ELECTROMAGNETIC-FIELD SHOCK WAVES AND THEIR CUMULATION

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Submitted to JETP editor February 11, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) 33, 442-450 (August, 1957)

We consider electromagnetic waves with narrow transition regions between the initial and final states, and in particular, a converging cylindrical wave. It is found that as such a wave converges its amplitude increases without bound. A qualitatively new phenomenon, cumulation, is found. This is the occurrence of infinitely strong fields, at finite distances from the axis, on the front of the wave reflected from the cylindrical axis. This property is not peculiar to electromagnetic phenomena, but is related to the cylindrical geometry. Acoustical waves have the same property, but for them this solution is valid only for weak waves, whereas this restriction does not apply to electromagnetic-field waves.

1. FORMATION OF WAVES AND THICKNESS OF THE WAVE FRONT

CONSIDER semi-infinite space filled with a perfect conductor and bounded (along the $x = 0$ plane) by a vacuum with a stationary magnetic field H_{0y}. Let a plane shock wave be emitted by this conductor, with its whole surface instantaneously attaining the velocity u toward the field; then an electromagnetic shock wave with velocity c propagates into the field.

Since the field flux between the surface of the ideal conductor and a line at $x = \infty$ is constant, the magnetic field satisfies an equation similar to the equation of conservation of matter:

$$H_0 c = H_1 (c - u).$$

Here $H_1 = H_{0y}$, the field behind the wave. Thus $H_1 = H_0 c/(c - u)$. When H changes on the wave, so does E, and $\Delta E_z = -\Delta H_y$ (where Δ denotes the change at the wave front), so that behind the front $E = -H_0 u/(c - u)$.

We note that in the coordinate system moving with the moving wall, there is no electric field. Indeed, on going from one coordinate system to another, the Lorentz force is conserved, that is

$$\mathbf{E} + [\mathbf{u}\times\mathbf{H}]/c = \mathbf{E}' + [\mathbf{u}'\times\mathbf{H}']/c.$$

In the coordinate system moving with the moving wall (denoted by a prime) $u' = 0$, so that

$$\mathbf{E}' = \mathbf{E} + \frac{1}{c}[\mathbf{u}\times\mathbf{H}] = -\mathbf{k}\,\frac{H_0 u}{c-u} + \frac{\mathbf{k}}{c}\,u H_1 = 0.$$

Let us further calculate the pressure of the field on the moving wall. The pressure due to the magnetic field H_0 on the wall is $p_0 = H_0^2/8\pi$, so that the pressure on the moving wall is $p_1 = H'^2/8\pi$. On going over from one system of coordinates to another, the quantity $H^2 - E^2$ is conserved, so that $H'^2 - E'^2 = H_1^2 - E_1^2$, and since $E' = 0$, we have

$$p_1 = (H_1^2 - E_1^2)/8\pi.$$

Let us evaluate the thickness of the electromagnetic-field wave front due to a shock wave from a perfect conductor. This thickness does not vanish, since the front of the shock wave moving with a velocity $D \approx 10^6$ cm/sec is spread over a thickness of the order of the lattice constant $a \approx 10^{-8}$ cm of the conductor, or over a time of the order of $\tau = a/D \approx 10^{-14}$ sec. This causes waves spread out in the vacuum over a thickness of the order of $\ell \approx \tau c \approx 3 \times 10^{-4}$ cm, which is extremely small. In actuality the spread is determined by the finite electrical resistance of real conductors. This may be of several orders greater than the above; let us consider it in more detail.

In order to evaluate the thickness of the wave front in the vacuum, let us first consider the wave in the conductor, and then knowing the time characteristic of its spread we can find the thickness of the wave in the vacuum.

Consider a shock wave moving through a real conductor containing a field H_0, this shock wave increasing the density of the material by a factor δ. Let us assume that behind the wave the matter everywhere has the same velocity and density. Let us consider the electromagnetic field accompanying this wave.

In the coordinate system in which **E** is along the z axis, **H** is along the y axis, and **j** is along the z axis (where **j** is the current density), Maxwell's equations with $\epsilon = \mu = 1$ can be written

$$\frac{1}{c}\frac{dE}{dt} = \frac{dH}{dx} - \frac{4\pi}{c}j, \quad \frac{1}{c}\frac{dH}{dt} = \frac{dE}{dx}, \quad j = \lambda\left(E + \frac{uH}{c}\right). \tag{1.1}$$

Here λ is the conductivity, and u is the mass velocity of the substance (in front of the shock wave $u = 0$).

For a steady wave propagating with a velocity D, the quantities H, E, and j depend only on $q = x - Dt$. Then Eq. (1.1) leads to

$$-\frac{D}{c}\frac{dE}{dq} = \frac{dH}{dq} - \frac{4\pi\lambda}{c}\left(E + \frac{uH}{c}\right), \quad -\frac{D}{c}\frac{dH}{dq} = \frac{dE}{dq}. \tag{1.2}$$

From this we obtain

$$d^2E/dq^2 = (-dE/dq)\,4\pi\lambda(D-u)/(c^2 - D^2).$$

Let us write the solution of this equation in the form

$$E = \begin{cases} A_1 \exp\left\{-\frac{4\pi\lambda D}{c^2 - D^2}q\right\} + B_1 & \text{for } q > 0, \\ A_2 \exp\left\{-\frac{4\pi\lambda(D-u)}{c^2 - D^2}q\right\} + B_2 & \text{for } q < 0. \end{cases} \tag{1.3}$$

The second of Eqs. (1.2) gives

$$H = \begin{cases} H_1 + \frac{c}{D}A_1\left(1 - \exp\left\{-\frac{4\pi\lambda D}{c^2 - D^2}q\right\}\right) & \text{for } q > 0, \\ H_1 + \frac{c}{D}A_2\left(1 - \exp\left\{-\frac{4\pi\lambda(D-u)}{c^2 - D^2}q\right\}\right) & \text{for } q < 0. \end{cases} \tag{1.4}$$

Here H_1 is the field on the shock front. It is obvious that H cannot change discontinuously on the front, since this would mean an infinite current density, which cannot occur if λ is finite.

The solution given by (1.3) and (1.4) contains five constants which are determined by the following conditions: in the unperturbed region $H(q = +\infty) = H_0$, and $E(q = +\infty) = 0$; far behind the front $E(q = -\infty) \neq \infty$; on the shock wave E is continuous, or $E(+0) = E(-0)$.

The last condition can be obtained by considering a rectangular circuit of dimension a by b, whose sides of length a are parallel to the wave front. For such a circuit

$$\int_{-\infty}^{+\infty}\left(\oint \mathbf{j}\,d\mathbf{l}\right)dt = \text{const}\int_{-\infty}^{\infty}\frac{d\Phi}{dt}dt = \text{const}\,\Delta\Phi,$$

where Φ is the magnetic flux. But $\oint \mathbf{j}d\mathbf{l} = j_1 a - j_2 a$ (where j_1 is the current in the left member, and j_2 is that in the right member of length a of the circuit, and there is no current in the members of length b). But in view of the fact that the process is stationary,

$$\int_{-\infty}^{+\infty} j_1 dt = \int_{-\infty}^{+\infty} j_2 dt, \text{ i.e. } \Delta\Phi = 0, \quad H_0 ab = a\frac{b}{\delta}H(-\infty)$$

so that finally $H(-\infty) = H_0\delta$; this means that far behind the wave front the magnetic field strength has increased by the same factor as the matter density.

Without going through the simple operations, we present only the final result:

$$H = \begin{cases} H_0[1 + (\delta - 1)e^{-q/l}] & \text{for } q > 0, \\ H_0\delta & \text{for } q < 0, \end{cases} \quad E = \begin{cases} -(D/c)H_0(\delta - 1)e^{-q/l} & \text{for } q > 0, \\ -(D/c)H_0(\delta - 1) & \text{for } q < 0, \end{cases} \tag{1.5}$$

$$j = \begin{cases} -(\lambda D/c)\, H_0 (\delta - 1)\, e^{-q/l} & \text{for } q > 0, \\ 0 & \text{for } q < 0, \end{cases} \qquad l = \frac{c^2 - D^2}{4\pi\lambda D}$$

The total current is

$$I = \int_{-\infty}^{+\infty} j\, dq = -H_0 (\delta - 1) \frac{c^2 - D^2}{4\pi c}.$$

As can be seen from (1.5), the perturbation in the magnetic field leads the shock wave by an effective length ℓ or a time $\tau = \ell/D$. On coming to the boundary of the conductor and the vacuum, this wave causes an electromagnetic wave in the vacuum with the same characteristic time of spread, or with a front thickness* $b = \tau c$. For the usual conductors this quantity is much greater than the spread related to that of the shock wave in the matter itself; for instance, in copper (at room temperature $\lambda = 5.8 \times 10^{17}\,\text{sec}^{-1}$) $b = 14.8$ cm (we have taken $D = 5$ km/sec). We note that the thickness ℓ of the wave in the copper itself is 2.5μ.

The wave front can be decreased greatly by cooling the conductor to decrease its resistance. For instance, when copper is cooled to $-253°$C, its resistance is decreased by a factor of 170, and the spread of the wave in the vacuum decreases to 14.8 cm/170 = 0.9 mm.

We note that the total current I accompanying the shock wave in the conductor is independent of the conductivity, and that behind the shock front the current vanishes, so that the conductivity plays no role behind the shock wave. Thus if one is able to achieve a narrow front by increasing the conductivity by cooling, this result will not be affected even if the high conductivity is destroyed by the shock wave itself (for instance, because of heating).

In the following sections we shall consider a converging cylindrical electromagnetic-field wave, assuming it to be a mathematically rigorous discontinuity.

2. CONVERGING CYLINDRICAL FIELD WAVE

In principle a converging wave can be obtained by suddenly turning on a circular current simultaneously over the whole surface of a cylinder, by reflecting a plane or diverging wave from a curved mirror, or by moving a cylindrical conducting surface in a magnetic field towards or away from its axis.

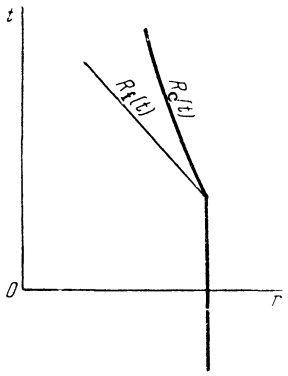

FIG. 1. The formation of a converging cylindrical electromagnetic-field shock wave. The vacuum lies to the left of the heavy line.

In the case of cylindrical symmetry, Maxwell's equations (H is parallel to the x axis, and E is circular, i.e., perpendicular to r and x) become

$$\frac{1}{c}\frac{\partial E}{\partial t} = -\frac{\partial H}{\partial r}, \quad \frac{1}{c}\frac{\partial H}{\partial t} = -\frac{1}{r}\frac{\partial (rE)}{\partial r}. \tag{2.1}$$

Eliminating H or E from these equations, we obtain, respectively,

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \frac{1}{\partial r}\left[\frac{1}{r}\frac{\partial (rE)}{\partial r}\right] = 0, \tag{2.2}$$

$$\frac{1}{c^2}\frac{\partial^2 H}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial H}{\partial r}\right] = 0. \tag{2.3}$$

The equation for H is simply the wave equation for the cylindrical case. That for E is not the wave equation.

Let us consider the variation of the cylindrical wave amplitude as it moves towards the axis, first treating a wave within a cylindrical cavity of radius R_c in an ideal conductor. The motion of the conductor and the wave in this case are shown in Fig. 1.

The total flux of H in the cavity is conserved, so that

$$\Phi = 2\pi \int_0^{R_c} rH\, dr = \text{const}.$$

*Here we have obtained only the width in the vacuum corresponding to the field perturbation in front of the shock wave. The amplitude of this part of the wave and its profile can be found only by the complete solution of the problem of the wave leaving the conductor.

Differentiating this identity twice with respect to time, using (2.3), and $dR_f/dt = -c$, and $dR_c/dt = U$ (where R_f is the radius of the wave front), gives

$$c\left[2R_f\frac{d(H_f-H_0)}{dt}-c(H_f-H_0)\right]+c^2R_c\left(\frac{\partial H}{\partial r}\right)_c+\frac{d(H_cR_cU)}{dt}=0.$$

The second group of terms, which refers to the boundary, vanishes. This is easily seen by considering a field due not to a sharp shock wave, but to one which is somewhat spread out. Then the first group of terms vanishes, and the second, which contains only quantities referring to the boundary, does not change.* Therefore the first group also vanishes, which leads to

$$H_f - H_0 = \text{const}\cdot R_f^{-1/2}. \tag{2.4}$$

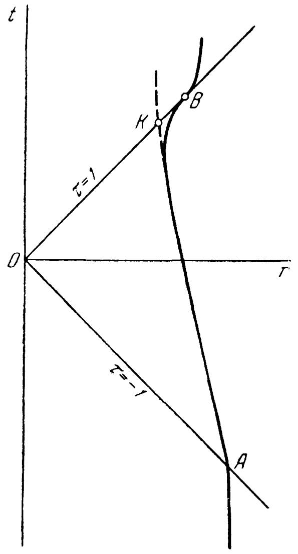

FIG. 2.

Thus the amplitude of the converging cylindrical wave increases without bound as $R_f^{-1/2}$ in its approach toward the axis. We have obtained this result for the special case of a wave within a cavity in an ideal conductor, although it is valid for any cylindrical shock wave, since waves from the surface of the cylinder do not catch up to the shock wave; therefore the behavior of the wave amplitude on the front is determined entirely by its magnitude (that is the initial amplitude at the point at which the wave is formed) and is independent of other field changes on the boundary of the cylinder.

On the cylindrical wave front $E_f - E_0$ also changes according to

$$E_f - E_0 = \text{const } R_f^{-1/2}. \tag{2.5}$$

Equations (2.4) and (2.5) are not approximations valid only for large or small amplitudes; they describe the behavior of a field with the same accuracy as do Maxwell's equations.

For a complete description of H and E behind the shock wave we must solve (2.2) and (2.3).

Henceforth we shall consider only the particular solution corresponding to a converging wave, which describes the limiting behavior of a field close to the axis and close to the time of focusing, which means that we shall solve the self-similar problem.

3. SELF-SIMILAR SOLUTION FOR A CONVERGING CYLINDRICAL WAVE

Since we know how the wave amplitude varies, we may immediately assert that the self-similar solution is of the form

$$E = \mathcal{E}_0\sqrt{\frac{r_0}{r}}\,e(\tau), \qquad H = H_0 + \mathcal{H}_0\sqrt{\frac{r_0}{r}}\,h(\tau), \tag{3.1}$$

where $\tau = ct/r$. Here t is the time calculated from the instant of focusing (before focusing, $t < 0$), and $\mathcal{E}_0$ and $\mathcal{H}_0$ are the amplitudes of the electric and magnetic fields on the wave front at the time when the wave front is at a distance r_0 from the axis. For a wave moving towards the axis, $\mathcal{H}_0 = -\mathcal{E}_0$. On a converging wave front, $\tau = -1$, so that

$$e(-1) = 1, \quad h(-1) = 1. \tag{3.2}$$

A diagram of the phenomenon in the r, t plane is shown in Fig. 2.

Inserting (3.1) into Maxwell's equations (2.2) and (2.3), we obtain equations for the new functions

$$e''(1-\tau^2) - 2\tau e' + 3e/4 = 0, \quad h''(1-\tau^2) - 2\tau h' - h/4 = 0. \tag{3.3}$$

We solve these equations by means of a power series in $(1+\tau)$ with the conditions (3.2) for the interval $-1 \le \tau \le 1$ (between the incident and reflected waves), obtaining

$$e(\tau) = 1 - \sum_{n=1}^{\infty}\frac{(2n+1)!!\,(2n-3)!!}{2^{3n}(n!)^2}(1+\tau)^n, \tag{3.4}$$

*This reasoning is due to B. P. Mordvinov.

$$h(\tau) = \sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{n!}\right]^2 \frac{(1+\tau)^n}{2^{3n}} \tag{3.5}$$

(for $n = 0$ we take $(2n-1)!! = +1$).

These series converge for $\tau < 1$, and diverge for $\tau \geq 1$, so that as a result of reflection from the axis there arises a wave whose amplitude is unbounded not only at the cylindrical axis, but also at a finite distance from it.

Such a cumulation is a qualitatively new phenomenon. In converging waves previously considered, unbounded amplitudes arose only for small times and in small regions of space (at the focus). In our case, however, unbounded amplitudes arise at all points of space, not only at the cylindrical axis.

We note that nowhere in the solution have we assumed that we are dealing with a compression wave, so that all the results obtained refer also to a rarefaction wave, which can be formed, for instance, by the sudden expansion of a cavity in a conductor. In this case the field strength on the wave front $H_f = H_0 - \Delta H$ will decrease as the amplitude ΔH increases, vanishing at some distance r and thereafter approaching $-\infty$.

We wish, further, to find a solution of the equations for e and h in the interval $1 < \tau < \infty$, that is behind the wave reflected from the axis. We write this solution in the form of a power series in $1/\tau$ with the condition $e = h = 0$ when $\tau = \infty$ (that is with the condition $E(r=0) \neq \infty$ and $H(r=0) \neq \infty$), obtaining

$$e = \frac{a_0}{\tau^{3/2}} \sum_{n=0}^{\infty} \frac{(4n+1)!!}{2^{4n} n!\,(n+1)!}\,\frac{1}{\tau^{2n}}, \quad h = \frac{b_0}{\tau^{1/2}}\left[1 + \sum_{n=1}^{\infty} \frac{(4n-1)!!}{2^{4n}(n!)^2}\,\frac{1}{\tau^{2n}}\right]. \tag{3.6}$$

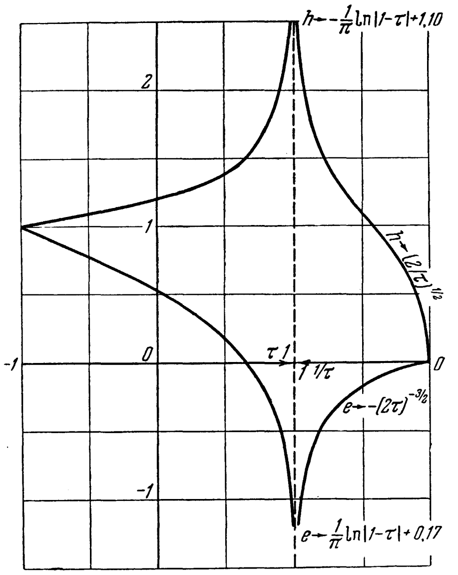

FIG. 3. The self-similar solution for a converging cylindrical electromagnetic-field wave: $H = H_0 - \text{const} \times h/\sqrt{r}$, $E = \text{const} \times e/\sqrt{r}$.

These series converge for $\tau > 1$, but diverge for $\tau \leq 1$, so that close to the reflected wave and behind its front E and H increase without limit.

The as yet unknown quantities a_0 and b_0 can be obtained from the following two conditions: first, from Maxwell's equations, and second, from the condition that $e + h$ are conserved on passing through the wave front reflected from the axis (see below). Let us consider each condition separately.

Inserting E and H in their self-modeling form (3.1) into the second of Maxwell's equations (2.1), we obtain

$$dh/d\tau + \tau\, de/d\tau - e/2 = 0.$$

Inserting the expressions for $de/d\tau$, $dh/d\tau$, and e behind the wave into this expression, we obtain

$$b_0 = -4a_0. \tag{3.7}$$

Let us now consider the second condition. On the front of a diverging wave, $\Delta E = \Delta H$, or $E_2 - H_2 = E_1 - H_1 = \text{const}$. In our case

$$E - H = \mathcal{E}_0 \sqrt{\frac{r_0}{r}}\, e - \mathcal{H}_0 \sqrt{r_0/r}\, h - H_0 = \mathcal{E}_0 \sqrt{r_0/r}\,(e+h) - H_0 = \text{const},$$

so that $e + h = \text{const}$. Using (3.4) and (3.5), we find that in front of the wave

$$e + h = 2\left[1 - \sum_{n=1}^{\infty} \frac{(2n-1)!!\,(2n-3)!!}{2^{2n}(n!)^2}\right]. \tag{3.8}$$

This series converges, so that the invariant $e + h$ has a finite value on the wave front.

Using the solution (3.6) for the region behind the front, as well as relation (3.7), we find that in that region

$$e + h = -3a_0\left[1 + \sum_{n=1}^{\infty} \frac{(4n-1)!!}{2^{4n} n!\,(n+1)!}\right]. \tag{3.9}$$

Comparing (3.8) and (3.9) we obtain a_0 in the form of the ratio of two series; numerical calculation gives $a_0 = 0.3536$. It would seem that the accurate solution is

$$a_0 = -1/2\sqrt{2}; \text{ so that } b_0 = \sqrt{2}.$$

Let us consider in more detail the divergence of e and h close to the reflected wave front. Introducing the new variables $z = \tau - 1$ and $y = e/e'$ into Eq. (3.2) for e, we are led to

$$(1 - y')\, y^{-2} z (2 + z) + 2(1 + z)/y - {}^3/_4 = 0.$$

When $|z| \ll 1$ and $|y| \ll 1$, this equation can be simplified, becoming $dy/dz = 1 + y/z$. We then have

$$y = z(\ln z + \text{const}) = e/e'$$

and finally

$$e = A_0 + A_1 \ln|1 - \tau|. \tag{3.10}$$

This equation describes the asymptotic behavior in the neighborhood of the diverging wave front on both its sides.

Let us determine the coefficient of the logarithm in this expression. We shall choose A_0 and A_1 such that the difference between the exact expression e_{ex} as given by (3.6) and the asymptotic one e_a as given by (3.10) vanish for $\tau = 1$. Let us write e_{ex} and e_a in the form of power series in $(1 + \tau)$:

$$e_{ex} = 1 + \sum_{n=1}^{\infty} a_n (1 + \tau)^n, \quad e_a = A_0 + A_1 \ln 2\left(1 - \frac{1+\tau}{2}\right) = A_0 + A_1 \ln 2 - A_1 \sum_{n=1}^{\infty} \frac{(1+\tau)^n}{2^n n},$$

$$e_{ex} - e_a = 1 - A_0 - A_1 \ln 2 + \sum_{n=1}^{\infty} (a_n + A_1/2^n n)(1 + \tau)^n.$$

The series in the last expression should converge when $\tau = 1$. From this it follows that

$$\lim_{n\to\infty} (1 + a_n n 2^n / A_1) = 0,$$

since otherwise the series diverges as $\Sigma\, 1/n$. Therefore

$$A_1 = -\lim_{n\to\infty} 2^n n a_n = \lim_{n\to\infty} \frac{(2n+1)!!\,(2n-3)!!}{2^{2n} n!\,(n-1)!} = \frac{1}{\pi}$$

(in order to calculate the limit, it is convenient to express the factorials by Stirling's formula).

The coefficient of $\ln|1 - \tau|$ in the asymptotic expression for e behind the front is found in the same way, and is $A_2 = -a_0 2\sqrt{2}/\pi$. With the exact equation $a_0 = -\frac{1}{2}\sqrt{2}$, we have $A_2 = A_1 = 1/\pi$, which means that the coefficient of the logarithm in the asymptotic expressions for e on both sides of the wave front are the same. It is clear that the coefficient in the asymptotic expression for h differs from this only in its sign (e + h is finite on the front).

Using the same method to calculate A_0 and the analogous quantity for h, we obtain asymptotic expressions for e and h which are found to be of the same form on both sides of the waves, and are given by

$$e = \frac{1}{\pi} \ln|1 - \tau| + 0.17, \quad h = -\frac{1}{\pi} \ln|1 - \tau| + 1.10. \tag{3.11}$$

Close to the cylindrical axis after focusing,

$$e \to -1/2\sqrt{2}\,\tau^{3/2} = -(r/2ct)^{3/2}, \quad h \to (2/\tau)^{1/2} = \sqrt{2r/ct}.$$

From this it is seen that the electric field close to the axis is

$$E = -\mathcal{E}_0 \sqrt{r_0/r}\,(r/2ct)^{3/2} = -\mathcal{E}_0 r_0^{1/2} r/(2ct)^{3/2},$$

so that $E \equiv 0$ on the axis. The magnetic field is

$$H = H_0 + \mathcal{H}_0 \sqrt{2r_0/ct},$$

so that on the axis $H - H_0$ decreases as $t^{-1/2}$ after focusing.

The fundamental quantitative results of the self-similar solution are shown in Fig. 3, which gives graphs of $e(\tau)$ and $h(\tau)$ calculated from the formulas for the exact solution. The same figure gives the asymptotic formulas describing e and h in the neighborhood of the reflected wave and the axis.

4. FURTHER REMARKS

(a) Region of Applicability of the Self-Similar Solution

As can be seen from the solution itself, it is applicable for waves of arbitrary amplitude, not only for strong waves as in gas dynamics. The solution gives an exact description of all the phenomena in a finite (r, t) region if, for instance, the surface of an ideal conductor surrounding a cylindrical cavity moves with $r(t)$ corresponding to the solution obtained. If, however, this surface is suddenly caused to move and thereafter moves in an arbitrary way, the self-similar solution is not valid in the whole (r, t) region, but describes only the behavior close to the axis at the moment of focusing.

(b) Reflection of a Diverging Wave from a Conductor

It is not difficult to see that the radius $r(t)$ of a cylinder within which the total flux Φ is conserved has, in the case of the self-similar solution and a compression wave, the form of the $r(t)$ curve shown in Fig. 2: in view of the unbounded values of H on the diverging wave, this curve should be tangent to the line $r = ct$, which means that the conducting surface should move away from the axis with the velocity of light c (behind the front the velocity decreases again).

If before $r = ct$ the conductor does not start moving sharply in the other direction, but moves as shown by the dotted line, then a wave of infinite amplitude will be reflected from it at point K. We have not considered the further behavior of such a wave as it converges, although it may be of interest.

(c) Cylindrical Acoustical Wave

Let us consider a weak converging cylindrical shock wave for which $u \ll c$, where c is the velocity of sound. The solution of this problem reduces to solving the wave equation for the pressure

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = 0.$$

This equation for p is exactly the same as (2.5) for H in the electromagnetic wave problem. Therefore their self-similar solutions are also identical, and on the wave front reflected from the axis the pressure is also infinite. Thus this qualitative property of cumulation is not peculiar to the electromagnetic nature of the waves, but is a property of converging cylindrical waves described by the wave equation.

We note that for a spherical acoustical wave one may obtain a solution without the assumption that it is self-similar, and this makes it possible to verify the assertion that the self-similar solution describes the limiting form of the solution at the center close to the focus. A test shows that the self-similar solution for a sphere is indeed identical with the limiting form of the general solution, although in this case no singularities occur on the wave reflected from the center. (The amplitudes of the incident and reflected waves are equal, but opposite in sign; between the waves the pressure is maintained at every point.)

In conclusion we thank V. P. Mordvinov, D. G. Lominadze, A. B. Govorkov, and A. A. Bunatian, with whom we have had extremely valuable discussions of the material contained in the present article.

Translated by E. J. Saletan
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