ON THE KINETIC THEORY OF AN ELECTRON GAS IN THE PRESENCE OF BOUNDARIES

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The kinetic equations of an electron gas in a plane diode are used to show how one can construct a distribution function satisfying given conditions of emission and reflection at the electrodes for the one-dimensional case, neglecting the collision integral. Corrections to Langmuir's result, important for small anode potentials, are obtained.

PHYSICAL processes occurring in vacuum tubes are determined by the interactions of charged particles and physical conditions on the boundaries (for instance, on electrodes).

Particles of different energies satisfy different physical boundary conditions and we must thus use kinetic onsiderations in our calculations. We give below a method of integrating the kinetic equation for charged particles, suitable for one-dimensional problems. As an example we shall consider the flow of a direct electron current through a plane diode. The collective interactions of the electrons will be taken into account by using a self-consistent field; the individual interactions between electrons — collisions — will not be taken into account.

With these assumptions the distribution function $f(x, v)$ of the electron gas satisfies the kinetic equation

$$v \frac{\partial f}{\partial x} - \frac{u'(x)}{m} \frac{\partial f}{\partial v} = 0. \tag{1}$$

The potential energy $u(x)$ of an electron in a self-consistent field must be determined from Poisson's equation

$$d^2u/dx^2 = -4\pi e^2 n(x), \quad n(x) = \int_{-\infty}^{\infty} f(x,v)\, dv. \tag{2}$$

We formulate the boundary conditions for the distribution function for the most typical case, namely, a region bounded by two surfaces (in the case of a diode, the anode and the cathode).

The solution of the kinetic equation in the presence of two boundaries involves well-known mathematical difficulties. If the boundary values of the distribution function at both boundaries are arbitrary, the problem is incompatible. Actually, the kinetic equation together with data on one of the boundaries corresponds to the usual Cauchy problem, and determines uniquely the solution in the whole region, and in particular on the second boundary.

This problem was solved by Vainshtein[1] for a particular case (corresponding to no reflection of particles from the electrodes), where one can limit oneself to giving the distribution function on part of the boundary, for instance, on the cathode for positive velocities and on the anode for negative velocities. That distribution function was formulated by introducing discontinuous functions.

In the general case, the difficulties just mentioned result from incorrectly specifying the boundary conditions. The physical processes on the boundaries, viz. emission, reflection, and absorption of the particles, are mathematically formulated in the form of functional equations for its boundary values, but not in terms of the boundary values themselves. Below it is shown that for such boundary conditions one can find a solution of the kinetic equation.

The boundary conditions in the case of a diode must describe the emission of electrons from the cathode [with a distribution function $f_e(mv^2/2)$] and the elastic reflection of the electrons, characterized by a reflection coefficient $R(mv^2/2)$ that depends on the electron energy.

Such reflection, indeed, exists and is a quantum effect, connected with the passage of charged particles through a potential jump on the metal-vacuum boundary. Below we shall construct the solution for any function $R(mv^2/2)$.*

The presence of emission and reflection on the cathode ($x = 0$) leads to the boundary condition

$$f(0, v > 0) = f_e(mv^2/2) + R_c(mv^2/2)\, f(0, v < 0); \tag{3}$$

Reflection at the anode ($x = a$) leads to the condition

$$f(a, v < 0) = R_a(m v^2/2)\, f(a, v > 0). \tag{4}$$

The boundary conditions for the Poisson equation are determined by giving the potentials of the cathode and the anode,

$$u(0) = 0, \quad u(a) = u_a. \tag{5}$$

For a given density of the electron gas (that is, for a given distribution function) Eq. (2) together with the boundary conditions (5) determines the distribution of the potential inside the diode; therefore, additional data of the field on one of the electrodes is not only unnecessary, but even, in general, incompatible with equation (5).

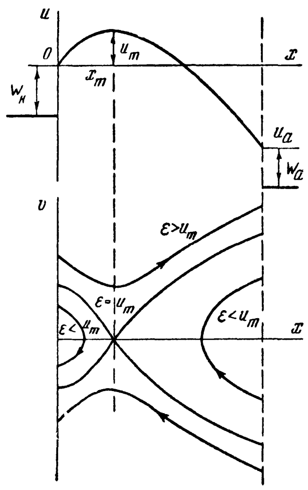
Our problem reduces thus to solving the differential-functional equations (1) and (2) with the boundary conditions (4) and (5).

The solution of Eq. (1) is a function of the electron energy.

$$f(x, v) = f(\varepsilon), \quad \varepsilon = m v^2/2 + u(x). \tag{6}$$

It would appear at first glance that $f(\epsilon)$ is an even function of the velocity and thus leads to zero current and is useless for our problem. To prove the opposite, let us consider the lines of constant energy in the (x, v) plane, along which $f(\epsilon)$ stays constant (see figure). The form of the function $u(x)$ corresponds to a large emission current which leads to a potential barrier of height u_m near the cathode.

The lines ϵ = constant go for $\epsilon \leq u_m$ continuously from regions of positive to regions of negative velocities. Since the function $f(x, v) = f(\epsilon)$ stays constant along these lines, it is indeed an even function of the velocity. However, for $\epsilon > u_m$ each line comprises two unconnected parts, one of which lies in the region of positive, and the other in the region of negative velocities. On each part $f(\epsilon)$ stays constant, but the values are not the same, meaning that $f(\epsilon)$ is, generally speaking, not an even function of the velocity for $\epsilon > u_m$.



The distribution function can thus be written in the following form,

$$f(x, v) = \begin{Bmatrix} f_1(\varepsilon), & x \leqslant x_m \\ f_2(\varepsilon), & x \geqslant x_m \end{Bmatrix} \varepsilon \leqslant u_m, \quad v \gtrless 0; \quad f(x, v) = \begin{Bmatrix} f_+(\varepsilon), & v > 0 \\ f_-(\varepsilon), & v < 0 \end{Bmatrix} \varepsilon \geqslant u_m. \tag{7}$$

These functions are simply determined by the boundary conditions. Substitution of f_1 into (3) and of f_2 into (4) gives

$$\left.\begin{aligned} f_1(\varepsilon) &= \frac{f_e(\varepsilon)}{1 - R_c(\varepsilon)}, & x \leqslant x_m \\ f_2(\varepsilon) &= 0 & x \geqslant x_m, \end{aligned}\right\} v \gtrless 0, \ \varepsilon \leqslant u_m. \tag{8}$$

Substituting f_+ and f_- into (3) and (4) yields a set of linear equations with a solution of the form

$$\left.\begin{aligned} f_+(\varepsilon) &= \frac{f_e(\varepsilon)}{1 - R_c(\varepsilon)\, R_a(\varepsilon - u_a)}, & v > 0 \\ f_-(\varepsilon) &= \frac{f_e(\varepsilon)\, R_a(\varepsilon - u_a)}{1 - R_c(\varepsilon)\, R_a(\varepsilon - u_a)}, & v < 0 \end{aligned}\right\} \varepsilon \geqslant u_m. \tag{9}$$

Equations (8) and (9) give us the complete solution of the problem of finding the distribution function of an electron gas, moving in a given field $u(x)$ with boundary conditions (3) and (4). We emphasize that

*To get an estimate we use the formula $R(\epsilon) = [\sqrt{1 + \epsilon/W} - \sqrt{\epsilon/W}]^4$ (see Ref. 2), which does not take into account the influence of the field at the surface of the metal on the reflection; W is here the total work needed to get the electron out of the metal.

the distribution functions (8) and (9) which we have found show a discontinuity on the line $\epsilon = u_m$. If we neglect reflection, Eqs. (8) and (9) go over into the solution given by Vainshtein[1] for the case when there is no emission from the anode.*

We can construct the function f_e for the electrons emitted by the cathode by assuming for the electrons inside the metal a Fermi distribution and by taking into account their reflection from the metal-vacuum boundary with the same coefficient R_K

$$f_e\left(\frac{mv^2}{2}\right) = \begin{cases} Ae^{-mv^2/2\Theta}[1 - R_c(mv^2/2)], & v > 0, \\ 0, & v < 0, \end{cases} \qquad (10)$$
$$A = (m^2\Theta / 2\pi^2\hbar^3)\, e^{-\varphi/\Theta}.$$

From Eqs. (8) to (10) we obtain the following expression for the current density in the diode,†

$$j = \frac{eA}{m}\int_{u_m}^{\infty} e^{-\varepsilon/\Theta} D_1(\varepsilon)\, d\varepsilon \simeq \frac{eA\Theta}{m} e^{-\frac{u_m}{\Theta}} [D_1(u_m) + \Theta D_1'(u_m) + \cdots],$$
$$D_1(\varepsilon) = [1 - R_c(\varepsilon)][1 - R_a(\varepsilon - u_a)] / [1 - R_c(\varepsilon) R_a(\varepsilon - u_a)]. \qquad (11)$$

Here and henceforth we expand the integrals in a power series of the parameter $y_m^{-1} \equiv \Theta/m$; the terms written out are sufficient for the case $y_m \gg 1$.

The distribution functions (7) to (9) are expressed in terms of the potential energy $u(x)$ which must be found from Eq. (2). An evaluation of the density of the electron gas from Eqs. (2) and (9) to (11) gives

$$n(x) = A\sqrt{2\pi\Theta/m}\, e^{\eta - y_m}\Phi(\eta^{1/2}) + n_2(\eta), \quad x \leqslant x_m; \quad (\eta\ n(x) = n_2(\eta)$$
$$\equiv A\sqrt{\frac{\Theta}{2m}}\, e^{-u/\Theta}\int_{\eta}^{\infty} e^{-\xi} D_2[u_m + \Theta(\xi - \eta)]\,\xi^{-1/2} d\xi \simeq \frac{1}{2}\sqrt{\frac{2\pi\Theta}{m}}\, D_2(u_m)\, e^{\eta - y_m}\left\{1 - \Phi(\eta^{1/2}) + \Theta\frac{D_2'(u_m)}{D_2(u_m)}\right. \qquad (12)$$
$$\left.\times\left[\sqrt{\frac{\eta}{\pi}}\, e^{-\eta} + \left(\frac{1}{2} - \eta\right)(1 - \Phi(\eta^{1/2}))\right]\right\}, \quad x \geqslant x_m; \quad \eta(x) = \frac{u_m - u(x)}{\Theta}; \quad D_2(\varepsilon) = \frac{[1 - R_K(\varepsilon)][1 - R_a(\varepsilon - u_a)]}{1 - R_K(\varepsilon) R_a(\varepsilon - u_a)}.$$

where Φ is the probability integral.

Substituting this expression for the density into Eq. (2) and integrating we find the dependence $x(\eta, y_m)$ of the coordinate on the potential. This dependence differs from Langmuir's results[4] through correction terms, connected with reflection. We calculate also the distance a between the electrodes from $y_a \equiv -u_a/\Theta$ and y_m (instead of y_m we can introduce as parameter the field on one of the electrodes). In this way we obtain a relation between y_a and y_m from which we can find their interdependence $y_m(y_a)$. Substituting this into Eqs. (7) to (9), (11), and (12) we can express $f(x, v)$, $n(x)$, $u(x)$ and j in terms of two parameters, u_a and Θ, i.e., we have obtained a complete solution of the diode problem.

The dependence of y_m on y_a is approximately of the form

$$y_m = \ln\alpha - \frac{\varphi}{\Theta} - \ln\frac{(y_a + \ln\alpha - \varphi/\Theta)^{3/2}}{D_2(\Theta\ln\alpha - \varphi)}, \qquad \alpha = (9/2^{3/2}\pi)\, e^2 m^{3/2}\Theta^{1/2} a^2\hbar^{-3} = 6{,}93\cdot 10^{15}\,\Theta^{1/2}\,(\mathrm{eV})\, a^2(\mathrm{cm}). \qquad (13)$$

This formula enables us to calculate the barrier height at the cathode for given cathode temperature and anode potential; for instance, for a tungsten cathode, $\Theta = 2320°$ K, a = 1 cm, u_m decreases from 1.8 to 0.5 ev if the anode potential increases from 1 to 160 v.

The diode characteristic is obtained by substituting expression (13) into (11),

$$j = \frac{1 - R_a(u_m - u_a)}{1 + R_a(u_m - u_a)}\,\frac{\sqrt{2}\,\Theta^{3/2}}{9\pi m^{1/2} a^2}\,(y_a + y_m)^{3/2}\left[1 + \frac{3\sqrt{\pi}}{2}(y_a + y_m)^{-1/2}\right]; \qquad (14)$$

The expression within the square brackets on the right hand side is the series expansion of the function D_1 and D_2 up to terms of the order $(y_a + y_m)^{-1}$.

Equation (14) takes the reflection of the electrons from the anode into account only through the factor $(1 - R_a)/(1 + R_a)$, and thus differs from the "3/2-law."[4] For $u_m - u_a < W_a$ this factor is nearly equal to unity which means that reflection is unimportant. However, for $u_m - u_a > W_a$ reflection appreciably

*However, the value given in Ref. 1 for the distribution function on the line of discontinuity is wrong.

†If $\epsilon - u_a > W_a$, the functions $D_1(\epsilon)$ and $D_2(\epsilon)$ will be the transmission coefficients of the cathode.

decreases the current; this in the above example of an anode potential of 1 v the current is less than the value for the "3/2-law" by 20%.

Equations (13) and (14) enable us to calculate the diode characteristic directly, and not in parametric form as is usually done.[5]

As we have shown above, the present method of integrating the kinetic equation for charged particles makes it possible to construct a distribution function satisfying boundary conditions of emission and reflection on two boundaries and to find the self-consistent field.

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CORRELATION OF POLARIZED QUANTA IN THE CASE OF NONCONSERVATION OF PARITY

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The angular correlation between the electron and the circularly polarized γ-quantum emitted in a cascade β-transition is considered in the case when parity is not conserved. The effect of the nuclear coulomb field is neglected.

As it has been shown by Lee and Yang[1] (see also the review by Shapiro[2]), parity nonconservation in β-decay implies an angular correlation between the circularly polarized γ-quantum and the electron. We calculated of this effect for allowed transitions, without taking into account the influence of the nuclear Coulomb field. If the β-decay is followed by a γ-transition, the probability for the γ-quantum to be emitted at an angle θ with respect to the direction of emission of the electron is equal to:

$$w(\theta) = 1 - (\mu\alpha v/c)\cos\theta, \tag{1}$$

where $\mu = \pm 1$ corresponds to right-hand or left-hand polarization of the γ-quantum, v is the election velocity, and α a coefficient depending on the interaction constants, on the momenta of the nuclei, and on the multipole order of the γ-quantum:

$$\alpha = \frac{2B\sqrt{j_2(j_2+1)}\,\delta_{j_2 j_1} + D\,[2 + j_2(j_2+1) - j_1(j_1+1)]}{2\sqrt{j_2(j_2+1)}\,(A\,\delta_{j_2 j_1} + C)}\;\frac{j_2(j_2+1) + L_1(L_1+1) - j_3(j_3+1)}{2L_1(L_1+1)\sqrt{j_2(j_2+1)}}. \tag{2}$$

Here j_1, j_2 and j_3 are the total angular momenta of the initial nucleus and of the excited and of the ground-state of the final nucleus, while L_1 is the multipole order of the γ-quantum. The coefficient α does not depend on the type of radiation (electric or magnetic), but only on its multipole order. The