from the comparison of the experimental and theoretical curves:

1. The one-dimensional theory describes satisfactorily the character of the distribution of transverse momentum components.*

2. Best agreement of both distributions is obtained† for $T = \mu_\pi c^2/k$. The values $T = \mu_\pi c^2/2k$ and $T = 3\mu_\pi c^2/2k$ are already difficult to reconcile with experimental data, although the scarcity of the latter does not permit to rule these values out. The value $T = \mu_\pi c^2/k$ is in agreement with previous indications, based on the analysis of other experimental results (for composition of showers see Refs. 6 and 7 and for energy spectrum see Ref. 3).

3. Since the value of temperature is connected with the value of the interaction cross-section of secondary particles[6] (evidently π-mesons) we can conclude that in the order of magnitude the latter equals the geometrical cross-section of the nucleon $(\hbar/\mu_\pi c)^2$.

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SOVIET PHYSICS JETP VOLUME 6, NUMBER 1 JANUARY, 1958

ON THE RELATION BETWEEN "ACCIDENTAL" DEGENERACY AND "HIDDEN" SYMMETRY OF A SYSTEM

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Submitted to JETP editor January 16, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) 33, 200-203 (July, 1957)

An example is discussed which confirms the connection between "accidental" degeneracy and "hidden" symmetry of a system. The symmetry of a two-dimensional oscillator is studied, and the relation is found between the quantization of the oscillator and that of a certain operator of the type of an angular momentum.

SUPPOSE the operator H is invariant with respect to some group G of transformations. Then the application of these transformations results in the expression of the eigenfunctions of the operator H belonging to an eigenvalue E_n in terms of each other, and thus gives a certain representation D_n of the group G.[1] As

*The cited calculations have shown that the introduction of the conical stage[2] of hydrodynamical motion predicts that the mean value of $p_\perp$ is $\gtrsim Mc$ (where M is the nucleonic mass), contradicting the histogram shown in the figure. It is possible that the conical stage of burst is necessary for the description of particle interaction at considerably higher energies than the energy of the shower of Ref. 4 ($\sim 5 \times 10^{12} - 10^{13}$ ev).

†It should be noted that since we neglected the possible influence of the hydrodynamical transverse components the given values of T are, strictly speaking, the upper limits of the values.

a rule this transformation is irreducible. In the opposite case (D_n reducible) one speaks of an "accidental" degeneracy. The study of several systems with "accidental" degeneracy, as carried out by Fock[2] (hydrogen atom), Demkov[3] (three-dimensional oscillator), and Baker[4] (n-dimensional oscillator), has shown that these systems possess in addition to the obvious symmetry group G also a higher "hidden" symmetry group G^*. On the application of the transformations of the group G^* the eigenfunctions belonging to the eigenvalue E_n give an irreducible representation D_n^* of the group G^*.

In the present paper we consider a further example confirming the connection between "accidental" degeneracy and "hidden" symmetry of a system. In addition, the symmetry group of the two-dimensional oscillator is studied and a relation is found between the quantization of the two-dimensional oscillator and the quantization of a certain operator of the type of an angular momentum.

1. Let us consider a system with its Hamiltonian of the form

$$H = -\Delta - 2/r, \tag{1}$$

where

$$\Delta = \sum_{i=1}^{f} \partial^2/\partial x_i^2, \quad r = \Big(\sum_{i=1}^{f} x_i^2\Big)^{1/2}. \tag{2}$$

The obvious symmetry group of the operator (1) is the group d_f of orthogonal transformations of the f-dimensional space. Separating variables in hyperspherical coordinates, one can readily verify that the eigenfunctions and eigenvalues of the operator (1) are given by

$$\psi_{n,k} = Ar^k \exp(-p_0 r)\, F(-n+k,\ 2k+f-1,\ p_0 r/2)\, Y_k^{(f)},\quad E_n = -1\Big/\Big(n+\frac{f-1}{2}\Big)^2,\quad p_0 = 1\Big/\Big(n+\frac{f-1}{2}\Big),$$
$$k = 0, 1, 2, \ldots n;\quad n = 0, 1, 2, \ldots, \tag{3}$$

where F is the confluent hypergeometric function and $Y_k^{(f)}$ denotes the set of f-dimensional hyperspherical functions belonging to the characteristic value $-k(k+f-2)$ of the angular part of the Laplacian operator.

Noting that the functions $Y_k^{(f)}$ for fixed k give an irreducible representation $D_k^{(f)}$ of the group d_f and that E_n does not depend on k, one easily sees that the representation D_n corresponding to E_n is reducible and decomposes into a sum of irreducible representations:

$$D_n = \sum_{k=0}^{n} D_k^{(f)}. \tag{4}$$

Thus we have to do with an "accidental" degeneracy.

To find the "hidden" symmetry group of our system, following Fock[2] we write the equation for the eigenvalues of the operator (1) in the p representation:

$$(\mathbf{p}^2 + p_0^2)\,\Phi(\mathbf{p}) = \pi^{-(f+1)/2}\,\Gamma\Big(\frac{f-1}{2}\Big)\int \frac{\Phi(\mathbf{p}')\,(d\mathbf{p}')}{|\mathbf{p}-\mathbf{p}'|^{f-1}},\quad p_0^2 = -E. \tag{5}$$

Regarding the components of the vector $\mathbf{p}/p_0$ as the stereographic projection of a point of an $(f+1)$-dimensional unit hypersphere, and introducing the new function

$$Y = B\,(\mathbf{p}^2 + p_0^2)^{(f+1)/2}\,\Phi(\mathbf{p}), \tag{6}$$

we get the equation

$$Y(M) = \frac{\Gamma\Big(\frac{f-1}{2}\Big)}{2\pi^{(f+1)/2} p_0}\int \frac{Y(M')\,d\Omega'}{\{2(1-\cos\gamma)\}^{(f-1)/2}}, \tag{7}$$

where $d\Omega'$ is the surface element of the $(f+1)$-dimensional unit hypersphere and γ is the "angle" between the points M and M′ of the hypersphere.

The invariance of Eq. (7) with respect to the group of orthogonal transformations $d_{(f+1)}$ of the $(f+1)$-dimensional space is obvious. Thus the "hidden" symmetry group of the operator (1) is the group $d_{(f+1)}$.

The solutions of Eq. (7) are the $(f+1)$-dimensional hyperspherical functions $Y^{(f+1)}$ and the eigenvalues are as given in Eq. (3):

$$p_0 = 1\Big/\Big(n+\frac{f-1}{2}\Big). \tag{8}$$

For given n the functions $Y_n^{(f+1)}$ give the irreducible representation $D_n^{(f+1)}$ of the group d_{f+1}.

Since the system in question is the f-dimensional analogue of the hydrogen atom, our results are the same as those of Fock[2] for $f = 3$.

2. In Ref. 4 it is shown that the symmetry group of an n-dimensional oscillator is the group of n-dimensional unitary transformations. There is a well known connection between two-dimensional unitary transformations and the three-dimensional rotation group,[1] so that it seems not without interest to examine the symmetry of the two-dimensional oscillator, the more so because there turns up by the way a connection between the quantization of the oscillator and that of a certain operator of the type of an angular momentum.

According to Ref. 4, the Hamiltonian of the two-dimensional oscillator, written in the form

$$H = {}^1/_2\,(a_1^* a_1 + a_2^* a_2) + 1, \tag{9}$$

where

$$a_1 = x + \frac{\partial}{\partial x},\quad a_1^* = x - \frac{\partial}{\partial x},\quad a_2 = y + \frac{\partial}{\partial y},\quad a_2^* = y - \frac{\partial}{\partial y}, \tag{10}$$

is invariant with respect to transformations of the form

$$a_k' = \sum_{l=1}^{2} U_{kl} a_l,\quad k = 1,2, \tag{11}$$

where $U_{k\ell}$ is a unitary matrix.

We note that under rotations $\Omega(\alpha)$ in the x, y plane and under Fourier transformation Φ of the function $\varphi(x, y)$ with respect to the coordinate x,

$$\Phi\varphi(x, y) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix\xi}\varphi(\xi, y)\,d\xi, \tag{12}$$

the operators a_k are transformed according to Eq. (11) with the matrices

$$\Omega(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix},\quad \Phi = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}. \tag{13}$$

Let us confine ourselves to transformations (11) that are close to the identity and have determinant unity. They can be written in the form

$$U = E + \frac{i}{2}\sum_{l=1}^{3} \alpha_l \sigma_l, \tag{14}$$

where E is the unit matrix and σ_ℓ are the Pauli matrices.

The group of transformations (14) is isomorphic to the three-dimensional rotation group,[1] and the corresponding infinitesimal operators are

$$I_k \equiv iJ_k = \frac{i}{2}\sigma_k,\quad k = 1, 2, 3. \tag{15}$$

Using the following relations (valid for small α_k):

$$E + \frac{i}{2}\alpha_1\sigma_1 = \Phi^{-1}\Omega\left(\frac{\alpha_1}{2}\right)\Phi,\quad E + \frac{i}{2}\alpha_2\sigma_2 = \Omega\left(\frac{\alpha_2}{2}\right),\ E + \frac{i}{2}\alpha_2\sigma_3 = \Omega\left(\frac{\pi}{4}\right)\Phi^{-1}\Omega\left(\frac{\alpha_3}{2}\right)\Phi\Omega\left(-\frac{\pi}{4}\right), \tag{16}$$

and also using the definitions of the operators Ω and Φ (Eq. 12), we can express the operators J_k of Eq. (15) in terms of x and y (cf. Ref. 3):

$$J_1 = {}^1/_2\,(xy - \partial^2/\partial x\,\partial y),\quad J_2 = (1/2i)\,(x\partial/\partial y - y\partial/\partial x),\quad J_3 = {}^1/_4\,(x^2 - y^2 - \partial^2/\partial x^2 + \partial^2/\partial y^2). \tag{17}$$

The operators (17) commute with the Hamiltonian (9) and satisfy the commutation relations of the angular momentum operators.

The eigenfunctions of the Hamiltonian (9) belonging to the eigenvalue $E_n = n + 1$ are given by

$$\psi_{n_1 n_2}(x, y) = (\pi 2^{n_1+n_2} n_1! n_2!)^{-1/2} \exp\left(-\frac{x^2+y^2}{2}\right) H_{n_1}(x) H_{n_2}(y),\quad n_2 = n - n_1,\quad n_1 = 0, 1, 2, \ldots, n,\quad n = 0, 1, 2, \ldots, \tag{18}$$

where the H_n are Hermite polynomials. Introducing the notations

$$j = (n_1 + n_2)/2 = n/2,\quad m = (n_1 - n_2)/2,\quad m = -j, -j+1, \ldots j;\quad j = 0, {}^1/_2, 1, {}^3/_2, \ldots, \tag{19}$$

and using the properties of Hermite polynomials, one can easily verify that the functions

$$Y_{jm}(x, y) = \psi_{n_1 n_2}(x, y) = \{\pi 2^{2j}(j+m)!\,(j-m)!\}^{-1/2} \exp\left(-\frac{x^2+y^2}{2}\right) H_{j+m}(x) H_{j-m}(y) \tag{20}$$

satisfy the relations

$$\mathbf{J}^2 Y_{jm} \equiv (J_1^2 + J_2^2 + J_3^2) Y_{jm} = j(j+1) Y_{jm}, \quad J_3 Y_{jm} = m Y_{jm}, \quad (J_1 \pm i J_2) Y_{jm} = \sqrt{(j \mp m)(j \pm m + 1)}\, Y_{jm\pm 1}, \tag{21}$$

which establishes the connection between the quantizations of the operators (9) and (17).

It must be pointed out that the definition of an "angular momentum" operator by Eq. (17) combines the integral and half-integral values of the quantum number j . We also note the appearance in this case of an operator K :

$$K = {}^1\!/_2 H = {}^1\!/_4\left(x^2 + y^2 - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right), \quad K^2 - {}^1\!/_4 = \mathbf{J}^2, \quad K Y_{jm} = (j + {}^1\!/_2) Y_{jm}, \tag{22}$$

analogous to the Dirac operator[5]

$$K = (\mathbf{L}\sigma + 1)\beta. \tag{23}$$

It follows from Eq. (21)[1] that under the transformations (14) the eigenfunctions (20) of the two-dimensional oscillator that belong to the eigenvalue $E_n = n + 1 = 2j + 1$ transform according to the irreducible representation D_j of the three-dimensional rotation group. Thus the group of transformations (14), isomorphic to the three-dimensional rotation group, suffices to explain the degeneracy of the eigenvalues of the operator (9) and can be regarded as the symmetry group of the two-dimensional oscillator.

In the n-dimensional case one can also restrict the discussion to the unitary transformations that are close to the identity and have determinant unity, but this does not lead to any interesting consequences.

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