SOVIET PHYSICS JETP VOLUME 5, NUMBER 6 DECEMBER 15, 1957

Angular Operators For Nuclear Reactions

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(Submitted to JETP editor March 14, 1957)
J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 1536-1546 (June, 1957)

An orthogonal and normalized system of invariant angular operators is found; these are matrices in terms of which the scattering amplitude can be expressed. The angular operators correspond to transitions with definite total angular momentum and parity, and also definite values of other quantum numbers in the initial and final states, and they completely determine the angular distributions and polarizations of the particles in these transitions. The explicit form of the operators is presented for reactions in which the spin of the system is 0, ½, and 1, and for analogous reactions with γ-quanta, in particular for the scattering and energy loss of mesons against nucleons, for the scattering of photons and nucleons by nucleons, and for the reactions $\mathfrak{N} + \mathfrak{N} \to \pi + D$, $\gamma + D \to p + n$, and $\gamma + D \to \pi + D$.

1. INTRODUCTION

LET US CONSIDER in the center-of-mass system a reaction of the type $a + b \to a' + b'$, in which particle a with spin s is incident in the direction $\mathbf{k}$ on particle b with spin σ, the result being the formation of particle a' with spins s', emerging in the direction $\mathbf{k}'$, and of particle b' with spin σ' ($\mathbf{k}$, $\mathbf{k}'$ are unit vectors). The analysis of such a reaction can be conveniently carried out by means of the scattering matrix $S(\mathbf{k}'\alpha', \mathbf{k}\alpha)$, which takes the wave function $\psi_i(\mathbf{k}\alpha)$ of the initial state over into the wave function $\psi_f(\mathbf{k}'\alpha')$ of the final state

$$\psi_f(\mathbf{k}'\alpha') = S(\mathbf{k}'\alpha', \mathbf{k}\alpha)\,\psi_i(\mathbf{k}\alpha).$$

Hereinafter we shall be interested only in the angular and spin dependences of the scattering matrix, so that in S and in ψ_i, ψ_f there are written out only the angular variables $\mathbf{k}$, $\mathbf{k}'$ and the spin variables α, α' of the initial and final states.

The complete theoretical calculation of the scattering matrix requires a knowledge of the detailed mechanism of interaction of the particles involved in the reaction. But by using the invariance of the Hamiltonian system with respect to rotations and reflections in space one can separate out those properties of the scattering matrix that do not depend on the mechanism of the interaction. For this purpose we expand S in terms of eigenfunctions* $\psi_{JM\pi\nu}(\mathbf{k}\alpha)$ of the operators for the total angular momentum, for one of its components, and for the reflection operator, in the initial and final states. Besides the total angular momentum J, its component M, and the parity π, these functions are characterized by further quantum numbers ν, since the operators for the total angular momentum, for its component, and for reflection can have several eigenfunctions for given $JM\pi$. In addition, we provide these functions with indices i, f, which determine the type and states of the particles in the initial and final channels of the reaction. Then

$$S(\mathbf{k}'\alpha', \mathbf{k}\alpha) = \sum_{J'M'\pi'\nu'}\sum_{JM\pi\nu}\psi^f_{J'M'\pi'\nu'}(\mathbf{k}'\alpha')\,S^{fi}_{J'M'\pi'\nu',\,JM\pi\nu}\,\psi^{i*}_{JM\pi\nu}(\mathbf{k}\alpha).$$

In virtue of the invariance of S under rotations and reflections the matrix elements of S are diagonal in J, M, and π and do not depend on M:

$$S^{fi}_{J'M'\pi'\nu';\,JM\pi\nu} = S^{f}_{J\pi\nu'\nu}\,\delta_{J'J}\delta_{M'M}\,\delta_{\pi'\pi}.$$

Thus

$$S(\mathbf{k}'\alpha', \mathbf{k}\alpha) = \sum_{J\pi\nu'\nu} S^{fi}_{J\pi\nu'\nu}\sum_{M}\psi^f_{JM\pi\nu'}(\mathbf{k}'\alpha')\,\psi^{i*}_{JM\pi\nu}(\mathbf{k}\alpha). \tag{A}$$

The matrix elements $S^{fi}_{J\pi\nu'\nu}$ are functions of the total energy of the system, and their calculation requires a knowledge of the detailed mechanism of the interaction. The operators

$$L^{fi}_{J\pi\nu'\nu}(\mathbf{k}'\alpha', \mathbf{k}\alpha) = \sum_{M}\psi^f_{JM\pi\nu'}(\mathbf{k}'\alpha')\,\psi^{i*}_{JM\pi\nu}(\mathbf{k}\alpha) \quad (1)$$

are functions of the angular variables $\mathbf{k}$, $\mathbf{k}'$, and ma-

*The functions $\psi_{JM\pi\nu}$ are quantities that transform on space rotations according to the $(2J + 1)$-dimensional irreducible representation of the three dimensional rotation group. We shall call them J-vectors. For $J = ½$ these are spinors, for $J = 1$, vectors, for $J = 2$, tensors of the second rank, *etc.* For the general definition and the properties of J-vectors see Ref. 1, where they are called L-vectors.

trices with respect to the spin variables α, α'. We shall call these operators $L^{fi}{}_{J\pi\nu'\nu}$ angular operators or polynomials of the reaction $a + b \to a' + b'$, since they completely determine the angular distribution and polarization of the particles in transitions with given $J\pi\nu'\nu$.[2,9] As can be seen from their definition, the angular operators are invariant under rotations and reflections in the ordinary space. In this connection it must be noted that the parity π of a state is determined not only by the orbital motion, but also by the intrinsic parity of the particles. Since the internal wave functions of the particles will not be written out explicitly, the operators L will be either scalars or pseudoscalars, depending on whether the product of the intrinsic parities of all particles taking part in the reaction is positive or negative. The angular operators are Hermitian:

$$L^{fi*}_{J\pi\nu'\nu}(\mathbf{k}'\alpha', \mathbf{k}\alpha) = L^{if}_{J\pi\nu\nu'}(\mathbf{k}\alpha, \mathbf{k}'\alpha').$$

Since the J-vectors $\psi^i{}_{JM\pi\nu}$ are orthogonal and normalized[1] by the condition

$$\sum_\alpha \int d\mathbf{k}\, \psi^{i*}_{J'M'\pi'\nu'}(\mathbf{k}\alpha)\, \psi^{i}_{JM\pi\nu}(\mathbf{k}\alpha) = 4\pi\delta_{J'J}\delta_{M'M}\delta_{\nu'\nu}$$

(and similarly for the $\psi^f{}_{JM\pi\nu}$), the angular polynomials are orthogonal and normalized in the following way

$$\sum_{\alpha'} \int d\mathbf{k}' L^{fi'*}_{J'\pi'\nu''\nu'''}(\mathbf{k}'\alpha', \mathbf{k}''\alpha'')\, L^{fi}_{J\pi\nu'\nu}(\mathbf{k}'\alpha', \mathbf{k}\alpha) = 4\pi\delta_{J'J}\delta_{\nu'\nu''} L^{i'i}_{J\pi''\nu'''\nu}(\mathbf{k}''\alpha'', \mathbf{k}\alpha). \tag{2}$$

We now consider two methods for analyzing the reaction $a + b \to a' + b'$, which differ in the choice of the quantum numbers ν, ν' which describe the system in its initial and final states. We shall distinguish the equations corresponding to these choices by the numbers I and II.

I. The initial state of the system is specified by the total angular momentum J, the parity π, the total angular momentum j of the incident particle a, and its orbital angular momentum l. The angular momentum J is the vector sum of the angular momentum j of particle a and the spin σ of particle b: $\mathbf{J} = \mathbf{j} + \boldsymbol{\sigma}$; the angular momentum j of the particle is the vector sum of its spin s and orbital angular momentum l: $\mathbf{j} = \mathbf{s} + \mathbf{l}$. The final state of the system is described by analogous quantities with primes. Thus $\nu \equiv jl$, $\nu' \equiv j'l'$.

II. The initial state of the system is specified by the total angular momentum J, the parity π, the spin S of the system, and the orbital angular momentum l. The angular momentum J is the vector sum of the spin S and the orbital angular momentum l: $\mathbf{J} = \mathbf{S} + \mathbf{l}$. The spin S of the system is the vector sum of the spins s and σ: $\mathbf{S} = \mathbf{s} + \boldsymbol{\sigma}$. The final state of the system is described by the analogous quantities with primes. Thus $\nu \equiv Sl$, $\nu' \equiv S'l'$.

For these two cases the J-vectors ψ can be represented in the form

$$\psi_{JMjl}(\mathbf{k}\alpha\gamma) = \sum_\mu C^{jM-\mu,\,\sigma\mu}_{JM} D_{jlM-\mu}(\mathbf{k}\gamma)\, Q_{\sigma\mu}(\alpha), \quad \psi_{JMSl}(\mathbf{k}\beta) = \sum_\mu C^{lM-\mu,\,S\mu}_{JM} Y_{lM-\mu}(\mathbf{k})\, Q_{S\mu}(\beta),$$

where $C^{jM-\mu,\sigma\mu}_{JM}$ are the Clebsch-Gordan coefficients, and Y_{lm}, D_{jlm}, $Q_{\sigma\mu}$, and $Q_{S\mu}$ are J-vectors that are eigenfunctions of the operators of the orbital angular momentum, the total angular momentum of the particle, the spin of the nucleus, and the spin of the system; Y_{lm} are ordinary spherical harmonics, and $D_{jlm}(\mathbf{k}\gamma)$ are spherical harmonics with spin

$$D_{jlm}(\mathbf{k}\gamma) = \sum_\mu C^{lm-\mu,\,s\mu}_{jm} Y_{lm-\mu}(\mathbf{k})\, Q_{s\mu}(\gamma) = \sum_\mu D^{s\mu}_{jlm}(\mathbf{k})\, Q_{s\mu}(\gamma), \tag{3}$$

$l = |j - s|, \ldots, j + s$; $Q_{\sigma\mu}$, $Q_{s\mu}$, and $Q_{S\mu}$ are spin functions depending on the respective spin variables α, γ, β. All of these functions are orthogonal and normalized in the following way:

$$\int Y^*_{l'm'}(\mathbf{k})\, Y_{lm}(\mathbf{k})\, d\mathbf{k} = 4\pi\delta_{l'l}\delta_{m'm},$$

$$\sum_\gamma \int D^*_{j'l'm'}(\mathbf{k}\gamma)\, D_{jlm}(\mathbf{k}\gamma)\, d\mathbf{k} = 4\pi\delta_{j'j}\delta_{l'l}\delta_{m'm}, \qquad \sum_\alpha Q^*_{\sigma\mu'}(\alpha)\, Q_{\sigma\mu}(\alpha) = \delta_{\mu'\mu}.$$

Consequently the angular operators for the transitions $Jjl \to Jj'l'$ and $JSl \to JS'l'$ are given by

$$L_{Jj'l'jl}(\mathbf{k}'\alpha'\gamma', \mathbf{k}\alpha\gamma) = \sum_{\mu'\lambda'\mu\lambda} L^{\mu'\lambda',\mu\lambda} Q_{\sigma'\mu'}(\alpha')\, Q_{s'\lambda'}(\gamma')\, Q^*_{\sigma\mu}(\alpha)\, Q^*_{s\lambda}(\gamma), \tag{4.I}$$

$$L_{JS'l'Sl}(\mathbf{k}'\beta', \mathbf{k}\beta) = \sum_{\mu'\mu} L^{\mu'\mu} Q_{S'\mu'}(\beta')\, Q^*_{S\mu}(\beta), \quad \text{or} \tag{4.II}$$

$$L^{\mu'\lambda',\mu\lambda}_{Jj'l'jl}(\mathbf{k}'\mathbf{k}) = \sum_M C_{JM}^{j'M-\mu',\,\sigma'\mu'} C_{JM}^{jM-\mu,\,\sigma\mu} D^{s'\lambda'}_{j'l'M-\mu'}(\mathbf{k}')\, D^{s\lambda*}_{jlM-\mu}(\mathbf{k}), \tag{5.I}$$

$$L^{\mu'\mu}_{JS'l'Sl}(\mathbf{k}'\mathbf{k}) = \sum_M C_{JM}^{l'M-\mu',\,S'\mu'} C_{JM}^{lM-\mu,\,S\mu} Y_{l'M-\mu'}(\mathbf{k}')\, Y^*_{lM-\mu}(\mathbf{k}). \tag{5.II}$$

It is obvious that $L^{\mu'\lambda',\mu\lambda}$ and $L^{\mu'\mu}$ are matrix elements of the operators L in the spin-function representation and correspond to various components μ, μ', λ, λ', of the spins of the particles (I), or to various components μ, μ', of the spin of the system (II), in the initial and final states. The products of spin functions Q in (4) can be expressed in terms of the Hermitian operators T connecting the spin space of particles a, b with that of the particles a', b' (*cf.* Ref. 3, p. 70). Therefore Eqs. (4) and (5) make it possible to find the angular polynomials L in explicit form in terms of invariant products of these operators T and the vectors **k**, **k'**. This is done below for reactions with the spin of the system equal to 0, ½, and 1, and for analogous reactions with γ-quanta. For definiteness we shall talk about the scattering and photoproduction of mesons on nucleons, the scattering of photons and nucleons by nucleons, and also the reactions $\mathfrak{N} + \mathfrak{N} \to \pi + D$, $\gamma + D \to p + n$, and $\gamma + D \to \pi + D$.

2. SCATTERING OF MESONS BY NUCLEONS

For this reaction $S = S' = ½$, so that for this reaction Eq. (5.II) is

$$L^{\mu'\mu}_{Jl'l}(\mathbf{k}'\mathbf{k}) = \sum_M C_{JM}^{l'M-\mu';\,{}^1\!/_2\mu'} C_{JM}^{lM-\mu;\,{}^1\!/_2\mu} Y_{l'M-\mu'}(\mathbf{k}')\, Y^*_{lM-\mu}(\mathbf{k}).$$

Since $Q_{½\mu}(\alpha) = \delta_{\alpha\mu}$, it is not hard to see that the angular polynomial $L_{Jl'l}(\mathbf{k}'\alpha', \mathbf{k}\alpha)$ can be put in the form

$$L = L^{{}^1\!/_2{}^1\!/_2}\frac{1}{2}(1+\sigma_z) + L^{{}^1\!/_2-{}^1\!/_2}\frac{1}{2}(\sigma_x + i\sigma_y) + L^{-{}^1\!/_2{}^1\!/_2}\frac{1}{2}(\sigma_x - i\sigma_y) + L^{-{}^1\!/_2-{}^1\!/_2}\frac{1}{2}(1-\sigma_z). \tag{6}$$

If we choose the z axis along k, then $Y^*_{lM-\mu}(\mathbf{k}) \neq 0$ only for $M - \mu = 0$ (then $Y^*_{l0}(1) = \sqrt{2l+1}$), so that $M = \mu$, and $M - \mu' = \mu - \mu'$. Since μ and μ' take the values $\pm ½$, we have $M - \mu' = 0, \pm 1$. This means that the matrix L contains $Y_{l'0}(\mathbf{k}')$ and $Y_{l',\pm1}(\mathbf{k}')$. Let us choose the x axis in the plane of the vectors k, k'. Then

$$Y_{l'0} = \sqrt{2l'+1}\, P_{l'}(\mathbf{k}'\mathbf{k}),\; Y_{l',\pm1} = \mp\sqrt{\frac{2l'+1}{l'(l'+1)}}\, k'_x P_{l'}(\mathbf{k}'\mathbf{k}).$$

The result is

$$\begin{aligned} L = {} & \frac{1}{2}(1+\sigma_z)\sqrt{(2l+1)(2l'+1)}\, C_{J{}^1\!/_2}^{l'0;\,{}^1\!/_2{}^1\!/_2} C_{J{}^1\!/_2}^{l0;\,{}^1\!/_2{}^1\!/_2} P_{l'} \\ & + \frac{1}{2}(\sigma_x + i\sigma_y)\sqrt{\frac{(2l+1)(2l'+1)}{l'(l'+1)}}\, C_{J-{}^1\!/_2}^{l'-1;\,{}^1\!/_2{}^1\!/_2} C_{J-{}^1\!/_2}^{l0;\,{}^1\!/_2-{}^1\!/_2} k'_x P'_{l'} \\ & - \frac{1}{2}(\sigma_x - i\sigma_y)\sqrt{\frac{(2l+1)(2l'+1)}{l'(l'+1)}}\, C_{J{}^1\!/_2}^{l'1;\,{}^1\!/_2-{}^1\!/_2} C_{J{}^1\!/_2}^{l0;\,{}^1\!/_2{}^1\!/_2} k'_x P'_{l'} \\ & + \frac{1}{2}(1-\sigma_z)\sqrt{(2l+1)(2l'+1)}\, C_{J-{}^1\!/_2}^{l'0;\,{}^1\!/_2-{}^1\!/_2} C_{J-{}^1\!/_2}^{l0;\,{}^1\!/_2-{}^1\!/_2} P_{l'}. \end{aligned} \tag{7}$$

According to the rules of vector addition l, l', and J can be related to each other in the following ways:

$$J = l + 1/_2 = l' + 1/_2,\ J = l + 1/_2 = l' - 1/_2,\ J = l - 1/_2 = l' + 1/_2,$$
$$J = l - 1/_2 = l' - 1/_2.$$

Furthermore, since the parity of the meson-nucleon system must be conserved, we have $(-1)^{l+1} = (-1)^{l'+1}$. Therefore only two cases are possible: $J = l + 1/2 = l' + 1/2$ and $J = l - 1/2\ l' - 1/2$. In both cases $l' = l$. Taking the Clebsch-Gordan coefficients for these two cases from tables[3] and substituting them into Eq. (7), we get

$$\begin{aligned} &J = l + 1/_2 = l' + 1/_2,\ L = (l+1)P_l + i\sigma_y k'_x P'_l = (l+1)P_l - i(\boldsymbol{\sigma}[\mathbf{k}'\mathbf{k}])P'_l; \\ &J = l - 1/_2 = l' - 1/_2,\ L = lP_l - i\sigma_y k'_x P'_l = lP_l + i(\boldsymbol{\sigma}[\mathbf{k}'\mathbf{k}])P'_l, \end{aligned} \tag{8}$$

since $\sigma_y k'_x = \sigma_y k'_x - \sigma_x k'_y = -[\boldsymbol{\sigma}\mathbf{k}']_z = -(\sigma[\mathbf{k}'\mathbf{k}])$. It must be pointed out that the angular polynomials (8) were first obtained by Tamm, Gol'fand, and Fainberg[4] from other considerations. These polynomials correspond to the case in which the parity of the incident and scattered mesons is the same. It is not hard to show that if the parities of the incident and scattered mesons are different, the corresponding polynomials differ from those obtained above only by the factor $-(\sigma\mathbf{k}')$:

$$\begin{aligned} J = l + 1/_2 = l' - 1/_2,\ L_{Jl'l}(\mathbf{k}'\mathbf{k}) = -(\boldsymbol{\sigma}\mathbf{k}')[(l+1)P_l - i(\boldsymbol{\sigma}[\mathbf{k}'\mathbf{k}])P'_l], \\ J = l - 1/_2 = l' + 1/_2,\ L_{Jl'l}(\mathbf{k}'\mathbf{k}) = -(\boldsymbol{\sigma}\mathbf{k}')[lP_l + i(\boldsymbol{\sigma}[\mathbf{k}'\mathbf{k}])P'_l]. \end{aligned} \tag{8'}$$

Since $(\sigma\mathbf{k}')^2 = 1$, the differential cross-section remains invariant with respect to a change of parity of the incident or scattered mesons. This fact is known as Minami's theorem.[5]

3. THE PHOTOPRODUCTION OF MESONS ON NUCLEONS

In this case $\sigma = \sigma' = 1/2$, $s = 1$, $s' = 0$, so that $j' = l'$. Before applying Eq. (4.I), we note the following facts:

a) In the case under consideration, with $s = 1$, the tensor $D_{jlm}(\mathbf{k}, \gamma)$ which appeared in Eq. (4.I) is commonly called a spherical vector and denoted by $\mathbf{D}_{jlm}(\mathbf{k})$ (the components of this vector are the values of the tensor $D_{jlm}(\mathbf{k}, \gamma)$ at the three "points" γ). The state of a photon with momentum $\mathbf{k}$ and angular momentum j is described by a vector $\mathbf{D}_{jm}(\mathbf{k})$ which is in general a linear combination of three spherical vectors $\mathbf{D}_{jlm}(\mathbf{k})$, $l = j$, $j \pm 1$, given by Eq. (3). But the spherical vector $\mathbf{D}_{jm}(\mathbf{k})$ must satisfy the condition of transversality, $(\mathbf{D}_{jm}(\mathbf{k})\mathbf{k}) = 0$. It can be shown[1] that from the three spherical vectors $\mathbf{D}_{jlm}$ we can construct two transverse vectors $\mathbf{D}^{(0)}_{jm}$, $\mathbf{D}^{(1)}_{jm}$ and one longitudinal vector $\mathbf{D}^{(-1)}_{jm}$, with

$$\mathbf{D}^{(0)}_{jm} = \mathbf{D}_{jjm},\ \mathbf{D}^{(1)}_{jm} = i[\mathbf{D}_{jjm}\mathbf{k}].$$

The tensors $\mathbf{D}^{(1)}_{jm}$ and $\mathbf{D}^{(0)}_{jm}$ have parities $(-1)^j$ and $(-1)^{j+1}$ [as can be seen from Eq. (3)] and correspond to states of electric and magnetic types. In connection with what has been said, in Eq. (4.I) one must understand the tensor $\mathbf{D}_{jlm}$ to mean the tensor $\mathbf{D}^{(1)}_{jm}$ or the tensor $\mathbf{D}^{(-1)}_{jm}$, depending on the parity of the state.

b) It is usually agreed to characterize the state of a photon not by the component of the spin along a chosen axis, but by the direction of a polarization vector $\mathbf{e} \perp \mathbf{k}$. Then in Eq. (4.I) instead of the product $D^{s\lambda}_{jlm} Q_{s\lambda}$ one must take $(\mathbf{D}^{(1)}_{jm}\mathbf{e})$ or $(\mathbf{D}^{(0)}_{jm}\mathbf{e})$ for the electric and magnetic radiations respectively. In connection with this, in the orthogonality-normalization condition (2), instead of summation over the variable γ' one must carry out an integration over the directions of the vector $\mathbf{e}'$, taking into account the statistical weight, *i.e.*, take the integral over $d\mathbf{e}' \cdot 2/4\pi$.

c) Finally, it can be shown that

$$\mathbf{D}^{(1)}_{jm}(\mathbf{k}) = \frac{1}{\sqrt{j(j+1)}}\left(\frac{\partial}{\partial\mathbf{k}} - \mathbf{k}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{k}}\right)\right)Y_{jm}(\mathbf{k}),$$

$$\mathbf{D}^{(0)}_{jm}(\mathbf{k}) = -\frac{i}{\sqrt{j(j+1)}}\left[\mathbf{k}\frac{\partial}{\partial\mathbf{k}}\right]Y_{jm}(\mathbf{k}),$$

where the vector **k** is to be taken to be a unit vector

$$(\mathbf{e}\mathbf{D}^{(1)}_{jm}) = \frac{1}{\sqrt{j(j+1)}}\left(\mathbf{e}\frac{\partial}{\partial \mathbf{k}}\right) Y_{jm}(\mathbf{k}),$$
$$(\mathbf{e}\mathbf{D}^{(0)}_{jm}) = \frac{1}{\sqrt{j(j+1)}}\left(i\,[\mathbf{k}\mathbf{e}]\frac{\partial}{\partial \mathbf{k}}\right) Y_{jm}(\mathbf{k}), \tag{9}$$

In what follows we shall call the operator $(\mathbf{a}\partial/\partial\mathbf{k})$ the "polarization" operator. Then the construction of the operators for the reaction $\gamma + b \to a' + b'$ reduces to the "polarization" of the operators for the reaction $a + b \to a' + b'$, where a is a scalar or pseudoscalar particle with spin 0.

Thus we shall start with the formula

$$L^{\mu'\mu} = \frac{1}{\sqrt{j(j+1)}}\left(\mathbf{e}\frac{\partial}{\partial \mathbf{k}}\right)^* \sum_M C_{JM}^{l'M-\mu';\,{}^1\!/_2\mu'} C_{JM}^{jM-\mu;\,{}^1\!/_2\mu} Y_{l'M-\mu'}(\mathbf{k}')\, Y^*_{jM-\mu}(\mathbf{k}),$$

in which for the magnetic radiation one must replace **e** by $i[\mathbf{ke}]$. Therefore, if we recall the rule for vector addition: $J = j + 1/2 = l' + 1/2$, $J = j - 1/2 = l' + 1/2$, $J = j + 1/2 = l' - 1/2$, $J = j - 1/2 = l' - 1/2$, and the conservation of parity: $(-1)^j = (-1)^{l'+1}$, $(-1)^{j+1} = (-1)^{l'+1}$ for the electric and magnetic radiations respectively, the construction of the polynomials L reduces to the magnetic and electric "polarization" of the operators (8) and (8′) respectively. The angular polynomials have the following forms:

$$J = j + {}^1\!/_2 = l' + {}^1\!/_2,$$
$$\frac{1}{\sqrt{j(j+1)}}\{P'_j[-(\sigma[\mathbf{k}'\mathbf{s}]) - i(j+1)(\mathbf{k}'\mathbf{s})] - P''_j(\sigma[\mathbf{k}'\mathbf{k}])(\mathbf{k}'\mathbf{s})\},$$
$$J = j - {}^1\!/_2 = l' + {}^1\!/_2,$$
$$-\frac{1}{\sqrt{j(j+1)}}\{P'_j[(\sigma\mathbf{e}) + (j-1)(\sigma\mathbf{k}')(\mathbf{k}'\mathbf{e})] + P''_j[(\sigma\mathbf{k})(\mathbf{k}'\mathbf{e}) - (\sigma\mathbf{k}')(\mathbf{k}'\mathbf{e})(\mathbf{k}'\mathbf{k})]\}, \tag{10}$$
$$J = j + {}^1\!/_2 = l' - {}^1\!/_2,$$
$$\frac{1}{\sqrt{j(j+1)}}\{P'_j[(\sigma\mathbf{e}) - (j+2)(\sigma\mathbf{k}')(\mathbf{k}'\mathbf{e})] + P''_j[(\sigma\mathbf{k})(\mathbf{k}'\mathbf{e}) - (\sigma\mathbf{k}')(\mathbf{k}'\mathbf{e})(\mathbf{k}'\mathbf{k})]\},$$
$$J = j - {}^1\!/_2 = l' - {}^1\!/_2,$$
$$\frac{1}{\sqrt{j(j+1)}}\{P'_j[(\sigma[\mathbf{k}'\mathbf{s}]) - ij(\mathbf{k}'\mathbf{s})] + P''_j(\sigma[\mathbf{k}'\mathbf{k}])(\mathbf{k}'\mathbf{s})\},$$

where $\mathbf{s} = [\mathbf{ke}]$. From the conservation of parity it can be seen that the polynomials of the 1st and 4th types correspond to magnetic radiation, and those of the 2nd and 3rd types to electric radiation.

The angular distributions, averaged over the spins of the initial states and summed over the spins of the final states, are given by the quantity $\frac{1}{2}\,\mathrm{Sp}\,|L|^2$:

$$J = j + {}^1\!/_2 = l' + {}^1\!/_2,$$
$$\frac{1}{j(j+1)}\{[1 + j(j+2)(\mathbf{k}'\mathbf{s})^2]P'^2_j - j(j+1)(\mathbf{k}'\mathbf{s})^2 P_j P''_j\};$$
$$J = j - {}^1\!/_2 = l' + {}^1\!/_2,$$
$$\frac{1}{j(j+1)}\{[1 + (j^2-1)(\mathbf{k}'\mathbf{e})^2]P'^2_j - j(j+1)(\mathbf{k}'\mathbf{e})^2 P_j P''_j\};$$
$$J = j + {}^1\!/_2 = l' - {}^1\!/_2, \tag{11}$$
$$\frac{1}{j(j+1)}\{[1 + j(j+2)(\mathbf{k}'\mathbf{e})^2]P'^2_j - j(j+1)(\mathbf{k}'\mathbf{e})^2 P_j P''_j\};$$
$$J = j - {}^1\!/_2 = l' - {}^1\!/_2,$$
$$\frac{1}{j(j+1)}\{[1 + (j^2-1)(\mathbf{k}'\mathbf{s})^2]P'^2_j - j(j+1)(\mathbf{k}'\mathbf{s})^2 P_j P''_j\}.$$

From these formulas it can be seen that the angular distributions with the same J and j differ only by rotation by 90° around the axis **k**: in fact, on such a rotation $(\mathbf{k}'\mathbf{e})^2$ goes over into $(\mathbf{k}'\mathbf{s})^2$, and vice versa. On averaging over the polarizations

$$\overline{(\mathbf{k}'\mathbf{e})^2} = \overline{(\mathbf{k}'\mathbf{s})^2} = {}^1/_2\,(1 - (\mathbf{k}'\mathbf{k})^2)$$

and the angular distributions of reactions with the same J and j are identical, as was noted by Feld in Ref. 6.

4. SCATTERING OF PHOTONS BY NUCLEONS

In this case $\sigma = \sigma' = 1/2$, $s = s' = 1$. In the construction of the polynomials of this type we shall start from Eq. (4.I) with the same changes as in the case of the photoproduction of mesons – namely, in place of the products $D^{s\lambda}_{jlm}\,Q_{s\lambda}$ we shall use the scalar products (9) for the electric and magnetic radiations respectively. Thus we shall start with the formula

$$L^{\mu'\mu} = \frac{1}{\sqrt{j'(j'+1)\,j\,(j+1)}}\left(\mathbf{e}'\,\frac{\partial}{\partial\mathbf{k}'}\right)\left(\mathbf{e}\,\frac{\partial}{\partial\mathbf{k}}\right)^{*}\sum_{M} C_{JM}^{j'M-\mu';\ {}^1/_2\mu'}\,C_{JM}^{jM-\mu;\ {}^1/_2\mu}\,Y_{j'M-\mu'}(\mathbf{k}')\,Y^{*}_{jM-\mu}(\mathbf{k})$$

(in this form it corresponds to the absorption and emission of electric radiation). The construction of the polynomials is carried out just as in Secs. 2 and 3, taking account of the vector addition of the angular momenta: $J = j = 1/2 = j' + 1/2$, $J = j - 1/2 = j' + 1/2$, $J = j + 1/2 = j' - 1/2$, $J = j - 1/2 = j' - 1/2$, and the conservation of parity. As the result we get for the polynomials corresponding to the absorption of electric quanta the following expressions

$$\begin{aligned}
&J = j + {}^1/_2 = j' + {}^1/_2,\quad \frac{1}{j\,(j+1)}\{[(j+1)(\mathbf{e}'\mathbf{e}) - i\,(\boldsymbol{\sigma}\,[\mathbf{e}'\mathbf{e}])]\,P_j'\\
&+ [(j+1)(\mathbf{k}'\mathbf{e})(\mathbf{e}'\mathbf{k}) + i\,(\boldsymbol{\sigma}\mathbf{k})(\mathbf{k}'[\mathbf{e}'\mathbf{e}]) + i\,(\boldsymbol{\sigma}\mathbf{k}')([\mathbf{e}'\mathbf{e}]\,\mathbf{k}) - i\,(\boldsymbol{\sigma}\,[\mathbf{k}'\mathbf{k}])(\mathbf{e}'\mathbf{e})\\
&- 2i\,(\boldsymbol{\sigma}\,[\mathbf{e}'\mathbf{e}])(\mathbf{k}'\mathbf{k})]\,P_j'' - i\,(\boldsymbol{\sigma}\,[\mathbf{k}'\mathbf{k}])(\mathbf{e}'\mathbf{k})(\mathbf{k}'\mathbf{e})\,P_j'''\};\\
&J = j - {}^1/_2 = j' + {}^1/_2,\quad \frac{i}{j\sqrt{j^2-1}}\{-(j-1)[(\boldsymbol{\sigma}\mathbf{s}')(\mathbf{k}'\mathbf{e}) + (\boldsymbol{\sigma}\mathbf{k}')(\mathbf{s}'\mathbf{e})]\,P_j'\\
&+ [-(i-2)(\boldsymbol{\sigma}\mathbf{k}')(\mathbf{s}'\mathbf{k})(\mathbf{k}'\mathbf{e}) - (\boldsymbol{\sigma}\mathbf{k})(\mathbf{s}'\mathbf{e}) - (\boldsymbol{\sigma}\mathbf{e})(\mathbf{s}'\mathbf{k}) + (\boldsymbol{\sigma}\mathbf{k}')(\mathbf{s}'\mathbf{e})(\mathbf{k}'\mathbf{k})\\
&+ (\boldsymbol{\sigma}\mathbf{s}')(\mathbf{k}'\mathbf{e})(\mathbf{k}'\mathbf{k})]\,P_j'' - (\boldsymbol{\sigma}\,[[\mathbf{k}'\mathbf{k}]\,\mathbf{k}'])(\mathbf{k}'\mathbf{e})(\mathbf{s}'\mathbf{k})\,P_j'''\};\\
&J = j + {}^1/_2 = j' - {}^1/_2,\quad \frac{i}{(j+1)\sqrt{j\,(j+2)}}\{-(j+2)[(\boldsymbol{\sigma}\mathbf{s}')(\mathbf{k}'\mathbf{e}) + (\boldsymbol{\sigma}\mathbf{k}')(\mathbf{s}'\mathbf{e})]\,P_j'\\
&+ [-(j+3)(\boldsymbol{\sigma}\mathbf{k}')(\mathbf{s}'\mathbf{k})(\mathbf{k}'\mathbf{e}) + (\boldsymbol{\sigma}\mathbf{k})(\mathbf{s}'\mathbf{e}) + (\boldsymbol{\sigma}\mathbf{e})(\mathbf{s}'\mathbf{k}) - (\boldsymbol{\sigma}\mathbf{k}')(\mathbf{s}'\mathbf{e})(\mathbf{k}'\mathbf{k})\\
&- (\boldsymbol{\sigma}\mathbf{s}')(\mathbf{k}'\mathbf{e})(\mathbf{k}'\mathbf{k})]\,P_j'' + (\boldsymbol{\sigma}\,[[\mathbf{k}'\mathbf{k}]\,\mathbf{k}'])(\mathbf{k}'\mathbf{e})(\mathbf{s}'\mathbf{k})\,P_j'''\};\\
&J = j - {}^1/_2 = j' - {}^1/_2,\quad \frac{1}{j\,(j+1)}\{[j\,(\mathbf{e}'\mathbf{e}) + i\,(\boldsymbol{\sigma}\,[\mathbf{e}'\mathbf{e}])]\,P_j'\\
&+ [j\,(\mathbf{k}'\mathbf{e})(\mathbf{e}'\mathbf{k}) - i\,(\boldsymbol{\sigma}\mathbf{k})(\mathbf{k}'[\mathbf{e}'\mathbf{e}]) - i\,(\boldsymbol{\sigma}\mathbf{k}')([\mathbf{e}'\mathbf{e}]\,\mathbf{k}) + i\,(\boldsymbol{\sigma}\,[\mathbf{k}'\mathbf{k}])(\mathbf{e}'\mathbf{e})\\
&+ 2i\,(\boldsymbol{\sigma}\,[\mathbf{e}'\mathbf{e}])(\mathbf{k}'\mathbf{k})]\,P_j'' + i\,(\boldsymbol{\sigma}\,[\mathbf{k}'\mathbf{k}])(\mathbf{e}'\mathbf{k})(\mathbf{k}'\mathbf{e})\,P_j'''\}.
\end{aligned} \tag{12}$$

The polynomials corresponding to the absorption of magnetic quanta are obtained from (12) by the following replacements: in all the polynomials **e** is replaced by $-i[\mathbf{ke}]$, and in polynomials 1, and 4 $\mathbf{e}'$ is replaced by $i[\mathbf{k}'\mathbf{e}']$, while in polynomials 2 and 3 $i[\mathbf{k}'\mathbf{e}']$ is replaced by $\mathbf{e}'$.

The angular distributions for transitions of the 1st and 4th types, averaged over the spins of the initial states of the nucleon and summed over those of the final states, are given by

$$\frac{1}{j^2(j+1)^2}\{[j(j+2)(\mathbf{e}'\mathbf{e})^2+1]P_j'^2+2j(\mathbf{k}'\mathbf{e})(\mathbf{e}'\mathbf{k})(\mathbf{e}'\mathbf{e})P_j'P_j''$$
$$-j(j+1)(\mathbf{e}'\mathbf{k})^2(\mathbf{k}'\mathbf{e})^2P_j'P_j'''+[j(j+2)(\mathbf{k}'\mathbf{e})^2(\mathbf{e}'\mathbf{k})^2+1-(\mathbf{k}'\mathbf{k})^2]P_j''^2\};$$
$$\frac{1}{j^2(j+1)^2}\{[(j^2-1)(\mathbf{e}'\mathbf{e})^2+1]P_j'^2-2(j+1)(\mathbf{k}'\mathbf{e})(\mathbf{e}'\mathbf{k})(\mathbf{e}'\mathbf{e})P_j'P_j''$$
$$-j(j+1)(\mathbf{e}'\mathbf{k})^2(\mathbf{k}'\mathbf{e})^2P_j'P_j'''+[(j^2-1)(\mathbf{k}'\mathbf{e})^2(\mathbf{e}'\mathbf{k})^2+1-(\mathbf{k}'\mathbf{k})^2]P_j''^2\}.$$

The angular distributions for transitions of the 2nd and 3rd types are more cumbersome, and therefore are not written out here. We remark that the angular polynomials for the photoproduction of mesons and the scattering of photons by nucleons have been obtained previously in Ref. 2.

5. SCATTERING OF NUCLEONS BY NUCLEONS

In this case we shall use Eq. (4.II), in which $S = S' = 1$ for triplet transitions and $S = S' = 0$ for singlet transitions (as is well known, in the scattering of nucleons by nucleons the spin $\mathbf{S}$ of the system is conserved). The construction of the matrices L is again carried through as in Sec. 2; in doing so, we use instead of the matrices $\boldsymbol{\sigma}$ the matrices $\mathbf{S}$ of the spin of the system. Then the matrix L can be expressed in the form

$$L = L^{11}\frac{1}{2}(S_z^2+S_z)+L^{10}\frac{1}{\sqrt{2}}S_z(S_x+iS_y)+L^{1-1}\frac{1}{2}(S_x+iS_y)^2+$$
$$+L^{01}\frac{1}{\sqrt{2}}(S_x-iS_y)S_z+L^{00}(1-S_z^2)-L^{0-1}\frac{1}{\sqrt{2}}(S_x+iS_y)S\ +$$
$$+L^{-11}\frac{1}{2}(S_x-iS_y)^2-L^{-10}\frac{1}{\sqrt{2}}S_z(S_x-iS_y)+L^{-1-1}\frac{1}{2}(S_z^2-S_z).$$

By calculations taking account of the vector composition of angular momenta and of the conservation of parity we get:

A. *Triplet States*

$$\begin{gathered}
J=l+1=l'+1,\\
\frac{1}{l+1}\{(l+1)P_l+[2(\mathbf{k}'\mathbf{k})-(\mathbf{S}\mathbf{k}')(\mathbf{S}\mathbf{k})-(l+1)i(\mathbf{S}\mathbf{n})]P_l'-(\mathbf{S}\mathbf{n})^2P_l''\};\\
J=l+1=l'-1,\\
\frac{1}{\sqrt{(l+1)(l+2)}}\{[-2(l+1)^2+(2l+3)(l+1)(\mathbf{S}\mathbf{k}')^2]P_l\\
+[-2(l+2)(\mathbf{S}\mathbf{k}')(\mathbf{S}\mathbf{k})+(2l+3)(\mathbf{S}\mathbf{k}')^2(\mathbf{k}'\mathbf{k})+(l+2)i(\mathbf{S}\mathbf{n})\\
+2(\mathbf{k}'\mathbf{k})]P_l'-(\mathbf{S}\mathbf{n})^2P_l''\};\\
J=l-1=l'+1,\\
\frac{1}{\sqrt{l(l-1)}}\{[-2l^2+(2l-1)l(\mathbf{S}\mathbf{k}')^2]P_l+[2(l-1)(\mathbf{S}\mathbf{k}')(\mathbf{S}\mathbf{k})\\
-(2l-1)(\mathbf{S}\mathbf{k}')^2(\mathbf{k}'\mathbf{k})-(l-1)i(\mathbf{S}\mathbf{n})+2(\mathbf{k}'\mathbf{k})]P_l'-(\mathbf{S}\mathbf{n})^2P_l''\};\\
J=l-1=l'-1,\\
\frac{1}{l}\{-lP_l+[2(\mathbf{k}'\mathbf{k})-(\mathbf{S}\mathbf{k}')(\mathbf{S}\mathbf{k})+li(\mathbf{S}\mathbf{n})]P_l'-(\mathbf{S}\mathbf{n})^2P_l''\};\\
J=l=l'.\\
\frac{2l+1}{l(l+1)}\{l(l+1)P_l+[(\mathbf{S}\mathbf{k}')(\mathbf{S}\mathbf{k})-2(\mathbf{k}'\mathbf{k})]P_l'+(\mathbf{S}\mathbf{n})^2P_l''\}.
\end{gathered}\tag{13}$$

In these formulas $\mathbf{n} = [\mathbf{k}'\mathbf{k}]$. For the singlet states Eq. (5.II) reduces to $\sum_M Y^*_{lM}(\mathbf{k})\, Y_{lM}(\mathbf{k}')$.

B. *Singlet States*

$$J = l = l', \ S = 0, \ L = (2l + 1)\, P_l. \tag{14}$$

6. THE REACTION $\mathfrak{N} + \mathfrak{N} \to \pi + D$

Just as in Sec. 5 we shall use Eq. (4.II), in which we have for the triplet → triplet transitions $S = S' = 1$, and for singlet → triplet transitions $S = 0$, $S' = 1$. For pseudoscalar π mesons we get from the conservation of angular momentum and of parity four varieties in the case of triplet → triplet transitions: $J = l + 1 = l'$, $J = l = l' + 1$, $J = l = l' - 1$, $J = l - 1 = l'$, and for the singlet → triplet transitions two varieties: $J = l = l' + 1$, $J = l = l' - 1$. For the triplet → triplet transitions the spin functions in Eq. (4.II) can be conveniently expressed in terms of the spin matrices $\mathbf{S}$, as was done in Sec. 5.

Triplet → triplet:

$$\begin{gathered}
J = l + 1 = l', \\
\frac{1}{\sqrt{l+1}}\sqrt{\frac{2l+3}{l+2}}\{-(l+1)(\mathbf{S}\mathbf{k}')\,P_l - [(\mathbf{S}\mathbf{k}')(\mathbf{k}'\mathbf{k}) - i\,(l+2)(\mathbf{S}\mathbf{k}')(\mathbf{S}\mathbf{n})]\,P'_l - i\,(\mathbf{S}[\mathbf{n}\mathbf{k}'])(\mathbf{S}\mathbf{n})\,P''_l\};, \\
J = l = l' + 1, \\
\frac{1}{l}\sqrt{\frac{2l+1}{l+1}}\{[-(\mathbf{S}\mathbf{k}) + i\,(l-1)(\mathbf{S}\mathbf{n})(\mathbf{S}\mathbf{k}')]\,P'_l + i\,(\mathbf{S}\mathbf{n})(\mathbf{S}[\mathbf{n}\mathbf{k}'])\,P''_l\}, \\
J = l = l' - 1, \\
\frac{1}{l+1}\sqrt{\frac{2l+1}{l}}\{[-(\mathbf{S}\mathbf{k}) - i\,(l+2)(\mathbf{S}\mathbf{n})(\mathbf{S}\mathbf{k}')]\,P'_l + i\,(\mathbf{S}\mathbf{n})(\mathbf{S}[\mathbf{n}\mathbf{k}'])\,P''_l\}, \\
J = l - 1 = l', \\
\frac{1}{l}\sqrt{\frac{2l-1}{l-1}}\{l\,(\mathbf{S}\mathbf{k}')\,P_l - [(\mathbf{S}\mathbf{k}')(\mathbf{k}'\mathbf{k}) + i\,(l-1)(\mathbf{S}\mathbf{k}')(\mathbf{S}\mathbf{n})]\,P'_l - i\,(\mathbf{S}[\mathbf{n}\mathbf{k}'])(\mathbf{S}\mathbf{n})\,P''_l\}.
\end{gathered} \tag{15}$$

Here $\mathbf{n} = [\mathbf{k}'\mathbf{k}]$. In the case of the singlet → triplet transitions the spin functions can be conveniently expressed in terms of the matrices $\mathbf{T}$ (*cf.* Ref. 3, p. 70), the elements of which correspond to singlet ⇄ triplet transitions.

Singlet → triplet

$$\begin{gathered}
J = l = l' + 1, \ \sqrt{\frac{2l+1}{l}}\{[(\mathbf{T}\mathbf{k}) - (\mathbf{T}\mathbf{k}')(\mathbf{k}'\mathbf{k})]\,P'_l + l\,(\mathbf{T}\mathbf{k}')\,P_l\}, \\
J = l = l' - 1, \ \sqrt{\frac{2l+1}{l+1}}\{[(\mathbf{T}\mathbf{k}) - (\mathbf{T}\mathbf{k}')(\mathbf{k}'\mathbf{k})]\,P'_l - (l+1)(\mathbf{T}\mathbf{k}')\,P_l\}.
\end{gathered} \tag{16}$$

For the reaction $\mathfrak{N} + \mathfrak{N} \to \pi + D$ with pseudoscalar mesons, in the case of the singlet → triplet transitions the type $J = l = l'$ is forbidden because of parity. For completeness, however, we write out the corresponding angular operator

$$J = l = l', \ -\frac{2l+1}{\sqrt{l(l+1)}}\, i\,(\mathbf{T}[\mathbf{k}'\mathbf{k}])\,P'_l. \tag{17}$$

With this inclusion, the operators written out in Secs. 5 and 6 form a complete set of angular operators for reactions in which the total spin of the initial and final states is equal to zero or unity.*

7. THE REACTIONS $\gamma + D \to p + n$ AND $\gamma + D \to \pi + D$

1. Using Eqs. (4) and (5) and also the considerations presented in Sec. 3, one can show without difficulty that the basic formula for the construction of the angular operators for the reactions $\gamma + D \to p + n$ and $\gamma + D \to \pi + D$ is

$$L_{Jl'S'j1}(\mathbf{k}'\mathbf{k}) = \frac{1}{\sqrt{j(j+1)}}\left(\mathbf{a}\frac{\partial}{\partial \mathbf{k}}\right)^* \sum_M C_{JM}^{l'M-\mu';\, S'\mu'} C_{JM}^{jM-\mu;\, 1\mu}\, Y_{l'M-\mu'}(\mathbf{k}')\, Y^*_{jM-\mu}(\mathbf{k})\, Q_{S'\mu'}(\alpha')\, Q^*_{1\mu}(\alpha), \quad (18)$$

where $S' = 0$ or 1 for transitions into singlet or triplet states, and $\mathbf{a} = \mathbf{e}$ and $i[\mathbf{ke}]$ respectively for electric and magnetic quanta. Apart from the "polarization" operator $[j(j+1)]^{-1/2}(\mathbf{a}\partial/\partial\mathbf{k})$ the right-hand member of Eq. (18) agrees exactly with the formula (4.II) used in Secs. 5 and 6 for the construction of the operators for the reactions $\mathfrak{N} + \mathfrak{N} \to \mathfrak{N} + \mathfrak{N}$ and $\mathfrak{N} + \mathfrak{N} \to \pi + D$, if we just make the replacement in the latter formula of the orbital angular momentum l by the total angular momentum j of the photon. Therefore the angular operators for the reactions $\gamma + D \to p + n$ and $\gamma + D \to \pi + D$ are obtained from the operators of Sections 5 and 6 by the simple application of the "polarization" operator $[j(j+1)]^{-1/2}(\mathbf{a}\partial/\partial\mathbf{k})^*$. Here, however, one remark must be made. Equations (16) and (17) correspond to the singlet $\to$ triplet case. Therefore for the reaction $\gamma + D \to p + n$ one must go over from these formulas to the triplet $\to$ singlet case. For this purpose we can use the Hermitian adjoint relation $[L_{Jl'S'lS}(\mathbf{k}'\mathbf{k})]^+ L_{JlSl'S'}(\mathbf{k}\mathbf{k}')$ and the exchange of notations $lS\mathbf{k} \rightleftarrows l'S'\mathbf{k}'$. We get as the result:

Triplet → singlet

$$J = l + 1 = l'.\quad \sqrt{\frac{2l'+1}{l'}}\{[(\mathbf{Tk}') - (\mathbf{Tk})(\mathbf{k}'\mathbf{k})]P'_{l'} = l'(\mathbf{Tk})P_{l'}\},$$

$$J = l - 1 = l'.\quad \sqrt{\frac{2l'+1}{l'+1}}\{[(\mathbf{Tk}') - (\mathbf{Tk})(\mathbf{k}'\mathbf{k})]P'_{l'} - (l'+1)(\mathbf{Tk})P_{l'}\}, \quad (19)$$

$$J = l = l' \cdot \frac{2l'+1}{\sqrt{l'(l'+1)}}\, i(\mathbf{T}[\mathbf{kk}'])P'_{l'}. \quad (20)$$

Then, taking the conservation of parity into account for the reaction $\gamma + D \to p + n$, we get the angular operators in the form

$$\left.\begin{array}{ll} \frac{1}{\sqrt{j(j+1)}}\left(\mathbf{e}\frac{\partial}{\partial\mathbf{k}}\right)^* & (13)\ (\text{triplet} \to \text{triplet}) \\ \frac{1}{\sqrt{j(j+1)}}\left(\mathbf{e}\frac{\partial}{\partial\mathbf{k}}\right)^* & (20)\ (\text{triplet} \to \text{singlet}) \end{array}\right\}\ \text{absorption of an electric quantum}$$

$$\left.\begin{array}{ll} \frac{1}{\sqrt{j(j+1)}}\left(i[\mathbf{ke}]\frac{\partial}{\partial\mathbf{k}}\right)^* & (15)\ (\text{triplet} \to \text{triplet}) \\ \frac{1}{\sqrt{j(j+1)}}\left(i[\mathbf{ke}]\frac{\partial}{\partial\mathbf{k}}\right)^* & (19)\ (\text{triplet} \to \text{singlet}) \end{array}\right\}\ \text{absorption of a magnetic quantum}$$

and for the reaction $\gamma + D \to \pi + D$ we get them in the form

*If the spin of the system arises from the addition of the spins of particles of spin ½, then $\mathbf{S} = \frac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)$, $\mathbf{T} = \frac{1}{2}(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)$.

$$\frac{1}{\sqrt{j(j+1)}}\left(\mathbf{e}\frac{\partial}{\partial \mathbf{k}}\right)^* \quad (15),$$
$$\frac{1}{\sqrt{j(j+1)}}\left(i\,[\mathbf{k}\mathbf{e}]\,\frac{\partial}{\partial \mathbf{k}}\right)^* \quad (13).$$

We do not write these operators out in greater detail here.

8. CONCLUSION

A knowledge of the coefficients $S_{J\pi\nu'\nu}^{fi}$ in the expression of the scattering matrix in terms of the angular operators L [*cf.* the relation (A)] gives complete information about the scattering matrix. More complete evidence as to the behavior of these quantities can be obtained by the study of reactions with polarized particles. In this case the differential scattering cross-section is determined from the scattering matrix S by the formula

$$d\sigma/d\Omega = \mathrm{Sp}\,(S\rho S^+), \tag{21}$$

where ρ is the density matrix describing the polarization of the incident particles.[7,8] In the case of unpolarized particles $\rho = 1/(2s+1)$, and we have the usual definition of the differential cross-section. The polarization of the scattered particles is completely described by the density matrix ρ',

$$\rho' = S\rho S^+/\mathrm{Sp}\,(S\rho S^+), \tag{22}$$

so that the average value of any operator Ω in the spin space of the scattered particles is given by the relation

$$\langle\Omega\rangle' = \mathrm{Sp}\,(\rho'\Omega). \tag{23}$$

In virtue of the expansion (A) the differential cross-section $d\sigma/d\Omega$ and the average values of operators Ω can be obtained in terms of the coefficients $S^{fi}{}_{J\pi\nu'\nu}$ and known functions of the angles. Thus the coefficients $S^{fi}{}_{J\pi\nu'\nu}$ can be directly related to experimentally measured quantities.

The angular operators make it possible to carry out a generalized phase-shift analysis of the scattering amplitude (in this connection see Ref. 9), and to obtain the angular distributions and polarizations of the particles in the partial transitions. They are also very convenient for the separation of angular variables in various sorts of equations. For example, if the scattering amplitude obeys an integral equation of the Fredholm type, in which the integration is performed both over the energy and over angles, then, by expanding this amplitude, the kernel, and the inhomogeneous term of the equation in series of the angular operators and carrying out the integration over the angles, owing to the orthogonality condition (2) we obtain one-dimensional integral equations for the partial amplitudes.

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Translated by W. H. Furry
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