then, for $x' = 10$, even for $Z_1 = Z_2 = 100$, $R = 0, 64 \times 10^{-8}$ cm. However, at distances $R > 10^{-8}$ cm, the calculation of the interaction potential of atoms on the basis of a statistical model loses its meaning.

Thus, in the limits of accuracy of the Thomas-Fermi statistical model of the atom, the interaction between atoms at distances between atoms less than 10^{-8} cm can be described by the potential

$$U(R) = \frac{Z_1 Z_2 e^2}{R} \cdot \chi\left([\sqrt{Z_1} + \sqrt{Z_2}]^{2/3} \frac{R}{a}\right), \tag{10}$$

where $\chi(x)$ is the Thomas-Fermi screening function.

This fact, that the screening function can be expressed approximately as a function a single argument, allows us to compute (within a suitable interval of energy of relative motion and for suitable scattering angles) the effective differential scattering cross section at once for an arbitrary pair of colliding atoms.

In conclusion, I want to thank Academician M. A. Leontovich, Prof. A. B. Migdal and V. Galitskii for useful discussions of the research. I am very grateful to G. I. Biriuk for the computation of the integrals of Eq. (4).

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CYLINDRICAL SELF-SIMILAR ACOUSTICAL WAVES

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A one-parameter family of self-similar solutions for cylindrical motion is constructed in the acoustical approximation. This construction is accomplished by superposition of plane waves and is expressed in elementary form by quadratures. For motion with a finite pressure discontinuity on the wave front of a converging cylindrical wave, the results agree with those obtained previously.[1] It is found again that the pressure in the reflected wave is infinite. The maximum pressure is estimated and allowances are made for the deviations from the acoustical approximation for large amplitudes.

ZABABAKHIN and Nechaev[1] have treated the propagation of a weak cylindrical shock wave and its reflection from the axis in the acoustical approximation.* Their solution for the reflected wave has an unexpected property: the pressure on the front diverges logarithmically, and is the same before and behind the front, that is

$$p \sim \frac{a}{\sqrt{r_f}} \ln \left| \frac{b r_f}{r - r_f} \right| \quad \text{for} \quad |r - r_f| \ll r_f$$

(where p is the pressure change, and the solution is valid only for $p \ll p_0$).

No such singularity occurs when a spherical acoustical wave converges onto a center and is reflected, or for strong cylindrical and spherical shock waves.[2] It is therefore desirable to obtain the result of Zababakhin and Nechaev differently, by a method in which the necessity for their solution would become clearer.

*I take this opportunity to express my gratitude to the authors, who communicated their work to me before its publication.

It was found possible to obtain a whole family of self-similar cylindrical solutions of which, however, the most interesting is just that found by Zababakhin and Nechaev.

1. GENERAL SOLUTION OF THE CYLINDRICAL PROBLEM

The known solution of the acoustical problem for a plane wave propagating along the x axis,

$$p_{\rm pl} = f(x - ct), \quad u_x = (1/\rho c) f(x - ct), \quad u_y = u_z = 0, \tag{1}$$

shall be written in cylindrical coordinates with the polar axis directed at an angle ϕ to the x axis. On the polar axis $x = r \cos \phi$ so that on this line the pressure and velocities are given by

$$p_{\rm pl} = f(r\cos\varphi - ct), \quad u_r = (1/\rho c)\cos\varphi \cdot f(r\cos\varphi - ct) \quad u_\varphi = (1/\rho c)\sin\varphi \cdot f(r\cos\varphi - ct), \quad u_z = 0. \tag{2}$$

By superposing such plane waves with all possible values of ϕ we obtain the cylindrically symmetric solution depending on one arbitrary function*

$$p = p(r, t) = \frac{1}{2}\int_0 f(r\cos\varphi - ct)\, d\varphi; \tag{3}$$

$$u_r = u_r(r, t) = \frac{1}{2\rho c}\int_0^{2\pi} \cos\varphi \cdot f(r\cos\varphi - ct)\, d\varphi, \quad u_\varphi = u_z = 0. \tag{4}$$

It is convenient to change the notation by making the substitution $k = \cos \phi$, so that

$$p = \int_{-1}^{+1} f(rk - ct)\, dk / \sqrt{1 - k^2}; \tag{5}$$

$$u_r = \frac{1}{\rho c}\int_{-1}^{+1} f(rk - ct)\, k\, dk / \sqrt{1 - k^2}. \tag{6}$$

It is easy to see by direct substitution that the function $p(r, t)$ so defined satisfies the wave equation

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial p}{\partial r}, \tag{7}$$

2. SELF-SIMILAR SOLUTIONS

We shall choose the function $f(z)$, where

$$z = r\cos\varphi - ct = rk - ct,$$

in such a way that the constants entering into the definition of $f(z)$ cannot be combined to form a quantity with the dimensions of a length. Then the only quantity with these dimensions which enters into the expression for the pressure will be, with the exception of the independent variable r, the product ct. Then the solution will be self similar, i.e., of the form $t^n \psi(r/ct)$.

Specifically, let us choose $f(z)$ such that

$$f(z) = 0 \quad \text{for} \quad z > 0, \quad f(z) = a(-z)^n \quad \text{for} \quad z < 0. \tag{8}$$

On inserting (8) into (5), the limits of integration depend on the ratio between r and ct, and are the k interval either from -1 to $+1$ or that given by the vanishing of $f(z)$ for $z = rk - ct = 0$.

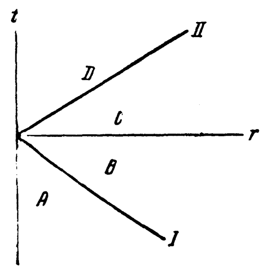

FIG. 1

It is easily seen that $t = 0$ is the time at which the wave is focused on the axis, so that t is the time calculated from the instant of focusing. The conventional rt diagram is shown in Fig. 1. Line I (or $r = -ct$) is the incident wave, and line II (or $r = +ct$) is the reflected wave. Region A (where $ct < 0$ and $r < -ct$) contains the unperturbed gas before the incident wave front. Region B (where $ct < 0$ and $r > -ct$) contains the gas behind the wave front. Region C (where $ct > 0$ and $r > ct$) contains the gas before the reflected

* The factor ½ in Eqs. (3) and (4) is introduced for convenience in writing (5) and (6) and those that follow.

wave front. Region D (where $ct > 0$ and $r < ct$) contains the gas behind the reflected wave front.

In region A, we have $z > 0$ and $f = 0$ throughout the interval $-1 < k < +1$, so that $p \equiv 0$ as is to be expected.

In region B we have $-1 < ct/r < 0$, so that

$$p = a(-ct)^n \int_{-1}^{ct/r} \left(\frac{r}{ct}k - 1\right)^n dk \big/ \sqrt{1-k^2}. \tag{9}$$

In region C we have $0 < ct/r < 1$, so that

$$p = a(ct)^n \int_{-1}^{ct/r} \left(1 - \frac{r}{ct}k\right)^n dk \big/ \sqrt{1-k^2}. \tag{10}$$

In region D we have $ct/r > 1$, so that

$$p = a(ct)^n \int_{-1}^{+1} \left(1 - \frac{r}{ct}k\right)^n dk \big/ \sqrt{1-k^2}. \tag{11}$$

As follows from dimensionality considerations, in all regions the solution is of the form

$$p = a(|ct|)^n \psi(r/|ct|) = a r_f^n \psi(S), \quad r_f = |ct|, \quad S = r/r_f, \tag{12}$$

which means that it is self similar for arbitrary values of n. We have used simple quadratures and have not integrated the differential equation, but have been able to obtain a whole family of self similar solutions depending on an exponent n each of which build up according to different laws and have different pressure profiles determined by the dimensionless function ψ.

It should be borne in mind that the function in Eq. (12) is different in the different regions A, B, C, and D, so that it depends not only on S but also on the sign of the time. This, of course, does not destroy the self modeling property of the solution.

3. CHOICE OF THE EXPONENT FROM INITIAL CONDITIONS

Let us find the pressure profile in the incident wave. According to Eq. (9), in region B we have

$$\psi_B(S) = \int_{-1}^{-1/S} (-kS - 1)^n dk \big/ \sqrt{1-k^2}, \quad S > 1. \tag{13}$$

If we let the variable of integration be $y = (-kS - 1)/(S - 1)$, we obtain

$$\psi_B(S) = (S-1)^{n+1/2}(S+1)^{-1/2} \int_0^1 y^n dy \Big/ \sqrt{(1-y)\left(1 + \frac{S-1}{S+1}y\right)}. \tag{14}$$

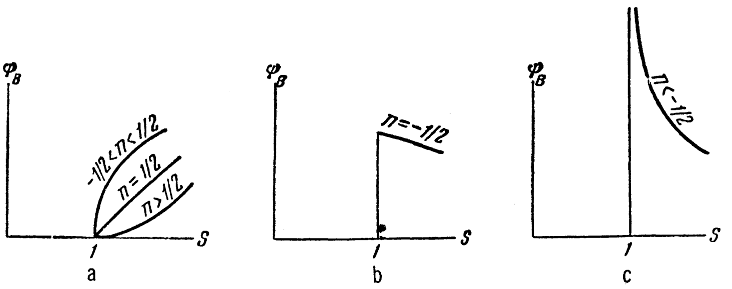

FIG. 2

In the neighborhood of the wave front with $S - 1 \ll 1$, the integral approaches a limit given by

$$\int_0^1 y^n (1-y)^{-1/2} dy,$$

so that the pressure profile is given by the factor $(S-1)^{n+1/2}$. When $n > -1/2$ we are dealing with a gradually increasing pressure (Fig. 2a), when $n = -1/2$, there is a finite discontinuity (Fig. 2b) and when $n < -1/2$ the pressure on the front is infinite (Fig. 2c).

Thus a shock wave with a steep front and a finite pressure discontinuity is described by the solution with index $n = -1/2$. As has been shown by Zababakhin and Nechaev, the amplitude increases as $r_f^{-1/2}$ as the wave converges (compare with Eq. (12)). One may have expected such a result from the fact that the acoustical energy of a unit volume is proportional to p^2, and the area of a cylindrical wave is proportional to r_f so that when $n = -\frac{1}{2}$ the acoustical energy $p^2 r_f \delta$ is constant in the layer of thickness δ about the wave front.

We note that these concepts are applicable* also when there is no finite discontinuity, or when $n \neq -\frac{1}{2}$: the pressure is proportional to r_f^n at similar distances (for a given S), but at equal distances from the front, that is when $S = 1 + \delta/r_f$ the pressure for any n is proportional to $r_f^{-1/2}$.

Let us return to the case $n = -\frac{1}{2}$, for which

$$\psi = \frac{1}{\sqrt{S+1}} \int_0^1 dy \Big/ \sqrt{y(1-y)\left(1 + \frac{S-1}{S+1} y\right)}. \tag{15}$$

When $S = 1$ we have

$$\psi(1) = \pi/\sqrt{2}, \quad p_f = a\pi / \sqrt{2r_f}. \tag{15a}$$

It is easily seen that behind the wave front the pressure drops with a finite derivative, as shown in Fig. 2b. Far from the front we have, for large S,

$$p \sim a/\sqrt{r_f S} = a/\sqrt{r}.$$

As one may have expected, the pressure far from the front is independent of the location of the front.

4. THE REFLECTED DIVERGING WAVE

By treating a cylindrical wave as a superposition of plane waves we can clarify the question of the uniqueness of the solution after reflection. This follows from the fact that none of the plane waves have singularities on crossing the axis at time $t = 0$.

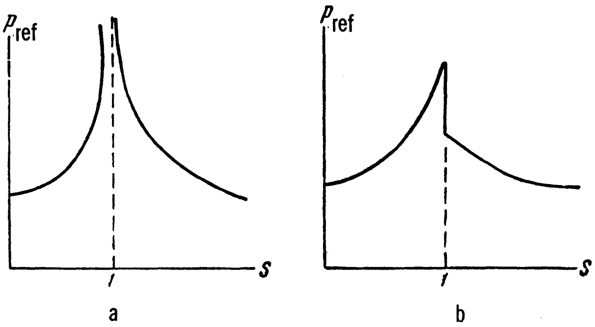

FIG. 3

Let us write the expressions in regions C and D for $n = -\frac{1}{2}$, namely

$$p = a r_f^{-1/2} \psi_C(S), \quad \psi_C = \int_{-1}^{1/S} dk / \sqrt{(1-k^2)(1-Sk)}, \quad S > 1; \tag{16}$$

$$p = a r_f^{-1/2} \psi_D(S), \quad \psi_D = \int_{-1}^{+1} dk / \sqrt{(1-k^2)(1-Sk)}, \quad S < 1. \tag{17}$$

When $S = 1$ both integrals approach

$$\int_{-1}^{+1} dk / (1-k)\sqrt{1+k};$$

and diverge logarithmically at the upper limit.

It is easily shown that when $|S - 1| = \epsilon \ll 1$ both expressions (16) and (17) give (up to terms which vanish as ϵ approaches zero and may be different for ψ_C and ψ_D)†

$$\psi_C = \psi_D = \frac{1}{\sqrt{2}} \ln \frac{32}{\epsilon}. \tag{18}$$

*The elementary concepts of geometric acoustics are applicable for the following reason. Close to a wave front on which the pressure or its derivatives undergo a discontinuity, harmonic analysis gives the pressure increase in terms of the asymptotic values of the short waves for which $\lambda \to 0$. Thus the condition $\lambda/r_f \ll 1$ for the applicability of geometric acoustics is fulfilled.

†To show this, let us set $S = (1-\epsilon)^{-1}$ in (16) and $S = (1+\epsilon)^{-n}$ in (17) and break up each integral in (16) and (17) into two parts J_1 from -1 to $1-a$ and J_2 from $1-a$ to the upper limit, where a is chosen such that $1 \gg a \gg \epsilon$. In J_1 we set $\epsilon = 0$ with an error of order ϵ/a, in which case the integral is elementary and equal to $2^{-1/2}\ln(8/a)$. In J_2 we may replace $\sqrt{1+k}$ by $\sqrt{2}$ with an error of the order of a. Then J_2 is elementary and given by (up to small terms of the order of ϵ and $\epsilon \ln \epsilon$) $J_2 = 2^{-1/2}\cosh^{-1}(2a/\epsilon)$ both for (16) and (17).

When $a \gg \epsilon$

$$J_2 = 2^{-1/2} \ln(4a/\varepsilon), \quad \psi = J_1 + J_2 = 2^{-1/2} \ln(32/\varepsilon),$$

so that a drops out of the expression for ψ.

This verifies the logarithmic divergence of the pressure in the reflected wave as found previously by Zababakhin and Nechaev.

If on the incident wave the pressure is p_{inc} on the wave front at a given distance r_f from the axis, then on the reflected wave

$$p_{ref} = \frac{p_{inc}}{\pi} \ln \frac{32 r_f}{|r - r_f|} \tag{19}$$

becomes infinite symmetrically before and behind the front $r_f = ct$ (see Fig. 3a).

This result, which has been obtained in the acoustical approximation, is known to be invalid if the pressure according to (19) is of the order of p_0, the absolute pressure of the gas in which the wave is propagating. A better evaluation can be obtained by finding the radius at which the incident plane wave can no longer be described by acoustical laws, that is

$$r_0 \sim a^2 / p_0^2, \tag{20}$$

where a is the constant in Eq. (15a).

In the region of applicability of the acoustical approximation, however, that is if $r_f \gg r_0$, the acoustical expression (19) is applicable only when $|r - r_f| > r_0$. From this some simple operations give the maximum pressure in the reflected wave of the order of

$$p_{ref}^{max} = (2 p_{inc}/\pi) \ln (p_0 / p_{inc}). \tag{21}$$

Here as in (19) p_{ref} and p_{inc} refer to the same distance from the axis.

Accounting for the deviation from the acoustical approximation also destroys the equality of the pressure at equal distances before and behind the wave front. Figure 3a gives a schematic diagram of the pressure profile when nonlinearity is not accounted for, and Fig. 3b gives the pressure profile with the deviations from the acoustical approximation.

When the reflected wave is sufficiently far from the center, $r_f \gg r_0$ and the pressure at the center can be calculated by the acoustical theory. It is clear that a perturbation in the entropy will not interfere with a smoothing out of the pressure over a region of the order of r_f. One then obtains

$$p(0, t) = a\pi / \sqrt{ct} = \sqrt{2}\, p_{inc}(-t). \tag{22}$$

We note finally that a similar method for constructing self similar solutions can be used for the spherical problem. In this case, however, when the pressure undergoes a finite discontinuity in the incident wave there will be only a finite discontinuity in the wave reflected from the center.

The difference between the cylindrical and spherical cases in the acoustical approximation is clearly related to the difference in wave propagation in spaces with even and odd numbers of dimensions, as has been mentioned by Courant.[3]

I take this opportunity to express my gratitude to E. I. Zababakhin and V. A. Aleksandrov.

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BOUND STATES IN POSITRONIUM

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It is shown that the bound states of an electron and positron may be studied by examining the poles of the photon propagator.

We consider the bound states of an electron and a positron. Starting from the Bethe-Salpeter equation in the ladder approximation, and taking Fourier transforms of the wave-function $\psi_{\alpha\beta}(1, 2)$ and of the functions S_F, D_F, we obtain the momentum-space equation

$$A_{n_1n_2}(\mathbf{p}, p_0) = -\frac{ie^2}{(2\pi)^4}\int\sum_{n_1'n_2'}\left\{\frac{[\bar{u}^{n_1'}(\mathbf{p}')\gamma_\mu u^{n_1}(\mathbf{p})][\bar{u}^{n_2}(\mathbf{p})\gamma_\mu u^{n_2'}(\mathbf{p}')]D_F(p-p')}{(\delta_{n_1}E_p+W/2-p_0)(\delta_{n_2}E_p-W/2-p_0)}\right.$$
$$\left.-\frac{[\bar{u}^{n_2}(\mathbf{p})\gamma_\mu u^{n_1}(\mathbf{p})][\bar{u}^{n_1'}(\mathbf{p}')\gamma_\mu u^{n_2'}(\mathbf{p}')]D_F(W)]}{(\delta_{n_1}E_p+W/2-p_0)(\delta_{n_2}E_p-W/2-p_0)}\right\}A_{n_1'n_2'}(\mathbf{p}', p_0'). \quad (1)$$

Here $n = 1, 2$ denote solutions of the Dirac equation with positive energy, $n = 3, 4$ those with negative energy, and $\delta_1 = \delta_2 = 1$, $\delta_3 = \delta_4 = -1$. The second term in Eq. (1) is the exchange term. We define the three-dimensional amplitude by

$$a_{n_1n_2}(\mathbf{p}) = \int A_{n_1n_2}(p)\,dp_0.$$

We obtain the adiabatic approximation[1] from Eq. (1) if we replace D_F by $\int D_F(p)\,\delta(p_0)\,dp_0$. The first non-adiabatic approximation is obtained by substituting the adiabatic expression for $A_{n_1n_2}(p)$ on the right of Eq. (1) and integrating with respect to p_0 and p_0'. Neglecting the minus particles, we have the old Tamm-Dancoff equation for positronium. We are interested in the contribution of the exchange term. After dropping tensor terms and integrating over angles, we obtain the equations for the triplet s-state $(\sigma_1\sigma_2 = 1)$,

$$a^\varepsilon(p) = \lambda\int\left\{\frac{K_1(p, p', \varepsilon W)\,a^\varepsilon(p')}{2E_p-\varepsilon W} + \frac{K_2(p, p', \varepsilon W)\,a^{-\varepsilon}(p')}{2E_p-\varepsilon W} - \frac{4\,[a^\varepsilon(p')+a^{-\varepsilon}(p')]}{W^2(2E_p-\varepsilon W)}\right\}p'^2\,dp', \quad (2)$$

Here $\epsilon = 1$ for the plus-component (a^{++}), and $\epsilon = -1$ for the minus-component (a^{--}); $\lambda = e^2/4\pi^2 = 1/137\pi$.

In the adiabatic approximation

$$K_1(p, p', \varepsilon W) = K_2 = \frac{1}{pp'}\ln\frac{p+p'}{|p-p'|}.$$

and in the first non-adiabatic approximation

$$K_1 = \frac{1}{pp'}\ln\frac{p+p'+E_p+E_{p'}-\varepsilon W}{|p-p'|+E_p+E_{p'}-\varepsilon W}, \quad K_2 = \frac{1}{pp'}\ln\frac{p+p'+E_p+E_{p'}}{|p-p'|+E_p+E_{p'}}.$$

We introduce the notation

$$\chi = \int 4\,[a^\varepsilon(p') + a^{-\varepsilon}(p')]\frac{p'^2dp'}{W^2}.$$

and look for a solution of Eq. (2) of the form

$$a^3(p) = \lambda\Gamma^\varepsilon(p, W)\,\chi/(2F_0-\varepsilon W) + g^\varepsilon(p).$$

Substituting this ansatz into Eq. (2), we find that Γ^ϵ satisfies