$$u_i = e_i \sqrt{\hbar/2\rho\omega}\; e^{ikr}$$

(**e** is the unit vector of the polarization of the sound wave). We note that the change in the chemical potential is taken into account by having $\lambda_{ij} - \bar{\lambda}_{ij}$, and not $\bar{\lambda}_{ij}$, enter into U_{12}.

Since $\hbar\omega \ll \mu_0$, $\hbar k \ll p_0$, then

$$n_1(1-n_2) - n_2(1-n_1) \approx -(\partial n/\partial \varepsilon)\,\hbar\omega,$$

$$\delta(\varepsilon_1 + \hbar\omega - \varepsilon_2) \approx \frac{s}{\hbar\omega v}\,\delta\left(\cos\theta - \frac{s}{v}\right),$$

where θ is the angle between the vectors **k** and $\mathbf{v} = \partial\epsilon/\partial\mathbf{p}$. Substituting this expression in Eq. (19), we get the value of γ given by Eq. (18). We note that the coincidence of the results of classical and quantum theory in the case $\omega\tau \gg s/v$ is essentially connected with the small value of s/v in comparison with unity.

In conclusion the authors express their gratitude for valuable discussions to L. D. Landau, I. M. Lifshitz and I. Ia. Pomeranchuk.

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Excitation of Rotational States in the Interaction Between Neutrons and Nuclei

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The excitation of rotational states in nuclei by neutrons is studied in the energy range from the threshold up to 1.5–2 Mev.

1. STATEMENT OF THE PROBLEM

WE SHALL INVESTIGATE interactions between neutrons and nonspherical nuclei and the excitation of rotational states, using an optical model which has been modified to take into account the nuclear deformation caused by the existence of rotational states.

On the usual optical model [1] the total scattering cross section is divided into two parts:

$$\sigma = \sigma_{se} + \sigma_c, \qquad (1)$$

where σ_{se} is the elastic scattering cross section for a spherically symmetrical complex potential, and σ_c includes both the cross section σ_{ce} for the formation of a compound nucleus with subsequent emission of particles of the same energy and the reaction cross section (Feshbach, Porter and Weisskopf call σ_c the cross section of compound nucleus formation, which is not quite correct because it includes the cross sections of direct expulsion processes and the excitation of collective motions which will be considered below). All these processes are described by the imaginary part of the complex potential.

For nonspherical nuclei this model must be modified as follows: 1) the complex potential must be nonspherical; 2) since the deformed nucleus is capable of rotational motion, the nonspherical potential of the optical model must be capable of a change of orientation.

In the interaction between a neutron and a nucleus different rotational levels can be excited, *i.e.*, the rotational velocity in the potential can change. We thus have a problem in which the variables which characterize the orientation of the nonspherical potential must be regarded as dynamical

variables. The Hamiltonian can then be put into the form

$$\hat{H} = -(\hbar^2/2m)\nabla^2 + \tilde{V}(\mathbf{r}, \theta_i) + \hat{T}_{\text{rot}}, \qquad (2)$$

where $\tilde{V}(\mathbf{r}, \theta_i)$ is the nonspherical complex potential, $\mathbf{r}$ is the neutron radius vector, θ_i are the Euler angles which define the directions of the nuclear deformation axes, and $\hat{T}_{\text{rot}}$ is the nuclear rotational energy operator which acts on the θ_i. Bohr[2] showed that for an axially symmetrical deformation the operator can be written in the form

$$\hat{T}_{\text{rot}} \equiv (\hbar^2/2J)[\hat{Q}^2 - Q_0(Q_0+1)], \qquad (3)$$

where $\hat{Q}^2$ is the operator of the square of the total nuclear angular momentum and $Q_0(Q_0+1)$ is its value in the unexcited state. By solving the Schroedinger equation

$$\hat{H}\psi(\mathbf{r}, \theta_i) = E\psi(\mathbf{r}, \theta_i), \qquad (4)$$

we can calculate the elastic scattering cross section σ_{se} for a nonspherical potential, the cross section σ_{rot} for the excitation of different rotational states, and the combined cross section $\tilde{\sigma}_c$ of the other processes which are associated with the imaginary part of the potential. On this model the total cross section is thus divided into three parts, *viz:*

$$\sigma = \sigma_{se} + \sigma_{\text{rot}} + \tilde{\sigma}_c. \qquad (5)$$

Just as σ_{se} is only the portion of the elastic scattering cross section that is not associated with the formation of a compound nucleus, σ_{rot} is only that portion of the excitation cross section which is not associated with the same process. However, the small value of the imaginary part of the potential, obtained by comparing the calculated and measured total neutron cross sections up to about 3 Mev, shows that for such energies the average probability of compound nucleus formation is small. It therefore seems that the excitation of rotational states in the interaction between a neutron and a nucleus does not proceed principally through the formation of a compound nucleus but rather through the direct transfer of energy to the collective motion. This means that in the indicated energy range σ_{rot} is the main portion of the excitation cross section for rotational states. We shall hereinafter limit ourselves to these energies.

The exact solution of (4) would evidently be very complicated. No solution has yet been obtained even for the simpler problem of scattering by a nonspherical potential without rotation. We shall therefore consider slightly deformed nuclei. More precisely, if

$$\tilde{V}(\mathbf{r}, \theta_i) = \tilde{V}_0(r) + \tilde{V}_1(\mathbf{r}, \theta_i);$$
$$V_0(r) = 0 \quad \text{for} \quad r > R_1; \qquad (6)$$
$$V_1(\mathbf{r}, \theta_i) = 0 \quad \text{for} \quad r < R_1, \; r > R_2,$$

we shall regard as small and shall neglect the quantities $(k\Delta R)^2$ and $(\Delta R/R_1)^2$, where $\Delta R = R_2 - R_1$, $k^2 = 2mE/\hbar^2$. If we do not claim high precision in calculating the cross section, these conditions are valid for the majority of nuclei in the given energy range. Whenever the conditions are not satisfied, our treatment will be of the nature of a limiting case.

We shall also assume that the nuclear deformations are axially symmetrical and thus $\tilde{V}_1(\mathbf{r}, \theta_i) = \tilde{V}(r, \vartheta)$, where ϑ is the angle between the radius vector and the deformation axis. This is well fulfilled for almost all deformed nuclei. Introducing for convenience

$$U(\mathbf{r}, \theta_i) = r\psi(\mathbf{r}, \theta_i); \; V_0(r) = (2m/\hbar^2)\tilde{V}_0(r);$$
$$V_1(r, \vartheta) = (2m/\hbar^2)\tilde{V}_1(r, \vartheta),$$

we obtain

$$[(d^2/dr^2) + k^2 - V_0(r) - V_1(r, \vartheta) - (\hat{l}^2/r^2) - (2m/\hbar^2)\hat{T}_{\text{rot}}]U(\mathbf{r}, \theta_i) = 0, \qquad (7)$$

where $\hat{l}^2$ is the operator of the square of the neutron angular momentum. Eq. (7) must be solved subject to the conditions

$$(k\Delta R)^2 \ll 1, \; (\Delta R/R_1)^2 \ll 1. \qquad (8)$$

We note that $k'\Delta R$ (where k' is the average wave vector of the neutron inside the nucleus) will not be regarded as small. Therefore the phase difference of the neutron waves traversing the nucleus parallel to the major and minor semi-axes can be large and the pattern of scattering by a nonspherical nucleus can thus differ essentially from that for a spherical nucleus. For real nuclei, for which we can expect to obtain a rotational spectrum, $k'\Delta R \gtrsim 1$; therefore for such nuclei the scattering is strongly influenced by the deformation and cannot be treated by perturbation theory.

We shall show that a solution can be obtained be-

cause when (8) is satisfied we can neglect in (7) the terms k^2, $2m\hat{T}_{\text{rot}}/\hbar^2$ and $\hat{l}^2/r^2$ in the region $R_1 < r < R_2$. The equation can then easily be solved in this region. The problem is then solved by smoothly joining this solution with the solutions for $r < R_1$ and $r > R_2$.

2. THE WAVE FUNCTION FOR $R_1 < r < R_2$

For the purpose of proving our last statement we write (7) in the form

$$U(\mathbf{r}, \theta_i) = U^0(\mathbf{r}, \theta_i) + \int G(\mathbf{r}, \theta_i; \mathbf{r}', \theta_i') V_1(\mathbf{r}', \vartheta') U(\mathbf{r}', \theta_i')\, dr'\, d\Omega'\, d\theta_i'; \quad (9)$$

$$U^0(\mathbf{r}, \theta_i) = \psi_{\mathbf{k}}(\mathbf{r})\, \tilde{D}^{Q_0}_{M_0 \varkappa}(\theta_i), \quad (10)$$

where $\tilde{D}^{Q_0}_{M_0\varkappa}$ is the eigenfunction of the operator $\hat{T}_{\text{rot}}$ for the ground state.

From the symmetry properties of the nuclear wave function it follows[2] that each state can be described by a superposition of the functions $\tilde{D}^{Q_0}_{M_0\varkappa}$ and $\tilde{D}^{Q_0}_{M_0-\varkappa}$. But it can be shown that this does not change the result.

Let $G(\mathbf{r}, \theta_i; \mathbf{r}', \theta_i)$ be the Green's function of (7) without the term $V_1(r, \vartheta)$. It is given by ($r, r' > R_1$):

$$G(\mathbf{r}, \theta_i; \mathbf{r}', \theta_i') = \sum_{Q, M} G_{k_Q}(\mathbf{r}, \mathbf{r}') \tilde{D}^{Q}_{M\varkappa}(\theta_i) \tilde{D}^{*Q}_{M\varkappa}(\theta_i'); \quad (11)$$

$$G_{k_Q}(\mathbf{r}, \mathbf{r}') = -\frac{1}{k_Q} \begin{cases} \sum_{l,m} j_l(k_Q r)\, h_l(k_Q r')\, Y_{lm}(\Omega)\, Y^*_{lm}(\Omega') & \text{for } r' > r; \\ \sum_{l,m} j_l(k_Q r')\, h_l(k_Q r)\, Y_{lm}(\Omega)\, Y^*_{lm}(\Omega') & \text{for } r' < r; \end{cases}$$

$$j_l(x) = \frac{1}{2i}\left[h_l(x) \exp(2i\delta_l^Q) - h_l^*(x)\right]; \quad h_l(x) = \sqrt{\frac{\pi x}{2}}\, H^{(1)}_{l+1/2}(x);$$

$$k_Q^2 = k^2 - (m/J)\left[Q(Q+1) - Q_0(Q_0+1)\right];$$

$$\psi_{\mathbf{k}}(\mathbf{r}) = \frac{4\pi}{k} \sum_{l,m} i^l j_l(kr)\, Y_{lm}(\Omega)\, Y^*_{lm}(\Omega_{\mathbf{k}}), \quad (10a)$$

where δ_l^Q is the complex scattering phase of a neutron with a wave vector k_Q by the complex potential $V_0(r)$.

For (9) in the region $R_1 < r < R_2$ and with (8) satisfied for all essential k_Q of the problem we can expand the functions $j_l(x)$ and $h_l(x)$ in (10) and (11) in powers of $k_Q(r - R_1)$, of which only the first two terms of the series are retained. In complete analogy to the calculations given in Ref. 3 we obtain for this similar problem

$$U(\mathbf{r}, \theta_i) = F^{(1)}(\Omega, \theta_i) + (r - R_1) F^{(2)}(\Omega, \theta_i) + \int_0^r (r - r') V_1(r', \vartheta)\, U(r', \Omega, \theta_i)\, dr';$$

$$F^{(1)}(\Omega, \theta_i) = U^0(R_1, \theta_i) + \int \left[G(\mathbf{R}_1, \theta_i'; \mathbf{R}_1', \theta_i')\, \xi(\Omega', \theta_i') + \frac{\partial G(\mathbf{R}_1, \theta_i; \mathbf{R}_1', \theta_i')}{\partial R_1'}\, \eta(\Omega', \theta_i') \right] d\Omega'\, d\theta_i'; \quad (12)$$

$$F^{(2)}(\Omega, \theta_i) = \frac{\partial U^0(\mathbf{R}_1, \theta_i)}{\partial R_1} + \int \left[\frac{\partial G(\mathbf{R}_1, \theta_i; \mathbf{R}_1', \theta_i')}{\partial R_1}\, \xi(\Omega', \theta_i') + \frac{\partial^2 G(\mathbf{R}_1, \theta_i; \mathbf{R}_1', \theta_i')}{\partial R_1 \partial R_1'}\, \eta(\Omega', \theta_1') \right] d\Omega'\, d\theta_i'. \quad (13)$$

$$G(\mathbf{R}_1, \theta_i; \mathbf{R}_1', \theta_i') = -\sum_{QMlm} \frac{1}{k_Q} \tilde{D}^{Q}_{M\varkappa}(\theta_i)\, \tilde{D}^{Q*}_{M\varkappa}(\theta_i') \times Y_{lm}(\Omega)\, Y^*_{lm}(\Omega')\, j_l(k_Q R_1) h_l(k_Q R_1');$$

$$\xi(\Omega, \theta_i) = \int_0^\infty V_1(r, \vartheta)\, U(\mathbf{r}, \theta_i)\, dr;$$

$$\eta(\Omega, \theta_i) = \int_0^\infty V_1(r, \vartheta)\, U(\mathbf{r}, \theta_i)\, (r - R_1)\, dr. \quad (14)$$

In the expansion it was considered that the joining conditions make $j_l(kR_1) \sim k/k'$ extremely small while $j_l' \sim 1$; therefore in the expansion of the

products $j_l h_l$ we consider terms of the order $j_l' h_l' (k\Delta R)^2$. Differentiating (12) twice with respect to r we obtain

$$d^2U(\mathbf{r}, \theta_i)/dr^2 = V_1(r, \vartheta)\, U(\mathbf{r}, \theta_i), \tag{15}$$

which is the desired proof.

In accordance with (15) we have

$$U(\mathbf{r}, \theta_i) = A(\Omega, \theta_i)\, \varphi_1(r, \vartheta) + B(\Omega, \theta_i)\, \varphi_2(r, \vartheta), \tag{16}$$

where φ_1 and φ_2 satisfy (15) and the conditions

$$\begin{aligned} &\varphi_1(R_1, \vartheta) = 1;\ \partial\varphi_1(R_1, \vartheta)/\partial R_1 = 0; \\ &\varphi_2(R_1, \vartheta) = 0;\ \partial\varphi_2(R_1, \vartheta)/\partial R_1 = 1. \end{aligned} \tag{17}$$

By joining (16) with the solutions for $r < R_1$ and $r > R_2$ we can obtain A and B and thus complete the solution. The usual method of joining requires that we obtain the derivative $\partial U/\partial r$, which is not desirable because when $U(\mathbf{r}, \theta_i)$ is calculated to terms of the order $k\Delta R$ inclusive, differentiation can reduce the accuracy by one degree. We therefore proceed as follows. According to (12)

$$A = F^{(1)},\quad B = F^{(2)}; \tag{18}$$

$U(\mathbf{r}, \theta_i)$ in the region $R_1 < r < R_2$ is therefore expressed in terms of $\xi(\Omega, \theta_i)$ and $\eta(\Omega, \theta_i)$, which in turn according to (14) depend on the behavior of $U(\mathbf{r}, \theta_i)$ for $R_1 < r < R_2$. Therefore by inserting (16) and (13) in (14) we obtain an equation for ξ and η.

In the next section it will be shown that the excitation amplitude can be expressed simply in terms of ξ and η. The equations for the latter are

$$\xi(\Omega, \theta_i) = F^{(1)}(\Omega, \theta_i)\, \alpha_1(\vartheta) + F^{(2)}(\Omega, \theta_i)\, \alpha_2(\vartheta), \tag{19}$$

$$\begin{aligned} \eta(\Omega, \theta_i) &= F^{(1)}(\Omega, \theta_i)\, \beta_1(\vartheta) + F^{(2)}(\Omega, \theta_i)\, \beta_2(\vartheta), \\ \alpha_{1,2}(\vartheta) &= \int V_1(r, \vartheta)\, \varphi_{1,2}(r, \vartheta)\, dr, \\ \beta_{1,2}(\vartheta) &= \int V_1(r, \vartheta)\, \varphi_{1,2}(r, \vartheta)\, (r - R_1)\, dr. \end{aligned} \tag{20}$$

For simple potentials $V_1(r, \vartheta)$ Eq. (15) can easily be solved; therefore we can assume that $\varphi_1(r, \vartheta)$, $\varphi_2(r, \vartheta)$ and thus $\alpha_{1,2}(\vartheta)$, $\beta_{1,2}(\vartheta)$ are known. The coefficients in (19) are thus determined and our problem is now the solution of the latter.

The pair of integral equations (19) can conveniently be converted into a pair of equations for the coefficients of the expansions of $\xi(\Omega, \theta_i)$ and $\eta(\Omega, \theta_i)$ in terms of the functions $Y_{lm}(\Omega)$ and $\tilde{D}^Q_{M\varkappa}(\theta_i)$, or more precisely for the quantities $\xi^I_{Ql;Q_0l_0}$ and $\eta^I_{Ql;Q_0l_0}$ defined by

$$\begin{aligned} \xi(\Omega, \theta_i) &= \frac{2\pi^{1/2}}{k} \sum_{\substack{l_0IQ \\ lm}} i^{l_0} \sqrt{2l_0 + 1}\, j_{l_0}(kR_1)\, C^{IM_0}_{QM;\, lm} \\ &\times C^{IM_0}_{Q_0M_0;\, l_00} Y_{lm}(\Omega) \tilde{D}^Q_{M\varkappa}(\theta_i) \xi^I_{Ql;Q_0l_0}; \\ \eta(\Omega, \theta_i) &= \frac{2\pi^{1/2}}{k} \sum_{\substack{l_0IQ \\ lm}} i^{l_0} \sqrt{2l_0 + 1}\, j_{l_0}(kR_1)\, C^{IM_0}_{QM;\, lm} \\ &\times C^{IM_0}_{Q_0M_0;\, l_00} Y_{lm}(\Omega) \tilde{D}^Q_{M\varkappa}(\theta_i) \eta^I_{Ql;Q_0l_0} \end{aligned} \tag{21}$$

$C^{IM}_{QM;lm}$ are the Clebsch–Gordan coefficients.

Inserting (21) into (19) we obtain

$$\begin{aligned} \xi^I_{Ql;Q_0l_0} &= K^I_{Ql;Q_0l_0} - \sum_{Q'l'} K^I_{Ql;Q'l'} \frac{j_{l'}h_{l'}}{k_{Q'}} [\xi^I_{Q'l';Q_0l_0} \\ &+ k_{Q'}\Phi_{l'}\eta^I_{Q'l';Q_0l_0}]; \end{aligned} \tag{22}$$

$$\begin{aligned} \xi^I_{Ql;Q_0l_0} &= \overline{K}^I_{Ql;Q_0l_0} - \sum_{Q'l'} \overline{K}^I_{Ql;Q'l'} \frac{j_{l'}h_{l'}}{k_{Q'}} [\xi^I_{Q'l';Q_0l_0} \\ &+ k_{Q'}\Phi_{l'}\eta_{Q'l';Q_0l_0}], \end{aligned}$$

$$\begin{aligned} K^I_{Ql;Q'l'} &= \sum_{\lambda} [(2l + 1)(2l' + 1)(2Q + 1)(2Q' + 1)]^{1/2} \\ &\times C^{\lambda 0}_{l0;\, l'0} C^{\lambda 0}_{Q\varkappa;\, Q'-\varkappa} W(ll'QQ';\lambda I)(-1)^{I-\varkappa} \tfrac{1}{2} [\alpha^{(\lambda)}_1 \\ &+ k_{Q'}\chi_{l'}\alpha^{(\lambda)}_2]; \\ j_l &= j_l(k_QR_1);\quad h_l = h_l(k_QR_1); \\ \Phi_l &= h_l'/h_l,\ \chi_l = j_l'/j_l; \end{aligned} \tag{23}$$

$W(abcd; ef)$ is the Racah coefficient;

$$\alpha^{(\lambda)}_{1,2} = \int_0^\pi \alpha_{1,2}(\vartheta)\, P_\lambda(\cos\vartheta) \sin\vartheta d\vartheta, \tag{24}$$

and $\overline{K}^I_{Ql;Q_0l_0}$ differs from $K^I_{Ql;Q_0l_0}$ through the replacement of $\alpha_{1,2}(\vartheta)$ by $\beta_{1,2}(\vartheta)$.

If instead of (22) we introduce $\zeta^I_{Ql;Q_0l_0}$ as defined by

$$\xi^I_{Ql;\,Q_0l_0} + k_Q\Phi_l\eta^I_{Ql,\,Q_0l_0} = \frac{k}{j_{l_0}h_{l_0}}\left[\delta_{ll_0}\delta_{QQ_0} + \frac{k_Q}{i_lh_l}\zeta^I_{Ql;\,Q_0l_0}\right], \quad (25)$$

then considering that $1/j_lh_l = \chi_l - \Phi_l$ we obtain

$$-k_Q(\chi_l - \Phi_l)\zeta^I_{Ql;\,Q_0l_0} = \delta_{QQ_0}\delta_{ll_0} + \sum_{Q'l'}(K^I_{Ql;\,Q'l'} + k_Q\Phi_l\bar{K}^I_{Ql;\,Q'l'})\zeta^I_{Q'l';\,Q_0l_0}; \quad (26)$$

$$\xi^I_{Ql;\,Q_0l_0} = -(k/j_{l_0}h_{l_0})\sum_{Q'l'}K^I_{Ql;\,Q'l'}\zeta^I_{Q'l';\,Q_0l_0},$$

$$\eta^I_{Ql;\,Q_0l_0} = -(k/j_{l_0}h_{l_0})\sum_{Q'l'}\bar{K}^I_{Ql;\,Q'l'}\zeta^I_{Q'l';\,Q_0l_0}. \quad (27)$$

In the next section the excitation amplitude will be expressed directly in terms of ζ; therefore we shall consider only (26). From the latter it follows that the ζ's are slightly dependent on the energy. Indeed since by virtue of the joining conditions $k\chi_l \sim k'$ is slightly energy dependent the same true for $k(\chi_l - \Phi_l) \sim k'$. Similarly

$$K^I_{Ql;\,Q_0l_0} + k_Q\Phi_l\bar{K}^I_{Ql;\,Q_0l_0} \approx K^I_{Ql;\,Q_0l_0} \approx \text{const.}$$

The complexity of the equations depends on how many of the $K^I_{Ql;Q'l'}$ are non-vanishing for given Q and l (on the number of interconnections). According to (23) this depends on how many coefficients $\alpha^{(\lambda)}_{1,2}$ must be taken into account when the $\alpha_{1,2}(\vartheta)$ are expanded in Legendre polynomials.

In the majority of cases which are of practical interest this number is small and a numerical solution can be obtained. In many practical cases (see Sec. 4) the nondiagonal elements in (26) are small compared with the diagonal elements. This equation can then easily be solved by perturbation theory with the result

$$\zeta^I_{Q_0l_0;Q_0l_0} = -[k(\chi_{l_0} - \Phi_{l_0}) + K^I_{Q_0l_0;\,Q_0l_0} + k_{Q_0}\Phi_{l_0}\bar{K}^I_{Q_0l_0;\,Q_0l_0}]^{-1};$$

$$\zeta^I_{Ql;\,Q_0l_0} = \frac{K_{Ql;\,Q_0l_0} + k_Q\Phi_l\bar{K}^I_{Ql;\,Q_0l_0}}{[k_Q(\chi_l - \Phi_l) + K^I_{Ql;\,Ql} + k_Q\Phi_l\bar{K}^I_{Ql;\,Ql}]}\zeta^I_{Q_0l_0;\,Q\hat{}_0l_0}; \quad (28)$$

$$Q \neq Q_0 \quad \text{or} \quad l \neq l_0.$$

This solution is unacceptable if any of the diagonal elements

$$C^I_{Ql} = k_Q(\chi_l - \Phi_l) + K^I_{Ql;\,Ql} + k_Q\Phi_l\bar{K}^I_{Ql;\,Ql}$$

is smaller than any of the nondiagonal elements, in spite of the fact that in general the nondiagonal elements are considerably smaller than the diagonal elements. But the equations can easily be corrected in this case. For this purpose we must in the equation containing this diagonal element retain the nondiagonal terms in the first approximation and append equations for the quantities $\zeta^I_{Q'l';Q_0l_0}$ which appear in these terms. In the nondiagonal elements it is only necessary to retain terms containing $\zeta^I_{Ql;Q_0l_0}$ and $\zeta^I_{Q_0l_0;Q_0l_0}$. By solving these equations we obtain the corrected value of $\zeta^I_{Ql;Q_0l_0}$. We then obtain instead of (28) expressions in which C^I_{Ql} is replaced by

$$C^I_{Ql} - \sum_{Q'l'}(1/C^I_{Q'l'})(K^I_{Ql;\,Q'l'} + k_Q\Phi_l\bar{K}^I_{Ql;\,Q'l'})(K^I_{Q'l';\,Ql} + k_{Q'}\Phi_{l'}\bar{K}^I_{Q'l';\,Ql}).$$

A general solution of (26) can also be obtained in the form of continuous fractions, incorporating the corrections in convenient form.

3. CALCULATION OF THE EXCITATION CROSS SECTION OF ROTATIONAL STATES

The amplitude of a nuclear transition to a state with angular momentum Q and projection M, with simultaneous scattering of the neutron into the solid angle $\Omega_{\mathbf{k}_Q}$ can be written as

$$f_{QM}(\Omega_{\mathbf{k}_Q}) = -\frac{1}{4\pi}\int\psi^{(-)*}_{\mathbf{k}_Q}(\mathbf{r})\,\tilde{D}^Q_{M\varkappa}(\theta_i)\,V_1(r,\vartheta) \times U(\mathbf{r},\theta_i)\,dr\,d\Omega\,d\theta; \quad (29)$$

$$\psi^{(-)}_{\mathbf{k}_Q}(\mathbf{r}) = \frac{4\pi}{k_Q}\sum_{lm}i^lY_{lm}(\Omega)\,Y^*_{lm}(\Omega_{\mathbf{k}_Q})\,j^*(k_Qr).$$

Expanding $\psi^{(-)}_{\mathbf{k}_Q}(\mathbf{r})$ in powers of $k_Q(r-R_1)$ and retaining the first two terms, inserting (21) and (27) into (30) and using the equality

$$\sum_{Q'l'} (K^I_{Ql;\,Q'l'} + k_Q \Phi_l \bar{K}^I_{Ql;\,Q'l'})\,\zeta^I_{Q'l';\,Q_0l_0} + k_Q(\chi_l - \Phi_l)\sum_{Q'l'} \bar{K}^I_{Ql;\,Q'l'}\zeta^I_{Q'l';\,Q_0l_0}$$
$$= -\delta_{QQ_0}\delta_{ll_0} - (k_Q/j_l h_l)\left[\zeta^I_{Ql;\,Q_0l_0} - \sum_{Q'l'} \bar{K}^I_{Ql;\,Q'l'}\zeta^I_{Q'l';\,Q_0l_0}\right],$$

we obtain

$$f_{QM}(\Omega) = 2\pi^{1/2}\sum_{ll_0I} \frac{i^{l_0-l}\sqrt{2l_0+1}}{h_{l_0}(kR_1)\,h_l(k_QR_1)} C^{IM_0}_{QM;\,lm} C^{IM_0}_{Q_0M_0;\,l_00}\, Y_{lm}(\Omega)\, f^I_{Ql;\,Q_0l_0}; \tag{30}$$

$$f^I_{Ql;\,Q_0l_0} = \zeta^I_{Ql;\,Q_0l_0} - \sum_{Q'l'} \bar{K}^I_{Ql;\,Q'l'}\zeta^I_{Q'l';\,Q_0l_0}. \tag{31}$$

With the use of (30) we easily calculate the excitation cross section

$$d\sigma_Q(\Omega)/d\Omega = [k_Q/(2Q_0+1)\,k]\sum_{M,M_0} |f_{QM}(\Omega)|^2$$
$$= \frac{k_Q}{k}\sum_{\substack{ll_0l'l'_0\\ II'n}} \frac{f^I_{Ql;\,Q_0l_0} f^{I*}_{Ql';\,Q_0l'_0}}{h_l(k_QR_1)\,h_{l_0}(kR_1)\,h^*_{l'}(k_QR_1)h^*_{l'_0}(kR_1)} N(QQ_0ll_0l'l'_0II'n)\,P_n(\cos\vartheta); \tag{32}$$

$$N(QQ_0ll_0l'l'_0II'n) = \frac{(2I+1)(2I'+1)}{2Q_0+1}[(2l+1)(2l'+1)(2l_0+1)(2l'_0+1)]^{1/2}$$
$$\times (-1)^{l_0+l'_0-Q-Q_0} C^{n0}_{l0;\,l_00} C^{n0}_{l_00;\,l'_00} W(ll'II';nQ)\,W(l_0l'_0II';nQ).$$

Following integration over the angles the result is considerably simplified and the total excitation cross section becomes

$$\sigma_Q = 4\pi\frac{k_Q}{k}\sum_{ll_0I}\frac{2I+1}{2Q_0+1}|f^I_{Ql;\,Q_0l_0}|^2\,|h_l(k_QR_1)\,h_{l_0}(kR_1)|^{-2}. \tag{33}$$

The $f^I_{Ql;Q_0l_0}$ like the $\zeta^I_{Ql;Q_0l_0}$ are only slightly dependent on energy, so that the energy dependence of the cross section is determined mainly by the $|h_l h_{l_0}|^{-2}$ factors, each of which is the product of the penetrabilities of the centrifugal barrier for an incident neutron with angular momentum l_0 and a scattered neutron with angular momentum l.

Since these factors decrease rapidly as l_0 and l increase, only a few terms of the summation in (27) are actually important. For example, when the first rotational level of an even-even nucleus ($Q_0 = 0$, $Q = 2$) is excited by 1–2 Mev neutrons the three important terms of the summation in (33) are those for $l = 0$, $l_0 = 2$; $l = 1$, $l_0 = 1$; $l = 2$, $l_0 = 0$. Eq. (33) is considerably simplified when we consider excitation near the threshold ($k_Q \to 0$). In this case $|h_l|^{-2} \to \delta_l$ and

$$\sigma_Q = 4\pi\frac{k_Q}{k}\frac{2Q+1}{2Q_0+1}\sum_{l_0}\frac{|f^Q_{Q0;\,Q_0l_0}|^2}{|h_{l_0}(kR_1)|^2}. \tag{34}$$

When we consider excitation of the first rotational level, in most cases $kR \ll 1$ and the sum in (34) is reduced to a single term:

$$\sigma_Q = 4\pi\frac{k_Q}{k}\frac{2Q+1}{2Q_0+1}\frac{(kR_1)^{2(Q-Q_0)}}{[(2Q-2Q_0-1)!!]^2}|f^Q_{Q0;\,Q_0Q-Q_0}|^2. \tag{35}$$

The formulae derived enable us to make a rough estimate of the magnitude of the excitation cross section without detailed calculation. The $f^I_{Ql;Q_0l_0}$ like the $\zeta^I_{Ql;Q_0l_0}$ have an average magnitude $\sim 1/k'$. Assuming

$$f^I_{Ql;\,Q_0l_0} \sim f/k', \quad k' = 1{,}4\cdot 10^{13}\ \text{cm}^{-1},$$

we obtain for the excitation cross section of the first rotational level of an even-even nucleus near the threshold

$$\sigma_2 \sim 3(k_2/k)\,|f|^2\cdot 10^{-28}\ \text{cm}^2$$

and for energies near 1 Mev $\sigma_2 \sim 2\times 10^{-25}|f|^2$ cm^2.

4. AN ELLIPSOIDAL SQUARE WELL MODEL

We shall consider in this section a square well of ellipsoidal shape; we shall assume

$$V_0(r) = \begin{cases} -k'^2(1+i\xi) & r < R_1 \\ 0 & r > R_1 \end{cases}$$

$$V_1(r,\vartheta) = \begin{cases} -k'^2(1+i\xi) & R_1 < r < R_1 + \Delta R\cos^2\vartheta \\ 0 & r < R_1;\ r > R_1 + \Delta R\cos^2\vartheta. \end{cases} \tag{36}$$

Let us consider a prolate nucleus. It was shown in Ref. 1 that ξ can assumed to be 0.03. Absorption in a layer of thickness ΔR is determined by the magnitude of $k'\Delta R\,\xi$, which is small when $k'\Delta R \sim 1$. The assumptions already made give $k'\Delta R \sim 1$ so that we shall neglect $k'\Delta R\xi$ compared with 1.

Subject to this condition, it follows (15) and (17) that

$$\varphi_1(r,\vartheta) = \cos k'(r - R_1),$$
$$\varphi_2(r,\vartheta) = \sin k'(r - R_1)/k, \tag{37}$$
$$R_1 < r < R_1 + \Delta R\cos^2\vartheta.$$

Furthermore, in accordance with (20)

$$\alpha_1(\vartheta) = -k'\sin(x\cos^2\vartheta);$$
$$\alpha_2(\vartheta) = -1 + \cos(x\cos^2\vartheta); \tag{38}$$
$$\beta_1(\vartheta) = -x\cos^2\vartheta\sin(x\cos^2\vartheta) - \alpha_2(\vartheta);$$

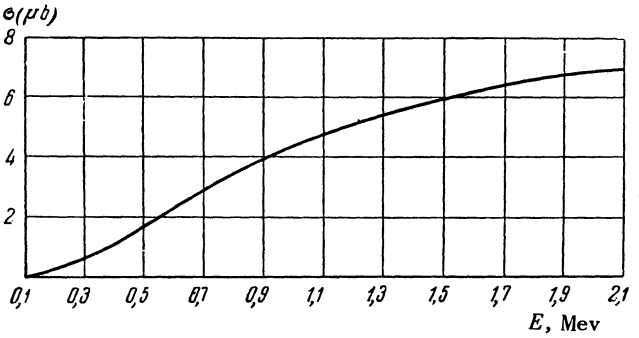
Inserting these expressions in (24) we obtain $\alpha^{(\lambda)}_{1,2}$ and $\beta^{(\lambda)}_{1,2}$ as integrals which are expressed simply in terms of Fresnel integrals. After calculating these integrals we obtain, for example

for $x = 1$:

$$\begin{aligned} &\alpha_1^{(0)} = -0{,}62\,k';\ \alpha_2^{(0)} = -0{,}2;\ \beta_1^{(0)} = -0{,}16;\ \beta_2^{(0)} = -0{,}09\ 1/k';\\ &\alpha_1^{(2)} = -0{,}23\,k';\ \alpha_2^{(2)} = -0{,}1;\ \beta_1^{(2)} = -0{,}1;\ \beta_2^{(2)} = -0{,}02\ 1/k';\\ &\alpha_1^{(4)} = -0{,}002\,k';\ \alpha_2^{(4)} = -0{,}006;\ \beta_1^{(4)} = -0{,}1;\ \beta_2^{(4)} = 0{,}048\ 1/k'; \end{aligned} \tag{39}$$

for $x = 2$:

$$\begin{aligned} &\alpha_1^{(0)} = -1\,k';\ \alpha_2^{(0)} = -0{,}34;\ \beta_1^{(0)} = -0{,}2;\ \beta_2^{(0)} = -0{,}23\ 1/k';\\ &\alpha_1^{(2)} = -0{,}31\,k';\ \alpha_2^{(2)} = -0{,}18;\ \beta_1^{(2)} = -0{,}09;\ \beta_2^{(2)} = 0{,}06\ 1/k';\\ &\alpha_1^{(4)} = -0{,}16\,k';\ \alpha_2^{(4)} = -0{,}03;\ \beta_1^{(4)} = -0{,}11;\ \beta_2^{(4)} = 0{,}045\ 1/k'. \end{aligned} \tag{40}$$



Energy dependence of the excitation cross section of the first rotational level of an even-even nucleus

Inserting (39) or (40) in (23) and (26) we see readily that for these values of $\alpha^{(\lambda)}_{1,2}$ and $\beta^{(\lambda)}_{1,2}$ the non-diagonal elements in (26) are considerably smaller than the diagonal elements, so that the formulas (28) can be used for the $\zeta^{I}_{Ql;Q_0l_0}$. Exceptions are certain values of R_1, *i.e.*, of the mass number A, for which some diagonal element is unusually small. For such values of A the equations must be

solved more exactly. For the values $k'R_1 = 11.2$; $k'\Delta R = 2$ ($\Delta R/R \sim 0.2$; $A \sim 190$) and using the values of $\alpha_{1,2}^{(\lambda)}$ and $\beta_{1,2}^{(\lambda)}$ from (40) we calculated first $\zeta_{2l;0l_0}^{l_0}$ (with (28) applying in this case) and then $f_{2l,0l_0}^{l_0}$ for the excitation of the first rotational level of an even-even nucleus with 0.1 Mev excitation energy.

As has already been indicated, for this case the three important terms in (33) are those with: $f_{22;00}^{0}$, $f_{21;21}^{1}$ and $f_{20;02}^{2}$. The figure shows the result of substituting the values of these quantities in (27).

5. EFFECT OF NUCLEAR DEFORMATION ON THE TOTAL NEUTRON SCATTERING CROSS SECTION

The existence of deformation and rotational levels in nuclei will affect both the elastic cross section and the total scattering cross section. Formulae for these quantities are easily obtained by using the results of the preceding sections.

For the elastic scattering amplitude we easily obtain instead of (30)

$$f_{Q_0M}(\Omega) = 2\pi^{1/2} \sum_{ll_0I} C_{Q_0M;lm}^{IM_0} C_{Q_0M_0;l_00}^{IM_0} Y_{lm}(\Omega) \left[\frac{1}{2ik} \delta_{ll_0} \left(\frac{h_l^*}{h_l} - 1 \right) + \frac{i^{l_0-l}}{h_l^* h_{l_0}} f_{Q_0l;Q_0l_0}^{I} \right] \sqrt{2l_0+1}. \tag{41}$$

The differential and the total elastic scattering cross section σ_{se} differ from (32) and (33) respectively through the replacement of $f_{Ql;Q_0l_0}^{I}$ by the expression in the square brackets in (41). The total cross section can be calculated by using (41) and the optical theorem:

$$(4\pi/k) \operatorname{Im} f_{Q_0M_0}(0) = \sigma = \sigma_{se} + \sigma_c, \tag{42}$$

where $f_{Q_0M_0}(0)$ is the forward scattering amplitude. Averaging $f_{Q_0M_0}(0)$ over the values of M_0 we obtain

$$\sigma = \frac{4\pi}{k} \sum_{I,l_0} \frac{2I+1}{2Q_0+1} \operatorname{Im} \left[\frac{1}{2ik} \left(\frac{h_{l_0}^*}{h_{l_0}} - 1 \right) + h_{l_0}^{-2} f_{Q_0l_0;Q_0l_0}^{I} \right] \tag{43}$$

At low energies ($kR \ll 1$) this expression becomes

$$\sigma = 4\pi (R_1^2 - 2R_1 \operatorname{Re} f_{Q_00;Q_00}^{Q_0}) + \frac{4\pi}{k} \operatorname{Im} f_{Q_00;Q_00}^{Q_0}. \tag{44}$$

It was shown in Ref. 1 that at low energies there exists a simple relation between the total scattering cross section and the ratio of the average level width of a compound nucleus to the average level spacing:

$$\sigma = 4\pi R'^2 + 2\pi^2 k^{-2} \Gamma / D. \tag{45}$$

Hence

$$\frac{\Gamma}{D} = \frac{2}{\pi} k \operatorname{Im} f_{Q_00;Q_00}^{Q_0}. \tag{46}$$

Feshbach, Porter and Weisskopf compared the value of Γ/D which was obtained experimentally with the value that was calculated on an optical model with a spherically symmetrical square potential. A marked deviation was found in the region $A \sim 150-160$. The theoretical curve shows a sharp peak in this region, whereas the experimental data do not reveal this peak. Following Bohr and Mottelson these authors state that the discrepancy results from the fact that nuclei of such mass numbers are highly deformed. The matter can investigated by using (46). We are interested in the position of the maximum of $f_{00;00}^{0}$. When $f_{00;00}^{0}$ is calculated by using (28), (31) and (40) we find that the maximum is only slightly shifted by comparison with a spherically symmetrical potential. But (28) cannot be used near the maximum because at the maximum the diagonal element of (26)

$$k(\chi_0 - \Phi_0) + K_{00;00}^{0} + k\Phi_0 K_{00;00}^{0}$$

is extremely small. When the solution of (28) is corrected as shown in Sec. 2 the maximum shows a greater shift. Thus for $k'\Delta R \sim 2$, it is shifted forward $A \sim 175$. This in itself is insufficient to explain the discrepancy. It must be noted, however, that the magnitude of the deformation varies strongly but not monotonically with A. The existence of a deformation in the region $A \sim 150-160$ transfers the maximum to another region where the deformation is either much greater or much smaller. Consequently, the maximum may either be nonexistent or much less steep.

In conclusion I wish to express my deep appre-

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Covariant Equation for Two Annihilating Particles

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The functional-derivative technique is used to investigate the annihilation (or production) of two interacting particles which may also exist in a bound state. Covariant equations have been found for the Green function (probability amplitude) which describes the annihilation of an electron and a positron into two quanta as well as for the Green function of the reverse process. The equations thus obtained have been used to solve the problem of interaction between the electron and positron during pair production (or annihilation) with account of radiative corrections.

RELATIVISTICALLY invariant equations for bound states were obtained by various authors[1-5]. Not enough attention, however, was paid to equations that take into account a possible annihilation of particles. In the present work the functional-derivative technique is applied to the solution of the problem concerning the annihilation (or production) of two interacting particles which may also exist in a bound state. While up to now functional equations were derived for the probability amplitudes (Green functions) describing transitions not accompanied by any change in the number of particles, in the present case functional equations have been set up for the probability amplitudes (Green functions) describing the annihilation or production of particles. The resulting equations are, therefore, of a different form. Starting with these equations, it is easy to obtain the wave equation of positronium, the possible annihilation of the electron and positron being taken into the account[6]. Such generalization of the method of functional derivatives to problems involving a change in the number of particles during the studied process enables us to calculate with any desired accuracy the probability of a two-photon (and in general, n-photon) annihilation of particles existing in a bound state. The results of previous works[7-9] dealing with the annihilation of two interacting particles in the S and P states are essentially reproduced if we limit ourselves to the first non-vanishing approximation. The contribution of Coulomb interaction in pair production is also accounted[10]. The proposed method, however, makes it also possible to find the radiative corrections for the above processes (*cf.*, Ref. 11 and 12). The investigation of radiative corrections for the probability of photoproduction and annihilation of positronium confirms the results of Ref. 13 with respect to the infra-red divergence in bound states of the particles.

1. DERIVATION OF THE EQUATION FOR THE GREEN FUNCTION OF TWO PARTICLES ANNIHILATING INTO TWO QUANTA

The Green function $G_2(x_1 x_2, \xi\xi')$ describing the transmutation of two photons into an electron-positron pair (and the two-photon annihilation of the particles as well) is defined, according to Ref. 14, in the following way: