whole group, since all the irreducible representations of the group of 3-rotations (and of any compact group) are finite-dimensional.

In conclusion we give a table of the irreducible representations of the class P_{Π}:

Representation	Unitarity. Dimensionality in the spin variable.	Fundamental Invariants $\Pi^2 = p^2_\mu > 0$ $\alpha = -s(s+1) - \Gamma_\sigma^2/\Pi^2$	Additional invariants
P_Π^a	Unitary infinite-dimensional	$\alpha = a > 0$	—
$P_\Pi'^a$	" "	$\alpha = a > \frac{1}{4}$	—
P_Π^{+l}	" "	$s = l = 0, 1, 2, \ldots$	$S_{\Gamma_0} = 1$
P_Π^{-l}	" "	$s = l = 0, 1, 2, \ldots$	$S_{\Gamma_0} = -1$
$P_\Pi'^{+l}$	" "	$s = l = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$	$S_{\Gamma_0} = 1$
$P_\Pi'^{-l}$	" "	$s = l = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$	$S_{\Gamma_0} = -1$
P_Π^s	Non-unitary; $2s + 1$	$s = 0, 1, 2, \ldots$	—
$P_\Pi'^s$	" "	$s = \frac{1}{2}, \frac{3}{2}$	—
P_Π^b	Non-unitary infinite-dimensional	$\alpha = b < 0 \quad s \neq 1, 2, 3, \ldots$	—
$P_\Pi'^b$	" "	$\alpha = b < \frac{1}{4} \quad s \neq \frac{1}{2}, \frac{3}{2}, \ldots$	—
P_Π^α	" "	α = complex	—
$P_\Pi'^\alpha$	" "	α = complex	—

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