FLOW INSTABILITY OF A SUPERFLUID FILM

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Instability of flow of a superfluid film with respect to the appearance of wave motion on its surface is shown to be theoretically possible. The value obtained for the critical velocity is appreciably greater than the experimental value. The problem of the shape of a He II film moving in a gravitational field has been solved by taking into consideration hydrodynamic forces, surface tension and Van der Waals forces.

THE phenomenon of superfluidity, discovered by Kapitza[1] in 1938, received its interpretation in the subsequent development by Landau[2] of a two fluid model of the quantum liquid. Landau derived the hydrodynamic equations of He II by starting out from the conservation laws. Later, Khalatnikov[3] obtained them from the Boltzmann equation for the excitation. It was established experimentally that the superfluidity is destroyed at sufficiently small velocities of flow. This should be connected with the instability of flow of the liquid with respect to the appearance of a different type of excitation. Unfortunately, it is not possible to investigate the appearance of instability in the general case and we must consider each specific type of excitation separately. Thus, it is necessary that the flow velocity exceed the sound velocity for the appearance of an individual phonon, while for the appearance of a roton it need exceed the quantity Δ/p_0 $\sim$ 60 m/sec. These values are appreciably larger than the real critical velocities. Actually, it is evident that the destruction of superfluidity is connected with the appearance in the liquid of vortex filaments of the Onsager type, as was shown by Feynman.[4] However, a quantitative criterion for the loss of stability can be obtained only for the case of the rotation of a cylinder in the superfluid.*

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* Consider the following simple calculations. A single vortex creates a velocity distribution $v_s = \hbar/mr$ in the superfluid component. The corresponding kinetic energy per unit length is

$$\int \rho_s \frac{v_s^2}{2} 2\pi r dr = \pi\rho_s \frac{\hbar^2}{m^2} \ln \frac{R}{a_0}$$

and the momentum per unit length

$$\int \rho_s v_s r \cdot 2\pi r\, dr = \pi\rho_s \frac{\hbar}{m} R^2$$

(m is the atomic mass of helium, R the radius of the vessel, and a_0 the distance between atoms). The part of the free energy per unit length of a column of fluid depending on Ω is

$$F = \frac{1}{4} \rho_n \pi R^4 \Omega^2 + \frac{I\Omega^2}{2} + \pi\rho_s \frac{\hbar^2}{m^2} \ln \frac{R}{a_0},$$

while its momentum is

$$M = \frac{1}{2} \rho_n \pi R^4 \Omega + I\Omega + \pi\rho_s \frac{\hbar}{m} R^2$$

(I is the moment of inertia of the vessel). In the state of thermodynamic equilibrium, the following quantity must have a minimum:

$$F - M\Omega = -\frac{1}{4} \rho_n \pi R^4 \Omega^2 - \frac{I\Omega^2}{2} + \pi\rho_s \frac{\hbar}{m} \left(\frac{\hbar}{m} \ln \frac{R}{a_0} - \Omega R^2 \right).$$

Formation of the first vortex is favorable when the last term becomes negative, whence

$$R^2 \Omega_{cr} = \frac{\hbar}{m} \ln \frac{R}{a_0}.$$

Extension of the formula for the rotation of a cylinder to an estimate of the critical velocity in the case of a slit of a film gives excellent order-of-magnitude agreement with experiment. In the present paper we consider one additional mechanism: the loss of stability in superfluid flow in a film, brought about by the appearance of an unusual vibration, and the transition of the flow to the wave regime reminiscent of the studies of Kapitza.[5] The phenomenon is analogous to the instability of a tangential disturbance.[6,6a] It reduces to the dissipation of energy in the normal component and to the appearance of a damping force. Although the values of the critical velocity computed in this way considerably exceed the experimental, consideration of the problem presents methodological interest. Moreover, there is the hope of isolating the mechanism experimentally, about which more will be said below.

We consider an incompressible superfluid which forms a layer of thickness h on the surface of a solid wall. As the unperturbed motion, we take $v_s^0 = U = \text{const}$, $v_n^0 = 0$ and look for a first approximation in the form of a traveling wave $e^{ikz - i\omega t}$. We draw the x axis perpendicular to the surface and consider $x = -h$ the bounding coordinate between solid and liquid. For the incompressible fluid, as is well known,[2] we can write the equations for the normal and superfluid components separately; for the normal component it suffices to use a linearized equation. The system of equations for superfluid hydrodynamics can be written in the following form:

$$\mathbf{v}_s = \nabla\varphi,\ \Delta\varphi = 0, \quad \partial \mathbf{v}_n / \partial t = \nu\Delta \mathbf{v}_n - (1/\rho_n)\nabla p_n, \ \ \operatorname{div} \mathbf{v}_n = 0. \tag{1}$$

Here $p = p_n + p_s$ is the total pressure, $p_s = -\rho_s \partial\varphi/\partial t - \rho_s v_s^2/2$, $\eta = \rho_n \nu$ is the viscosity of He II. The solutions satisfying the boundary conditions

$$v_{nx} = v_{nz} = 0,\ v_{sx} = \partial\varphi/\partial x = 0 \ \text{for}\ x = -h,$$

have the form

$$\varphi = ae^{i(kz-\omega t)}\cosh k(x+h) + yU_y + zU_z,$$

$$v_{nz} = e^{i(kz-\omega t)}\left\{A\left[\sinh k(x+h) - \frac{m}{k}\sinh m(x+h)\right] + B\left[\cosh k(x+h) - \cosh m(x+h)\right]\right\},$$

$$v_{nx} = e^{i(kz-\omega t)}\left\{-iA\left[\cosh k(x+h) - \cosh m(x+h)\right] - iB\left[\sinh k(x+h) - \frac{k}{m}\sinh m(x+h)\right]\right\}, \tag{2}$$

$$p_n = \rho_n \frac{\omega}{k} e^{i(kz-\omega t)}\left[A \sinh k(x+h) + A \cosh k(x+h)\right]; \quad m^2 = k^2 - \frac{i\omega}{\nu}.$$

We now introduce the quantity

$$\xi = be^{i(kz-\omega t)}, \tag{3}$$

which describes the vibration of the boundary, and write down a set of boundary conditions at the free surface (for $x = 0$). For simplicity, we shall not consider the effect of the vapor.

The rate of change of the x coordinate of the surface is equal to the velocity v_{nx} of the normal component:

$$\partial\xi/\partial t = v_{nx} \tag{4}$$

and, on the other hand, can be expressed by means of the velocity of the superfluid component

$$\frac{\partial\xi}{\partial t} = \frac{\partial\varphi}{\partial x} - U_z\frac{\partial\xi}{\partial z} - U_y\frac{\partial\xi}{\partial y} = \frac{\partial\varphi}{\partial x} - U_z\frac{\partial\xi}{\partial z}, \quad \text{since}\ \frac{\partial\xi}{\partial y} = 0. \tag{5}$$

Moreover, the component σ_{xz} of the momentum flux tensor should be equal to zero on the surface, whence

$$\eta(\partial v_{nz}/\partial x + \partial v_{nx}/\partial z) = 0. \tag{6}$$

Finally, the latter condition expresses the equality σ_{xx} with the total force acting on the liquid. After linearization, we have

$$\rho_s\frac{\partial\varphi}{\partial t} + \rho_s U_z\left(\frac{\partial\varphi}{\partial z} - U_z\right) - p_n + \rho\gamma\xi - \alpha\frac{\partial^2\xi}{\partial z^2} + 2\rho_n\nu\frac{\partial v_{nx}}{\partial x} = 0. \tag{7}$$

Here α is the surface tension, the first two terms correspond to the pressure of the superfluid, while the term $\rho\gamma\xi$ corresponds to the force which is connected with the Van der Waals interaction of the film with the substratum.* We note that only the U_z component of the velocity is involved, and will therefore put everywhere $U = U_z$.

Upon substitution of the solutions (2) (3) in the four conditions (4) to (7), we get a system of four homogeneous linear equations, which can be solved if their determinant vanishes. After evaluation of the determinant and rather lengthy transformations, we get an equation for ω:

$$\omega^2 - 2\frac{\rho_s}{\rho}Uk\omega - k^2\left[\left(\frac{\gamma}{k}+\frac{\alpha k}{\rho}\right)\tanh kh - \frac{\rho_s}{\rho}U^2\right] = -\frac{\rho_n}{\rho}\omega^2\left[\frac{k}{m}\frac{\tanh mh}{\sinh kh\cosh kh} + 4i\frac{\nu k^2}{\omega}\left(1-2\frac{k}{m}\tanh kh\tanh mh - \frac{1}{\cosh kh\cosh mh}\right)\right.$$
$$\left.-8\frac{\nu^2k^4}{\omega^2}\left(1-\frac{k}{m}\tanh kh\tanh mh - \frac{1}{\cosh kh\cosh mh}\right)\right]\Big/\left[1-(k/m)\coth kh\tanh mh\right]. \tag{8}$$

Even if only one of the roots ω of this equation has a positive imaginary part, the motion will be unstable. For the investigation of Eq. (8) we call attention to the fact that when the square bracket on the left side is equal to zero, then the equation certainly has the root $\omega = 0$. In order to establish this fact, we expand the numerator and replace the right hand side by $m^2 - k^2 = i\omega/\nu$. Neglecting terms $\sim \omega^2$, we get

$$2\frac{\rho_s}{\rho}Uk\omega + k^2\left[\left(\frac{\gamma}{k}+\frac{\alpha k}{\rho}\right)\tanh kh - \frac{\rho_s}{\rho}U^2\right] = -2i\frac{\rho_n}{\rho}\nu k^2\omega\,\frac{1+k^2h^2/\cosh^2 kh}{1-2kh/\sinh 2kh},$$

$$\omega = k^2\,\frac{\frac{\rho}{\rho_s}\left(\frac{\gamma}{k}+\frac{\alpha k}{\rho}\right)\tanh kh - U^2}{2i\frac{\rho_n}{\rho_s}\nu k^2\left(1+\frac{k^2h^2}{\cosh^2 kh}\right)\Big/\left(1-\frac{2kh}{\sinh 2kh}\right) - 2Uk}. \tag{9}$$

It is evident from the last equation, in particular, that if

$$U^2 \geqslant \min\frac{\rho}{\rho_s}\left(\frac{\gamma}{k}+\frac{\alpha k}{\rho}\right)\tanh kh, \tag{10}$$

then there always exists a region k where $\mathrm{Im}\,\omega > 0$ and, consequently, the motion is unstable. Investigation of the expansion of the right side of Eq. (8) under the limiting conditions $\nu k^2/\omega \ll 1$ and $\nu k^2/\omega \gg 1$ shows that the inequality (10) is not only a sufficient but is evidently a necessary condition of instability.

We can put the condition (10) in parametric form in terms of k, which corresponds to the minimum $U^2(k)$. Replacing the velocity by the volume flow $Q = \rho_s Uh/\rho$, and expressing γ in terms of a and h, we get the following expression for the transfer rate

$$Q^2 = \frac{\rho_s}{\rho}\left(\frac{3a}{kh^2}+\frac{\alpha kh^2}{\rho}\right)\tanh kh, \tag{11}$$

where k is determined from the condition

$$\frac{\sinh 2kh}{2kh} = \left(1+\frac{\alpha k^2h^4}{3a\rho}\right)\Big/\left(1-\frac{\alpha k^2h^4}{3a\rho}\right), \tag{12}$$

which expresses $\partial U^2/\partial k = 0$. Expanding Eq. (12) in a series, we can see that for $h > (a\rho/\alpha)^{1/2}$, Eq. (12) has only $k = 0$ as a root.

In this case the transfer rate is expressed by the formula

$$Q = (3a\rho_s/h\rho)^{1/2}.$$

For $h = (a\rho/\alpha)^{1/2}$, a root $k = 0$ appears, which quickly increases with increasing h. For sufficiently small h, $kh \gg 1$, and we can determine k by letting the denominator of the right side of Eq. (12) vanish; this yields:

$$kh^2 = (3a\rho/\alpha)^{1/2}.$$

In this case the discharge tends to the h-independent value

$$Q = (\rho_s/\rho)^{1/2}(12a\alpha/\rho)^{1/4}.$$

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* The thickness-dependent part of the chemical potential of the film is known from Ref. 7 to be $-am/h^3$ while the pressure is $(3am/h^4)(\rho/m)\xi$, where $\xi = \delta h$. Thus $\gamma = 3a/h^4$ ($a \sim 10^{-15}$ is connected with the thickness of the static film by the relation $gz = a/h^3$).

A plot of Q^2 vs. h is shown in Fig. 1.

For a complete solution of the problem we must solve simultaneously the equations (11) and (12) for the transfer rate and the equation for shape of the film in a gravitational field with account of the Van der Waals forces, the forces of surface tension, and the hydrodynamic pressure acting on it. In the case of stationary flow of the superfluid, we have, from the equation of Ref. 2

$$\partial \mathbf{v}_s / \partial t + \nabla (\mu + v_s^2 / 2) = 0$$

the relation $\mu + u_s^2/2 = \text{const}$ on the free surface. Eliminating the velocity dependence from the chemical potential μ, the gravitational forces, Van der Waals forces[7] and the surface tension, we get an equation which determines the form of the film:

$$\frac{\rho_s}{\rho}\frac{v_s^2}{2} + gz - \frac{a}{h^3} - \frac{\alpha}{\rho}\frac{d^2h}{dz^2}\left[1+\left(\frac{dh}{dz}\right)^2\right]^{-3/2} = 0. \qquad (13)$$

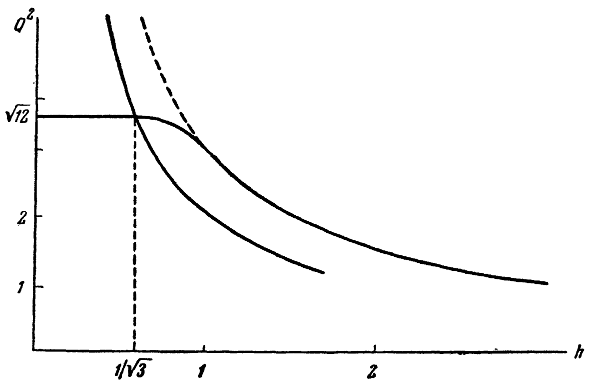

FIG. 1. Broken line shows the dependence of Q^2 [in units of $(\rho_s/\rho)(a\alpha/\rho)^{1/2}$] on h [in units of $(a\rho/\alpha)^{1/2}$]. The point of intersection of this curve with the curve of (19) determines the critical thickness and transfer rate in the film.

The constant here is so chosen that, upon neglect of the Van der Waals forces and the hydrodynamical term, we get an equation for the capillary meniscus, wherein measurement of the height is taken from the horizontal surface of the liquid. In a region located at a sufficient distance from the "bounding" meniscus,

$$z_0 = (2\alpha / \rho g)^{1/2},$$

and the form of the film is described by the equation of the capillary meniscus[6b]

$$h = \left(\frac{\alpha}{\rho g}\right)^{1/2}\left[\left(\cosh^{-1}\left(\frac{2}{z}\left(\frac{\alpha}{\rho g}\right)^{1/2}\right) - \cosh^{-1}\sqrt{2}\right) - 2\left(\sqrt{1-\frac{z^2\rho g}{4\alpha}} - \frac{1}{\sqrt{2}}\right)\right]. \qquad (14)$$

At heights much greater than z_0 we can neglect the effect of the surface tension; moreover, taking it into account that dh/dz is very small in this region, we can express v_s on the surface in terms of the discharge $Q = \rho_s v_s h/\rho$. We then obtain an equation describing the shape of the film in this region:

$$z = g^{-1}(a / h^3 - \rho Q^2 / 2\rho_s h^2). \qquad (15)$$

We can see from the shape that the thickness of the film decreases upon increase in the discharge. This result was obtained by Kontorovich.[8]

At points close to the boundary of the static meniscus, $|z - z_0| \ll z_0$, we can consider approximately that $gz = gz_0$ in Eq. (13). Moreover, the value of the derivative dh/dz is sufficiently small that we can neglect it in the term with the surface tension, and express v_s in terms of the discharge. As a result, we get the following equation for the solution of the combination of Eqs. (14) and (15):

$$\frac{\alpha}{\rho}\frac{d^2h}{dz^2} = \left(\frac{2\alpha g}{\rho}\right)^{1/2} - \frac{a}{h^3} + \frac{\rho Q^2}{2\rho_s h^2}, \qquad (16)$$

which does not contain z and which can therefore be integrated by successive quadratures. The result

$$z + C_2 = -\int^{h} dh\left\{2h\left(\frac{2\rho g}{\alpha}\right)^{1/2} + \frac{\rho}{\alpha}\frac{a}{h^2} - \frac{\rho^2 Q^2}{\rho_s \alpha h} + C_1\right\}^{-1/2} \qquad (17)$$

depends on two constants of integration, by the choice of which we can join the upper and lower solutions. Here we have a large parameter $A \sim 10^4$, which has the meaning of the ratio of the capillary constant z_0 to the thickness of the film at the height of the boundary of the capillary meniscus $H_0 = (a/gz_0)^{1/2}$. Consideration of conditions of continuity of z and dh/dz (written in non-dimensional form) at the point h_0 of the junction of the solution of (17) with the solution of (15) shows that for arbitrarily large A, the following conditions must be satisfied:

$$2h_0(2\rho g/\alpha)^{1/2} + \rho a/\alpha h_0^2 - \rho^2 Q^2/\rho_s \alpha h_0 + C_1 = 0, \qquad (18)$$

$$(2\rho g/\alpha)^{1/2} - \rho a/\alpha h_0^3 + \rho^2 Q^2/2\rho_s \alpha h_0^2 = 0. \qquad (19)$$

In these conditions there appears the parameter h_0, which is determined by the discharge Q [from Eq. (19)] and which is a multiple root of the expression found under the radical in the integral in (17). In the case of a multiple root, the integral is written in terms of elementary functions, which gives

$$z = z_0 - \sqrt{z_0 h + \frac{a}{2gh_0^2}} + h_0^2 \left(\frac{a\rho}{\alpha} + \frac{4h_0^3}{z_0}\right)^{-1/2} \cos^{-1} \frac{h + h_0 + a/gz_0h_0^2}{h - h_0}. \tag{20}$$

Joining of the solution in the region of the meniscus is done automatically; this is demonstrated by expanding Eq. (14) near z_0, giving the formula

$$h = (z - z_0)^2/z_0,$$

which coincides with that in (20) for $z_0 \gg h \gg h_0$ and $h_0 \gg (a\rho/\alpha)^{1/2}$. A graph of the function (2), which describes the shape of the surface, is given in Fig. 2.

The condition for stability can now be written in the form

$$f(h) - Q^2 > 0,$$

where f(h) is determined by Eqs. (11) and (12). Increase in the flow Q brings about a change in the function at the left side, for the condition of stability, by an amount

$$\left[\frac{df}{dh}\left(\frac{\partial h}{dQ^2}\right)_z - 1\right]\delta Q^2;$$

df/dh is always negative, since f is a monotonically decreasing function of h: $(\partial h/\partial Q^2)_z$ is also always negative, inasmuch as the thickness of the film decreases with increase in its velocity. As calculations with Eqs. (15) and (20) show, the minimum value of $(df/dh)(\partial h/\partial Q^2)_z$ is assumed at h_0, where the first condition of stability for growth of the discharge is disturbed. Simultaneous solution of the equation for h_0 (19) and the discharge [Eqs. (11) and (12)] gives, with considerable accuracy,

$$h_0 = (a\rho / 3\alpha)^{1/2}, \tag{21}$$

$$Q^2 = (\rho_s / \rho)(12 a\alpha / \rho)^{1/2}; \tag{22}$$

here kh ~ 3 and the error in Eqs. (21) and (22) is of the order $e^{-6} \sim 1/400$. For the value of $h_0 = (a\rho/3\alpha)^{1/2}$ that is obtained, the solution (20) is already unsuitable, for in this case the derivative

$$\frac{dh}{dz} = -\frac{h - h_0}{h}\sqrt{\frac{4h}{z_0} + \frac{a\rho}{\alpha h_0}}$$

is of the order of unity in a very small region $h \sim h_0$. This comes about as a result of the strong hydrodynamic compression of the film at large discharges Q. As a result, the meniscus disappears and the film should have a constant thickness h_0 along almost the entire height. An exact determination of the shape of the film under these conditions is extremely difficult, since, in addition to calculation of dh/dz in the capillary term of Eq. (13), we would also be obliged to solve it simultaneously with the equation for the potential flow of the superfluid component, in view of the fact that the velocity v_s on the free surface is no longer expressed in simple form in terms of the discharge. However, we can consider that this difficulty does not lessen the value of Eq. (22) for the transfer rate. Let us estimate the damping forces which arise in the presence of a developed wave regime. We shall consider that for $U \sim 2U_{cr}$, the amplitude of the waves on the surface is of the order of the thickness of the layer. Assuming that in order of magnitude $h dp/dz \sim \sigma_{xz}$; $\sigma_{yz} \sim \eta v_m/h$; $v_n \sim \omega h$, we get $h dp/dz \sim \rho_n \nu \omega$; we can estimate ω from Eq. (9):

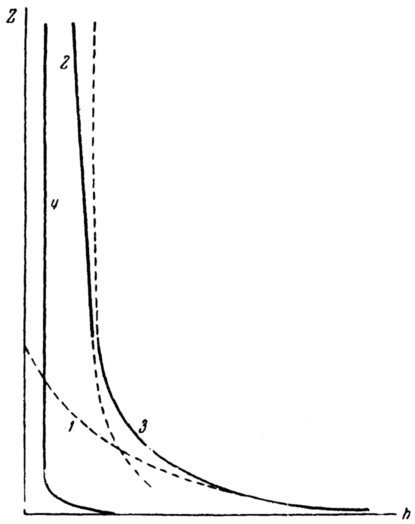

FIG. 2. Curve 1 corresponds to the shape of the capillary meniscus [Eq. (14)]. Curve 2 corresponds to Eq. (15). Curve 3 is the joined solution (20) (Q = 0). Curve 4 corresponds to the shape of the film for large discharges (22), when the meniscus disappears.

$$\omega \sim (U_{cr}^2 - U^2)/(\rho_n/\rho_s)\nu.$$

As a result we obtain

$$\frac{dp}{dz} \sim \frac{\rho_s U^2}{h} \approx \rho \frac{\rho}{\rho_s} \frac{Q^2}{h^3} \approx \rho \frac{(a\alpha / \rho)^{1/2}}{(a\rho / \alpha)^{3/2}} = \frac{\rho}{a} \left(\frac{\alpha}{\rho}\right)^2.$$

The ratio of this quantity to the actual hydrostatic pressure is of the order

$$(\alpha / \rho)^2 / ag \sim 10^{12},$$

i.e., loss of stability carries an extraordinary rigid character, and the value of the critical discharge is very close to Eq. (22).

Many measurements of the phenomenon of helium transfer along a film are described in the literature.[9,10] These give approximately the same temperature dependence of the volume discharge Q. They show that this dependence can be represented, within the limits of error, as ρ_s/ρ or $(\rho_s/\rho)^{1/2}$, wherein the first formula describes the experimental data somewhat better.

Equation (22) gives the temperature dependence of the discharge proportional to $(\rho_s/\rho)^{1/2}$, which does not contradict the experimental data. However, if we calculate a by the formula $gz = a/h^3$, considering that at $z = 1$ cm the thickness of the film $h \sim 2.0 \times 10^{-6}$ cm[11], we obtain $a \sim 8 \times 10^{15}$ and when the value $\alpha = 0.35$, measured along the rise in the capillary,[12] and $\rho = 0.145$ are substituted in the formula, we get for the discharge $Q \cong 6.9 \times 10^{-4}$ cm^2/sec, which is considerably larger than the experimentally observed value 1.7×10^{-4} cm^2/sec. The corresponding critical thickness of the film is of the order of atomic dimensions: $h \sim 3.4 \times 10^{-8}$ cm, and is extraordinarily small. The thickness of the film was measured simultaneously with the flow only in a single work,[13] in which $Q = 1.68 \times 10^{-4}$ cm^2/sec, and $h = 1.66 \times 10^{-6}$ cm for $T = 1.5°$.

The value of the thickness differs sharply from what has been expected; this points evidently to the action of a mechansim of destruction of the superfluidity, different from the described formation of a vortex. However, the effect of flow on the thickness of the film should be carefully noted, since the film ought to experience considerable hydrodynamic compression. An increase in the thickness of the flowing film of about 20% has already been observed by Burge and Jackson.[14]

In conclusion, we note that there is reason for assuming that if the formation of vortices is due to the destruction of superfluidity, then we can expect that this process requires an appreciable time (in the experiments of Hall and Vinen[15] vortices were formed within a time on the order of a minute).

In this case the mechanism just described takes place, yielding a much higher value of the discharge. According to our calculations, the capillary meniscus ought to disappear during the time of action of this mechanism. Also, during this time vortices evolve in the film and the discharge is established. The shape of the film will be described by Eqs. (15), (20), and (14) at all regions of height.

I regard it as my pleasant duty to express my deep gratitude to L. D. Landau and I. M. Khalatinkov for constant discussions on the work and for valuable advice.

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MANY-PHONON PROCESSES IN CRYSTALS

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We have calculated the differential cross section for the inelastic scattering of a neutron by a crystal with either emission or absorption of an arbitrary number of phonons. The first case is of interest when the temperature of the crystal is low, and the second when the neutron energy is very small and the crystal temperature sufficiently high. Formulae are obtained for crystals with an arbitrary spectrum of the normal vibrations. If we choose a specific spectrum, the calculations can be pursued to the end, leading to a simple final formula. We have also given the formula for the limiting case of the scattering of high energy neutrons by a free nucleus.

1. INTRODUCTION

To investigate the interaction of slow neutrons with crystalline substances, one uses the Debye model of a crystal. In that case the transfer of energy from the neutron to the crystal is treated as the excitation of one or several "phonons," that is, quanta of the thermal motion of the crystal. The transfer of energy from the crystal to the neutron corresponds to the absorption of phonons by the neutron. These processes have often been considered in the literature. Weinstock[1] derived formulae for the effective cross section for elastic and inelastic neutron scattering with the emission or absorption of one phonon (one-phonon process). The evaluation of processes involving simultaneously the emission and absorption of several phonons is in principle not difficult, but in practice very cumbersome.

Squires[2] in calculating the cross section for scattering of slow neutrons by Mg and Ni considered terms d_{mn} corresponding to processes where m phonons are emitted and n phonons absorbed. For $m + n \geq 2$ he did not take into account the interference between the waves scattered coherently from different atoms. Squires' calculations agree well with his own experiments. The neutron energy in those experiments was very small ($\lesssim 0.003$ ev), and it was therefore sufficient for the author to calculate several terms with the smallest values of m and n. In those cases where the number of phonons involved in the scattering can be large, the number of terms d_{mn} contributing to the cross section also becomes large; it becomes therefore impossible to evaluate the cross section by evaluating every term separately, as was done by Squires, and it is necessary to develop a method for summing the terms d_{mn}.

In the present paper we calculate the cross section for inelastic neutron scattering by evaluating only processes of identical character: either only emission, or only absorption of an aribrary number of phonons. In Squires' notation this corresponds to $\sum_{m=1}^{\infty} d_{m0}$ and $\sum_{n=1}^{\infty} d_{0n}$. Interference is not considered. The formulae obtained have practical value in two cases.

1. The case of low crystal temperatures and sufficiently large neutron energies. In this case absorp-