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PROPAGATION OF CASCADES IN A MULTI-LAYER MEDIUM

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The propagation of a cascade consisting of several types of particles in a medium composed of homogeneous layers R^λ is considered. The layer boundaries may be mobile. The cascade particles collide with the particles of the medium and in the process are absorbed, scattered, and produce new particles. The functions W^λ describing the distribution of particles of each type according to position and velocity are found under the assumption that the particle distribution functions V^λ in the layer R^λ when the layer occupies the whole space are known. The functions W^λ are represented as infinite series of integrals which, in general, show a good convergence. The integrands are certain products of the functions V^λ.

In the present article we shall make use of the notation and results of Ref. 1, in which a method of calculating the functions $V_{ij}(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v})$ was given. The functions V_{ij} determine the probabilities $V_{ij}\,d\mathbf{r}\,d\mathbf{v}$ that a particle of a given type A_j will be found at the time t to possess radius vector between $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ and velocity between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$ if the cascade is initiated by a single particle appearing with the velocity $\mathbf{u}$ at the time s and the point $\mathbf{q}$. It has been assumed that the medium is filling the whole space and that its properties are independent of time and place. In Ref. 1 it has been assumed that all new particles of the cascade are produced at the moment of collision. In Ref. 2, an analogous function V describing a neutron cascade — a cascade consisting of particles of a single type which may be produced with a delay — was found by means of the same method. It is easy to combine the two cases and to consider a cascade consisting of n types of particles which may be produced with a delay. In that case, too, we shall assume the functions V_{ij} to be known for the case of an infinite homogeneous medium. In the present, a method of solution of this problem will be given for the case of a multi-layer medium, i.e., a medium consisting of different homogeneous layers occupying adjoining regions $R^\lambda (\lambda = 1, 2 \ldots)$. Let E be the space of variables t, $\mathbf{r}$, $\mathbf{v}$, j. It consists of n 7-dimensional spaces $E_j (j = 1, 2 \ldots n)$. We shall assume that the boundaries are varying with time t. Furthermore, for the sake of symmetry, we shall assume that they may depend also on j and $\mathbf{v}$, i.e, that the R^λ are arbitrary regions in E.

The solution of this problem is important, for example, for the study of transient cosmic ray effects,[3,4] in the theory of nuclear reactors,[5] in calculations of radiation shielding,[6] etc. The problem amounts to solving the Boltzmann linear integral equation of the type (1.15) or (1.16)* in a multi-layer medium and represents a generalization of an analogous problem in the theory of parabolic differential equations.[7,8]

*Here and in the following (1.15) denotes the formula (15) of Ref. 1, etc.

If we put $P = (s, \mathbf{q}, \mathbf{u}, i)$ and $Q = (t, \mathbf{r}, \mathbf{v}, j)$ then the functions $V_{ij}(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v})$ can be written in an abbreviated form as $V(P, Q)$. We shall denote the function $V(P, Q)$ by $V^{\lambda}(P, Q)$ if the whole space is filled by the medium filling the region R^{λ}. We shall assume the latter functions to be given. We shall denote by $W(P, Q)$ the required function describing the cascade particle distribution in a multi-layer medium, and by $W^{\lambda}(P, Q)$ the expression of this function in the region R^{λ}. We shall also introduce the auxiliary functions $U^{\lambda}(P, Q)$. The function U^{λ} gives the particle distribution in the region R^{λ} if all other regions are filled with an ideally absorbing medium, so that any particle leaving R^{λ} disappears immediately. Evidently, $U^{\lambda}(P, Q) = 0$ if P lies within the region R^{λ}.

Before calculating the function W^{λ} we shall solve an auxilliary problem. The functions $V(P, Q)$ give the particle distribution at the time t according to particle type A, position $\mathbf{r}$ and velocity $\mathbf{v}$. It may be, however, required to solve an inverse problem of the particle distribution for a given value of a certain component of $\mathbf{r}$ or $\mathbf{v}$ according to the type A_j, time t, and the remaining components of $\mathbf{r}$ and $\mathbf{v}$. In general, we can make the following substitution for the variables t, $\mathbf{r}$, $\mathbf{v}$, j in E:

$$t = t_j(\tau, \sigma), \quad \mathbf{r} = \mathbf{r}_j(\tau, \sigma), \quad \mathbf{v} = \mathbf{v}_j(\tau, \sigma), \tag{1}$$

where σ is a 6-component variable, so that τ and σ are seven new variables and putting $P^* = (\tau, \sigma, j)$ we find the probability $V^*(P, P^*)\,d\sigma$ of finding a particle of the type A_j with σ in the interval between σ and $\sigma + d\sigma$ for given values of j and τ. Taking into account that the variable j varies in a discrete way we can see that, without any loss of generality, we have preserved its value carrying out the transformation (1).

Let K_0 be the original particle of the cascade and K one of the cascade particles existing at the time t. It has been shown[1] that corresponding to the latter particle is a chain C of particles connecting K_0 and K. All points Q in E which correspond to the particles of the chain C at the time intervals between the corresponding collisions, form a certain curve C which will be called the cascade development curve. The points in E corresponding to collisions divide this line into a finite number of arcs B, the velocity $\mathbf{v}$ varying continuously and j remaining constant along each arc. Every arc lies completely in one of the spaces E_j. By virtue of (1.1):

$$dt = \frac{d\mathbf{r}}{\mathbf{v}} = \frac{d\mathbf{v}}{\mathbf{F}_j(t, \mathbf{r}, \mathbf{v})};$$

clearly, a single development curve of a completely determined direction passes through every point of the spaces E and E_j Let $\mathbf{N}_j$ be a unit vector in E_j determining this direction. Let $\mathbf{n}_j$ and $\boldsymbol{\nu}_j$ be unit vectors normal to the surfaces $t = \text{const}$ and $\tau = \text{const}$ in E_j. It is evident that

$$\mathbf{N}_j = (1, \mathbf{v}, \mathbf{F}_j) \Big/ \sqrt{1 + v^2 + F_j^2}, \quad \mathbf{n}_j = (1, 0, 0)$$

and that the components of the vector $\boldsymbol{\nu}_j$ are given by the sixth-order determinants obtained from the matrix

$$\left(\frac{\partial t_j}{\partial \sigma}, \frac{\partial \mathbf{r}_j}{\partial \sigma}, \frac{\partial \mathbf{v}_j}{\partial \sigma}\right),$$

(which has 6 rows and 7 columns), divided by the square root Δ_j of the sum of their squares. Taking it into account that the mean length of the arcs B is always finite in E_j and that at least in the vicinity of the points of intersection of the surfaces $t = \text{const}$ and $\tau = \text{const}$ every development line passes through both surfaces, we obtain $V/\mathbf{N}_j\mathbf{n}_j = V/\mathbf{N}_j\boldsymbol{\nu}_j\Delta_j$ or, substituting for $\mathbf{N}_j$, $\mathbf{n}_j$, $\boldsymbol{\nu}_j$ their values given above, we obtain the following expression for the required probability:

$$V^*(P, P^*) = V(P, Q) \begin{vmatrix} 1 & \mathbf{v} & \mathbf{F}_j(t, \mathbf{r}, \mathbf{v}) \\ \frac{\partial t_j}{\partial \sigma} & \frac{\partial \mathbf{r}_j}{\partial \sigma} & \frac{\partial \mathbf{v}_j}{\partial \sigma} \end{vmatrix}, \tag{2}$$

where all the variables t, $\mathbf{r}$, $\mathbf{v}$ on the right-hand side of the equation should be expressed by τ and σ according to (1) (i and j do not change). If, for example, we want to find the probability that one particle of the type A_j traverses the area between x, y, and $x + dx, y + dy$ on the surface $z = \text{const}$ with a velocity between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$ within the time interval between t and $t + dt$ then, according to Eq. (2) we have:

$$V^*_{ij}(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v})\,dt\,dx\,dy\,d\mathbf{v} = V_{ij}(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v})\,v_z\,dt\,dx\,dy\,d\mathbf{v}. \tag{3}$$

Since the derivatives of the functions (1) with respect to τ do not enter into Eq. (2), it is clear that in order to find the value of the function V^* on a given surface $\tau = \text{const}$ it is necessary to know the functions (1) for that value of τ only.

Let R^μ and R^ν be two adjoining regions in E and let $S^{\mu\nu}$ be their common boundary surface, decomposing into surfaces $S_j^{\mu\nu}$ in the spaces E_j. Let $P^* = (\tau, \rho, \nu, k)$ be a point on the surface $S^{\mu\nu}$, and $F^\mu(P)$ and $G^\nu(P)$ be two functions of P defined in R^μ and R^ν respectively. We shall introduce the following notation:

$$F^\mu G^\nu = \sum_k \int_{S_k^{\mu\nu}} F^\mu(P^*) \begin{vmatrix} 1 & \mathbf{r} & \mathbf{F}_k(\tau, \rho, \mathbf{v}) \\ \frac{\partial \tau}{\partial \sigma_k} & \frac{\partial \rho}{\partial \sigma_k} & \frac{\partial \mathbf{v}}{\partial \sigma_k} \end{vmatrix} G^\nu(P^*)\, d\sigma_k, \tag{4}$$

where the integration is to be carried out over only those parts of the surfaces $S_j^{\mu\nu}$ where the vectors N_j, determining the direction of the development curves in E_j, are oriented from R^μ into R^ν. If the function $F^\mu(P)$ gives the distribution of cascade particles in the region R^μ and $G^\nu(P) = U^\lambda(P, Q)$ then, according to Eq. (2) and to the definition of the function U^λ, it is evident that the product $F^\mu U^\nu$ [in the sense of (4)] gives the distribution of cascade particles in the region R^ν, which are those of the particles from R^μ which passed into R^ν and the development curves of which lie totally in R^ν.

In the case when the surface $S^{\mu\nu}$ is the plane $z = z_0$, we can put simply $\sigma = (t, x, y, v)$, and from Eq. (4) we obtain

$$F^\mu U_j^\nu(t, \mathbf{r}, \mathbf{v}) = \sum_k \int_{-\infty}^{t} \iiiint\int_0^\infty F_k^\mu(\tau, \rho, \mathbf{v})\, v_z\, U_{kj}^\nu(\tau, \rho, \mathbf{v}, t, \mathbf{r}, v)\, d\tau\, d\xi\, d\eta\, dv_x\, dv_y\, dv_z \quad \text{for } \rho = (\xi, \eta, z_0). \tag{5}$$

We shall now pass to the solution of the main problem. Let a cascade originate in the point P of the region R^κ. We shall say that a cascade particle, present at the time t and represented in E by a point Q, is of the m-th order when its development curve, going from P to Q, passes through m regions R^λ. If the curve traverses a certain region several times, the region contributes a corresponding number of times to the order of the particle. Let $W_m^\lambda(P, Q)$ denote the required function $W^\lambda(P, Q)$ under the complementary condition that only particles of the m-th order are considered. We have then

$$W^\lambda(P, Q) = \sum_{m=1}^{\infty} W_m^\lambda(P, Q). \tag{6}$$

But it is evident that

$$W_1^\lambda = \begin{cases} U^\kappa(P, Q) & \text{for } \lambda = \kappa, \\ 0 & \text{for } \lambda \neq \kappa, \end{cases} \tag{7}$$

and taking (4) into account, we obtain

$$W_{m+1}^\lambda(P, Q) = \sum_\mu W_m^\mu(P, P^*)\, U^\lambda(P^*, P) \quad (m = 1, 2, \ldots). \tag{8}$$

Equations (6), (7), and (8) make it possible to calculate W^λ when U^λ is known. The functions W^λ are therefore obtained in the form of an infinite series of products of the type (4), i.e., integrals, the integrands of which are certain products of the functions U^λ. It should be noted that the sum (8) is to be extended only over those values of μ to which correspond regions R^μ adjoining to R^λ.

If there are, for example, only two regions R^0 and R' in the whole space and if the point P lies within R^0, we have

$$W_1^0 = U^0(P, Q),\ W_1' = 0, \quad W_{m+1}^0 = \begin{cases} 0 \\ W'_m U^0 \end{cases},\ W'_{m+1} = \begin{cases} W_m^0 U' & \text{for odd } m \\ 0 & \text{for even } m \end{cases}$$

and, consequently, we obtain the following infinite series for W^0 and W':

$$W^0 = U^0 + U^0 U' U^0 + U^0 U' U^0 U' U^0 + \ldots, \quad W' = U^0 U' + U^0 U' U^0 U' + U^0 U' U^0 U' U^0 U' + \ldots$$

These yield immediately the integral equations for W^0 and W':

$$W^0 = U^0 + U^0 U' W^0,\ W' = U^0 U' + U^0 U' W'. \tag{9}$$

In order to find the functions U^λ we shall denote by $\overline{R}^\lambda$ the region containing the whole space E outside R^λ. We shall assume that this region is filled with the same medium that fills R^λ and we shall denote by $\overline{U}^{\lambda}$ and $\overline{W}^{\lambda}$ the functions U and W corresponding to that space for the given medium. In that case, however, the whole space is filled by a single homogeneous medium and the functions W^{λ} and $\overline{W}^{\lambda}$ coincide with the functions V^{λ} and $\overline{V}^{\lambda}$ representing V^{λ} in R^{λ} and $\overline{R}^{\lambda}$. In this way we obtain for $R^0 \equiv R^{\lambda}$ and $R' \equiv \overline{R}^{\lambda}$ from Eq. (9)

$$V^{\lambda} = U^{\lambda} + U^{\lambda}\overline{U}^{\lambda}V^{\lambda}, \quad \overline{V}^{\lambda} = U^{\lambda}\overline{U}^{\lambda} + U^{\lambda}\overline{U}^{\lambda}\overline{V}^{\lambda}.$$

Putting $U^{\lambda}\overline{U}^{\lambda} = \overline{F}^{\lambda}$ we find

$$U^{\lambda} = V^{\lambda} - \overline{F}^{\lambda}V^{\lambda}, \quad F^{\lambda} = \overline{V}^{\lambda} - \overline{F}^{\lambda}\overline{V}^{\lambda}.$$

But from the second equation we obtain directly by integration

$$F^{\lambda} = \overline{V}^{\lambda} - \overline{V}^{\lambda 2} + \overline{V}^{\lambda 3} + \ldots$$

and substituting into the former equation, we obtain finally

$$U^{\lambda} = V^{\lambda} - \overline{V}^{\lambda}V^{\lambda} + \overline{V}^{\lambda 2}V^{\lambda} - \overline{V}^{\lambda 3}V^{\lambda} + \ldots \tag{10}$$

The functions U^{λ}, therefore, have been found.

If we substitute (10) into (7) and (8), and the result into (6), we obtain the functions W in the form of a convergent infinite series, the terms of which are products of the various powers of the known functions V^{λ} and $\overline{V}^{\lambda}$. All the products and powers should of course be understood in the sense of definition (4).

If we consider, for example, the particles of a cosmic ray shower, penetrating a number of plates R^{λ} none of the particles being scattered backwards, then Eqs. (8) and (10) are reduced to:

$$U^{\lambda} = \begin{cases} V^{\lambda} & \text{for } \lambda = \varkappa \\ 0 & \text{for } \lambda \neq \varkappa \end{cases}, \quad W^{\lambda+1}_{m+1} = W^{\lambda}_{m} U^{\lambda+1}.$$

Assuming $\kappa = 0$ and taking into account Eqs. (6) and (7), we obtain the simple relation

$$W^{\lambda} = V^0 V^1 V^2 \ldots V^{\lambda}.$$

We have assumed that the layers are homogeneous because in that case the functions V can be found. These have been assumed to be known. The above considerations are, however, valid when the layers are not homogeneous.

Bearing in mind first, that integration in Eq. (4) is carried out only over those values of σ for which the development curves are directed from R^{μ} into R^{ν} and, second, that a sharp return of the development curves is unlikely in practice and, third, that the cascade particles are often rapidly absorbed, it can be shown that, in general, the found infinite series converge well.

It should be noted that the products (4) are difficult to calculate since the integrands are complicated functions and the integrals sixtuple. In a number of important cases they can, nevertheless, be markedly simplified, namely:

1. If the medium R^{λ} is a layer of a thickness on the order of the particle mean free path then, instead of U^{λ} we can simply take the expression (1.23) for $m = 1$ and $V^0 = A$. Calculation of the product (4) is then reduced to a double integration (owing to the presence of δ-electrons).

2. If the layers are thick, then it is possible to use expressions of the type (1.59) instead of V^{μ} and V^{ν} in calculating $V^{\mu}V^{\nu}$. Taking it into account that a quadratic function of **r** stands in the exponent of the latter expression it becomes immediately clear that if the surface $S^{\mu\nu}_{j}$ is cylindrical, one integration is reduced to the Poisson integral, and if the surface is a plane two integrations are reduced to that type.

3. As a further approximation in the calculation of the products $V^{\mu}V$ where V^{μ} and V^{ν} are given by (1.59) we can substitute for ϵ, ϵ', ϵ'' and δ', δ''. . . in V^{ν} their values averaged in some sense over **u**. We can, for instance, take the average values over the surface $S^{\mu\nu}_{j}$, calculating by means of V^{μ} the probability of various values of **u** on this surface. The both factors in V^{μ} and V^{ν} will then become exponential functions of quadratic functions of **v**, so that if the surface $S^{\mu\nu}_{j}$ is independent of **u**, as it is practically always the case, then the integration over **u** is reduced to the Poisson integral or to the Gauss function, and the product will be represented by a double integral.

Uniting cases (2) and (3) we obtain that in a multi-layer medium with plane boundaries the products $V^{\mu}V^{\nu}$ are expressed by single integrals which are easy to calculate with the help of computing machines.

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CORRELATIONS IN THE DISTRIBUTION OF CASCADE PARTICLES

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A cascade is considered to consist of several types of particles moving in a generally inhomogeneous medium varying with time. The particles collide with the particles of the medium and in this process are absorbed, scattered, and produce new cascade particles. The functions, determining the distribution of the particles of each type in cascades initiated by a single particle of a given type appearing at a given time with known initial position and velocity, are assumed to be known. By means of these functions, the probability of the joint presence of a given number of cascade particles at a given instant in a given cell of the particle type-position-velocity space is found. The detecting probability is calculated for detectors having a sensitivity dependent upon the type and velocity of particles as well as upon their place and time of incidence.

LET A_j $(j = 1, 2, \ldots n)$ represent n various types of particles forming a cascade initiated by a single particle of a given type A_i with a given velocity $\mathbf{u}$, which has appeared at the instant s at a given point $\mathbf{q}$. Let

$$V(P, Q)\,dQ \quad (P = (s, \mathbf{q}, \mathbf{u}, i),\ Q = (t, \mathbf{r}, \mathbf{v}, j),\ dQ = d\mathbf{r}d\mathbf{v}) \tag{1}$$

be the probability that at the time t one particle of this cascade, of the type A_j has the radius vector between $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ and velocity between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$. This probability was found for the case of a homogeneous[1] and of a multi-layer[2] medium. In the following we shall assume that the functions V are known for any inhomogeneous and time-varying medium without referring to their actual expressions. Using the notation and results of Refs. 1 and 2 we shall solve several generalized problems concerning the correlations in particle distribution.

1. Let L be a natural number. We shall find the probability

$$V(P, Q_1, Q_2, \ldots Q_L)\,dQ_1 dQ_2 \ldots dQ_L \text{ or, in short } V(P, Q_l)\,dQ_l \tag{2}$$

that there are K_ℓ particles of the type $A_{j\ell}$ and with velocity between $\mathbf{v}_\ell$ and $\mathbf{v}_\ell + d\mathbf{v}_\ell$ having at the instants t_ℓ radius vector between $\mathbf{r}_\ell$ and $\mathbf{r}_\ell + d\mathbf{r}_\ell$ respectively.