Thermal Radiation from an Anisotropic Medium

F. V. BUNKIN
P. N. Lebedev Physical Institute, Academy of Sciences, USSR
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The radiation emitted by an element of volume of an anisotropic medium is examined on the basis of the electrodynamic theory of thermal radiation. A generalization of Kirchhoff's law is given. The thermal radiation from continuously varying magnetoactive media is considered. The case of weak gyrotropy is considered in some detail.

1. INTRODUCTION

KIRCHHOFF'S LAWS, which constitute the basis of the classical theory of thermal radiation, were established for an *isotropic* medium. Attempts to apply these laws directly to an anisotropic medium* encounter certain difficulties due mainly to the birefringent properties of such a medium. However, thermal radiation from anisotropic media has lately acquired practical importance—principally in connection with the development of radioastronomy. By way of example one may cite problems such as the role of thermal radiation (at radio frequencies) from the sun's corona in the general magnetic field of the sun[1–3], or radiation from sunspots. Another example, which concerns apparatus by itself, is that of the thermal emission from the ferrite used in the wave guides of modern instruments.

This paper formulates the problem of the thermal radiation of an anisotropic medium and solves that problem from the point of view of the electrodynamic theory of electric field fluctuations and thermal radiation as evolved by Rytov[4]. According to the basic concept of this theory, thermal fluctuations of the electric field in a medium may be described as the result of action by several extraneous random fields (or currents). In the same way these fields are also used to describe the random (thermal) radiation from each volume element of the medium.

In order to compute statistical averages of the energy values (which are quadratic in the extraneous field) one need know only the statistical characteristic of the extraneous random field, *i.e.*, the *correlation matrix* of the field components. The form of this matrix has been determined by Rytov[4] for an isotropic medium. Recently L. D. Landau and E. M. Lifshitz, in reporting on the papers by Callen *et al.*[5–8], have generalized the form of this matrix to media with arbitrary anisotropy*. In this case, when the medium experiences only electrical losses, the correlation matrix of extraneous random currents $j_\alpha(\mathbf{r})$

$$\overline{j_\alpha(\mathbf{r})\, j_\beta^*(\mathbf{r}')} = \frac{\hbar\omega^2}{8\pi^2}\cot\frac{\hbar\omega}{2\Theta}\,\frac{\overset{*}{\varepsilon}_{\beta\alpha}-\varepsilon_{\alpha\beta}}{2i}\,\delta(\mathbf{r}-\mathbf{r}')$$

$$= C_{\alpha\beta}\,\delta(\mathbf{r}-\mathbf{r}'), \tag{1.1}$$

where $\hbar$ is Planck's constant divided by 2π; $\Theta = kT$ the temperature in energy units; and $\epsilon_{\alpha\beta}$ is the dielectric permittivity tensor for the medium**.

Thermal radiation from a plasma situated in a constant magnetic field (magnetoactive medium***) is of practical importance. It is a medium of this type that we shall have in mind henceforth in this paper. Absorption in such a medium is caused, as is known, by collisions, ordinarily with $\nu/\omega \ll 1$ where ν is the collision frequency and ω is the wave frequency. The smallness of ν/ω implies the smallness of the absorption coefficient for an *ordinary* wave throughout all space available to it (see, for example, Ref. 11). In this case the maximum absorption occurs in the region of reflection, *i.e.*, where the refractive index is nearly zero.

The nature of the absorption coefficient for an *extraordinary* wave is more complicated on account of resonance absorption. The resonance is the strongest when the wave travels along the magnetic field and when $\omega = \omega_H$, ω_H being the gyrofrequency ($= eH/2mc$) of the plasma. The width of

*Under the meaning of "anisotropic media" we include both optically inactive and active (gyrotropic) crystals.

*These results have not yet been published. I wish to thank the authors for making their manuscript available to me.

**We point out that Levin[9] and the author[10] have used Eq. (1.1) before but without thorough substantiation.

***From a phenomenological point of view, ferrites located in a magnetic field can be assigned to this same type of category.

the resonance frequency band, in which the absorption is large, is on the order of ν. Outside of this comparatively narrow band, *i.e.*, where $|\omega - \omega_H| \gg \nu$, the absorption coefficient for the extraordinary wave then becomes of the same order of magnitude as in the case of an ordinary wave, ν/ω. The spatial distribution of the absorption coefficient is the same as for an ordinary wave, *i.e.*, with its maximum in the reflection area.

Therefore, if one disregards the resonance region that exists for longitudinal propagation, absorption in true magnetoactive media may be considered small. This is the principal argument in favor of the assumption made below that the radiating medium* is a weak absorber.

The subjects analyzed below have been examined in more detail in the author's dissertation[10].

2. INTENSITY OF THERMAL RADIATION FROM A VOLUME ELEMENT OF A MAGNETOACTIVE MEDIUM

Let us examine how a volume element of a magnetoactive medium radiates, *i.e.*, how one must generalize Kirchhoff's law,

$$\eta_\omega = \alpha_\omega I_\omega, \tag{2.1}$$

where η_ω and a_ω are respectively the emission and absorption coefficients of an isotropic medium, and I_ω is the intensity of the equilibrium thermal radiation in the given absorbing medium. As has been noted before[4], the concepts of equilibrium intensity and emission coefficient in an absorbing medium have definite meanings only when the absorption is so small that it is possible to disregard terms of the second order so far as losses are concerned. In this case Clausius' law is fulfilled

$$I_\omega = I_{0\omega} n^2, \tag{2.2}$$

where $I_{0\omega}$ is the intensity of the equilibrium radiation in vacuum, and n is the index of refraction for a *transparent* medium.

In the case of an anisotropic medium the problem is complicated first because two types of waves are propagated and secondly because the radiation intensity must depend on the angle formed by the direction of propagation and the axis of symmetry of the medium. Therefore, the generalized law (2.1) must specify both the distribution of the radiated energy for the two possible types of waves (polarizations) and the mentioned angular dependence.

Let us first write the components of the tensor $C_{\alpha\beta}$, which enters into the correlation matrix (1.1) of the extraneous random currents, for the case of a magnetoactive medium. If the magnetic field is directed along the Z axis, the tensor ε_{ik} is written as (see, for example Ref. 11, p. 326):

$$\varepsilon_{ik} = \begin{Vmatrix} \varepsilon & -ig & 0 \\ ig & \varepsilon & 0 \\ 0 & 0 & \eta \end{Vmatrix}. \tag{2.3}$$

When there is absorption, the quantities ε, η and g are, in general, complex. The explicit dependence of the components ε_{ik} on the parameters of the plasma do not concern us here.

Substituting (2.3) in (1.1) we obtain

$$\begin{gathered} C_{11} = C_{22} = \frac{\hbar\omega^2}{8\pi^2}\coth\frac{\hbar\omega}{2\Theta}\cdot\frac{\varepsilon^* - \varepsilon}{2i}, \\ C_{33} = \frac{\hbar\omega^2}{8\pi^2}\coth\frac{\hbar\omega}{2\Theta}\cdot\frac{\eta^* - \eta}{2i}, \\ C_{12} = -C_{21} = -\frac{\hbar\omega^2}{8\pi^2}\coth\frac{\hbar\omega}{2\Theta}\cdot\frac{g^* - g}{2}, \\ C_{13} = C_{31} = C_{23} = C_{32} = 0. \end{gathered} \tag{2.4}$$

We shall assume a weakly absorbing medium and therefore disregard terms of second order (*i.e.*, terms $\sim C^2_{\alpha\beta}$). Then when the emission coefficient is computed, the medium external to the radiating volume element dV can generally be treated as transparent.

The emission coefficient characterizes that portion of the total flow of energy from an element dV which diminishes only according to the exponential law as it travels away from the element, *i.e.*, that portion which is due to the wave field. However, from our point of view, the volume element dV is a dipole with a random moment of $d\mathbf{p} = \mathbf{j}\,dV/i\omega$ where $\mathbf{j}$ represents the density of the random extraneous currents. The wave field of the dipole in a magneto-

*In actuality, nearly longitudinal wave propagation may be found, for instance, in sun spots. Here, however, no resonance region is in fact observed because of the special nature of the inhomogeneity of the solar atmosphere whereby an increase in the radius ρ causes a monotonic decrease in both the electron concentration $N(\rho)$ and magnetic field intensity $H_0(\rho)$. The result is that extraordinary waves, whether they are generated by an element in the medium at resonance frequencies ($|\omega - \omega_H| \lesssim \nu$), or pass during emergence from the medium through a region of resonance absorption (ω_H decreases with increasing layer height) are unable to emerge outside[12].

active medium was previously found to be [see Bunkin[13], Eq. (5.3)]:

$$E_j^{(i)}(\mathbf{r}) = 4\pi k^6 A^{(i)}(\theta)(kr)^{-1}\exp\{ikr\psi^{(i)}(\theta)\}\, a_{jk}^{(i)}(\theta, \varphi)\, p_k, \tag{2.5}$$

where p_k represents the components of the dipole moment and where the $A^{(i)}$, $\psi^{(i)}$, $a_{ik}^{(i)}$ are determined by the components ε_{ik} and thus depend on polar angle θ (measured from the axis of symmetry, *i.e.*, from the direction of the external magnetic field) and on the azimuth angle φ of the radius vector $\mathbf{r}$, as well as being determined by the index of refraction n_i and its first and second derivatives [Bunkin[13], Eq. (5.4) and (5.5)]. The index i refers to one of the two possible types of waves ($i = 1$ for an ordinary wave, $i = 2$ for an extraordinary one).

From Eq. (2.5) for the dipole field and correlation matrix (1.1) and (2.4) for the current j, one can compute the emission coefficient η_ω for the medium under discussion. The following result is obtained

$$\eta_\omega = \eta_\omega^{(1)} + \eta_\omega^{(2)} + \eta_\omega^{(12)}, \tag{2.6}$$

where $\eta_\omega^{(i)}$ is the emission coefficient for i-type waves (the natural emission coefficient) and $\eta_\omega^{(12)}$ is the "interference" emission of thermal radiation. We are omitting the equations for the dependence of these coefficients on the values $A^{(i)}$, $a_{jk}^{(i)}$ and $\psi^{(i)}$, since our main interest is not in these equations but rather in presenting η_ω in such a form as to generalize Kirchhoff's law, Eq. (2.1). In this connection there arises a difficulty due to the presence of the interference emission $\eta_\omega^{(12)}$. However, as detailed examination reveals, the interference term proves quite inconsequential in practical problems. In clarifying this further we shall limit ourselves to a few remarks only.

The interference term describes the "fine structure" of the thermal radiation field. The difference in the propagation velocities of the ordinary and extraordinary waves causes the three-dimensional oscillatory character of the interference emission (see F. V. Bunkin[10]) thus

$$\eta_\omega^{(12)} = g^2 B(\theta, \omega)\cos\{kr[\psi^{(1)}(\theta, \omega) - \psi^{(2)}(\theta, \omega)] + \varphi(\theta, \omega)\}, \tag{2.7}$$

We have indicated an explicit dependence on frequency ω, since we allow for the presence of dispersion, *i.e.*, the dependence of ε_{ik} on ω. As is evident the emission coefficient $\eta_\omega^{(12)}$ is of second order in the gyrotropy parameter g. When $\theta = 0$ and $\theta = \pi/2$, the "amplitude" $B(\theta, \omega)$ becomes zero, *i.e.*, there is no interference emission in directions along and across the external magnetic field.

Because of the oscillatory dependence of $\eta_\omega^{(12)}$ on r and θ, this interference between the ordinary and extraordinary waves is totally lost when the investigated radiation is only slightly nonmonochromatic (slightly, that is, relative to the bandwidth of the receiver).

We note that the presence of interference between ordinary and extraordinary waves in thermal emission is precisely characteristic of a magnetoactive (gyrotropic in the general case) medium. In inactive crystals, such as uniaxial crystals, for which $g = 0$, there is no interference.

As for the natural emission of thermal radiation, the problem, as has been stated, is to write $\eta_\omega^{(i)}(\theta)$, expressed in terms of $A^{(i)}(\theta)$, $\psi^{(i)}(\theta)$, and $a_{jk}^{(i)}(\theta, \varphi)$, in such a form as to obtain a generalized Kirchhoff's law, *i.e.*, as

$$\eta_\omega^{(i)}(\theta) = \alpha_\omega^{(i)} I_\omega^{(i)}, \qquad (i = 1,2). \tag{2.8}$$

It is natural to take $I_\omega^{(i)}$ as the intensities of *equilibrium* thermal radiation in a *transparent* magnetoactive medium, in which case $I_\omega^{(i)}$ depends on the parameters ε, η, g and angle θ, and is given by Rytov[4], page 150*.

The following expression is then obtained for the coefficients $\alpha_\omega^{(i)} = \alpha_\omega^{(i)}(\theta)$,

$$\alpha_\omega^{(i)}(\theta) = 2kn_i(\xi_{0i})\,\varkappa_i(\xi_{0i})\cos(\xi_{0i} - \theta). \tag{2.9}$$

Here $n_i(\xi)$ and $\kappa_i(\xi)$ are related in the usual way to the real and imaginary parts of the complex index of refraction $n_i(\xi)[1-i\kappa_i(\xi)]$ for an i-type *plane* wave propagating at angle ξ to the z axis (terms

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*Rytov derives equations for equilibrium intensities $I_\omega^{(i)}$ due to a beam of plane waves whose *normals* form an angle θ with the z-axis. The *energy* flow vector of this beam forms a different angle, which Rytov designates by ξ. It must be borne in mind that our notation is different, we use ξ to represent the angle between the normal and the z-axis and θ to represent the angle of the energy flow. We find it expedient to express the intensity $I_\omega^{(i)}$ in terms of the angle θ.

$\sim \kappa_i^2$ are neglected). The angle $\xi_{0i} = \xi_{0i}(\theta)$ is determined from the equation

$$n_i'(\xi_{0i})/n_i(\xi_{0i}) = \tan(\xi_{0i} - \theta) \tag{2.10}$$

and is equal to the angle between the wave normal of a plane wave and the direction of the magnetic field, where the energy flow vector of this wave makes an angle θ with the magnetic field[11]. The components $C_{\alpha\beta}$ of the correlation tensor no longer enter in Eq. (2.9) because they are expressed in terms of $n_i(\xi_{0i})$ and $\kappa_i(\xi_{0i})$.

For an isotropic medium, $\xi_{0i} = \theta$, $n_1 = n_2$, $\kappa_1 = \kappa_2$ and are independent of direction, so that Eq. (2.9) now becomes the usual one for the absorption coefficient of an isotropic medium $a_\omega = 2kn\kappa$. Since in this case $I_\omega^{(2)} = I_\omega^{(1)} = \frac{1}{2}n^2 I_{0\omega}$ (see Rytov[4], p. 150), Eq. (2.8) reduces to Eq. (2.1). Thus, Eq. (2.8) and Eq. (2.9) are actually rational generalizations of Kirchhoff's law and the concept of absorption coefficient.

As has been mentioned, Eq. (2.8) is the solution to the problem of the division of the energy radiated by a point thermal source between the two possible types of waves, and this in turn makes it possible for one to determine the degree of polarization of this radiation (see next Section).

Let us pause briefly to examine the results of some numerical computations. In solving practical problems it is convenient to express the components of the tensor ε_{ik} in terms of the plasma parameters h, v, and q[11]. These are determined in the following manner,*

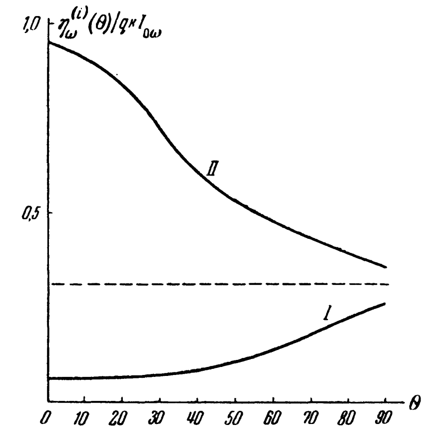
$$h = \omega_H/\omega, \quad v = (\omega_0/\omega)^2, \quad q = \nu/\omega, \tag{2.11}$$

where ω_H and ω_0 are respectively the gyromagnetic and critical frequencies of the plasma and ν is the collision frequency. The requirement of small absorption means the one must eliminate that frequency region ω, which satisfies the condition $|1-h| \lesssim q$. The expression for the absorption coefficient $a_\omega^{(i)}(\theta)$ is given by [with $n_i = n_i(\xi_{0i})$,

$$\alpha_\omega^{(i)}(\theta) = q\frac{k}{n_i}\cos(\xi_{0i} - \theta)\frac{(1-n_i^2)^2}{v(1-v)} \times \frac{2(1-v)^2(v-1+n_i^2) + h^2 v n_i^2 \sin^2\xi_{0i}}{2(1-v)(v-1+n_i^2) + h^2(1-n_i^2)\sin^2\xi_{0i}}. \tag{2.12}$$

*Al'pert, *et al*[11] use the parameter $u = h^2$ instead of h, and furthermore use s instead of q.

The curves in the figure illustrate the dependence of the dimensionless quantity $\eta_\omega^{(i)}(\theta)/qkI_{0\omega}$ on the angle θ when $v = 0.4$ and $h^2 = 0.3$. Numerals I and II refer to the ordinary and the extraordinary types of radiation respectively; the straight dotted line represents the emission coefficient of an isotropic medium ($v = 0.4$, $h = 0$).



It is obvious from the figure that the introduction of a magnetic field with $\omega_H \approx 0.55\omega$ increases the emissive power of each volume element of the plasma for extraordinary waves and decreases it for ordinary waves for all directions of emission. The total emissive power (for both wave types) when the field is present exceeds the emissivity when the field is absent. For other values of h (*i.e.*, for other frequencies with the same magnetic field) the ratio between $(\eta_\omega^{(1)} + \eta_\omega^{(2)})$ and $\eta_\omega^{(0)}$ the emission coefficient of an isotropic medium ($h = 0$) will, of course, be different, but invariably*

$$\eta_\omega^{(1)}(\theta) + \eta_\omega^{(2)}(\theta) \geqslant \eta_\omega^{(0)}. \tag{2.13}$$

Thus, when the magnetic field is applied, there is an increase in the output of thermal energy by each volume element of the plasma. Of course, this does not mean that there must of necessity be an increase also in the emissive power of the total volume of the plasma, for as the emission coefficient grows larger, reabsorption of the energy also increases (just as an increase in the resonance radiation of atoms is accompanied by an increase in their absorption). The general considerations indicating

*Further on it will be shown that when h is small,

$$\eta_\omega^{(1)}(\theta) + \eta_\omega^{(2)}(\theta) = \eta_\omega^{(0)}.$$

this circumstance were once set forth in connection with the discussion of the increase in the radiation of sun spots[14, 15].

3. POLARIZATION OF THE THERMAL RADIATION FROM A VOLUME ELEMENT

In the study of the thermal radiation of anisotropic media, questions concerned with polarization are of special interest in addition to the questions of radiation intensity. In general only partial polarization occurs, *i.e.*, the flow of radiation at any one point is the sum of the unpolarized (randomly polarized) and completely polarized (in general, elliptically polarized) components. The completely polarized portion of the flow is, of course, also random, but the field here fluctuates in such a way that the amplitude ratio and the difference in the phases of the two orthogonal projections remain constant.

In this Section we shall determine the degree of polarization of the thermal radiation emitted by a point source (volume element of the medium), *i.e.*, the ratio $p(\theta)$ of the intensity in a given direction of the totally polarized component to the total intensity. Moreover, we shall determine the degree of ellipticity (the ratio of semiaxes $a/b = \tan\beta(\theta)$ of the polarization ellipse of the electric field) and the position of the polarization plane, *i.e.*, the angle of inclination $\chi(\theta)$ of the major axis of the ellipse to a certain direction. These questions can be most simply solved by using Stokes' parameters Q, U, and V which must be expressed in terms of the components of the tensor $C_{\alpha\beta}$*. In this case we have for $\beta(\theta)$ and $\chi(\theta)$

$$\sin 2\beta = V\sqrt{Q^2+U^2+V^2},\ \tan 2\chi = U/Q, \quad (3.1)$$

where χ is the angle formed by the major axis of the ellipse and the ξ_1 axis. The double valuedness of these equations is eliminated by the following auxiliary condition: if of the two values for β the smaller in absolute magnitude is selected, then the sign of $\cos 2\chi$ should coincide with the sign of Q. The sign of β is understood to be counter-clockwise (from axis ξ_1 toward axis ξ_2).

Let us introduce an auxiliary Hermitian tensor

$$P_{ik}(\mathbf{r}) = \overline{E_i(\mathbf{r})\,E_k^*(\mathbf{r})}, \quad (3.2)$$

where $\mathbf{E}(\mathbf{r})$ represents the electric field. The determination of Stokes' parameters [where one utilizes the definitions* and Eq. (3.2)] leads to a connection between Q, U, V and P_{ik}. If, for example, the polarization in plane (x, y) is of interest, we have

$$Q = P_{11} - P_{22},\quad U = P_{12} + P_{21},$$
$$V = -i\,(P_{12} - P_{21}). \quad (3.3)$$

Returning to the question of the polarization of thermal radiation in a homogeneous medium, which is of interest to us, we make use of the representation of the field $\mathbf{E}(\mathbf{r})$ by the extraneous current $\mathbf{j}(\mathbf{r})$ (see Bunkin[13]),

$$E_i(\mathbf{r}) = \frac{4\pi k}{ic}\int_V T_{ik}(\mathbf{r},\ \mathbf{r}_1)\,j_k(\mathbf{r}_1)\,dV_1. \quad (3.4)$$

Substituting Eq. (3.4) into Eq. (3.2) and utilizing Eq. (1.1), we find

$$P_{ik} = \frac{16\pi^2k^2}{c^2}\int T_{i\alpha}(\mathbf{r},\ \mathbf{r}_1)\,\overset{*}{T}_{k\beta}(\mathbf{r},\ \mathbf{r}_1)\,C_{\alpha\beta}\,dV_1$$
$$= \int p_{ik}(\mathbf{r},\ \mathbf{r}_1)\,dV_1. \quad (3.5)$$

The additivity of components P_{ik} is obviously a consequence of the incoherence of the radiation from separate volume elements of the medium. The tensor $p_{ik}(\mathbf{r}, \mathbf{r}_1)dV_1$ characterizes the radiation polarization at point $\mathbf{r}$ due to the volume element dV_1, which is at point $\mathbf{r}_1$, and allows for both ordinary and extraordinary waves. Thus, in the general case we have

$$p_{ik} = p_{ik}^{(1)} + p_{ik}^{(2)} + p_{ik}^{(12)}, \quad (3.6)$$

where $p_{ik}^{(12)}$ is the interference term. For the same reasons as those given above in the discussion of the interference term in the energy flow, the term

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*If

$$\xi_1 = \xi_1^0 \sin(\omega t - \varepsilon_1),\qquad \xi_2 = \xi_2^{(0)}\sin(\omega t - \varepsilon_2)$$

are two mutually perpendicular components of the electric field, then by definition the Stokes parameters are[16–18]

$$Q = \overline{(\xi_1^{(0)})^2} - \overline{(\xi_2^{(0)})^2},\ U = 2\overline{\xi_1^{(0)}\ \xi_2^{(0)}\cos\delta},$$

$$V = \overline{2\xi_1^{(0)}\xi_2^{(0)}\sin\delta},\ \delta = \varepsilon_1 - \varepsilon_2.$$

Stokes introduced yet a fourth parameter $I = \overline{(\xi_1^{(0)})^2} + \overline{(\xi_2^{(0)})^2}$, to determine the intensity in isotropic media. In anisotropic media this parameter does not determine intensity and therefore is of no interest.

$p_{ik}^{(12)}$ may be disregarded. The components of the tensor $p_{jk}^{(i)}(i = 1, 2)$ are determined with the aid of Eq. (3.4).

In the wave region which is, of course, the only one of interest, we obtain

$$p_{jk}^{(i)} = B^{(i)} a_{j\alpha}^{(i)} (\theta, \varphi)\, a_{k\beta}^{(i)} (\theta, \varphi)\, C_{\alpha\beta}, \tag{3.7}$$

$$B^{(i)} = (16\pi^2 k^8/c^2\rho^2)\ |A^{(i)}(\theta)|^2, \rho = |\mathbf{r} - \mathbf{r}_1|, \tag{3.8}$$

where the functions $A^{(i)}$, $a_{jk}^{(i)}$ are the same as in Eq. (2.5).

By substituting Eq. (3.7) into Eq. (3.3) we establish the relationship of the Stokes parameters, which relate to the radiation from a volume element with the components of the tensor $C_{\alpha\beta}$. The expressions thus obtained for the parameters $Q^{(i)}(\theta)$, $U^{(i)}(\theta)$, and $V^{(i)}(\theta)$, which depict the polarization of the field in a plane orthogonal to an arbitrary direction of emission, are rather cumbersome and are not given here. These expressions show that thermal radiation from a volume element in waves of every type will be completely polarized, and in general elliptically. However, it is a matter of practical importance to determine the state of polarization of the total emission (*i.e.*, the flow of ordinary and extraordinary waves, assuming, of course, that there is emission of both types of waves, *i.e.*, that the refraction index at any given point be real for both waves). Since we disregard the interference term $p_{jk}^{(12)}$, then the Stokes parameters Q, U, V for the total beam are

$$Q = Q^{(1)} + Q^{(2)},\ U = U^{(1)} + U^{(2)},\ V = V^{(1)} + V^{(2)}. \tag{3.9}$$

In this case the flux is only partly polarized. The degree of polarization $p(\theta)$ can be evaluated by the equation

$$p(\theta) = \frac{|\eta_\omega^{(1)}(\theta) - \eta_\omega^{(2)}(\theta)|}{\eta_\omega^{(1)}(\theta) + \eta_\omega^{(2)}(\theta)} \tag{3.10}$$

$$= \frac{|\alpha_\omega^{(1)}(\theta) I_\omega^{(1)}(\theta) - \alpha_\omega^{(2)}(\theta) I_\omega^{(2)}(\theta)|}{\alpha_\omega^{(1)}(\theta) I_\omega^{(1)}(\theta) + \alpha_\omega^{(2)}(\theta) I_\omega^{(2)}(\theta)}.$$

In the case of weak gyrotropy, *i.e.*, when $h \ll 1$, the approximate expression for $\eta_\omega^{(i)}(\theta)$, which was obtained in Section 5 of this article, yields a very simple equation for $p(\theta)$, correct to terms $\sim h$

$$p(\theta) = 2h\left(1 + \frac{v}{4(1-v)}\right)|\cos\theta|. \tag{3.11}$$

As an example, let us examine the case of longitudinal ($\theta = 0$) thermal emission from a volume element. For the Stokes parameters we obtain

$$V^{(i)} = \mp 4g^2\eta^2 (C_{11} \pm iC_{12})\, B^{(i)}, \qquad Q^{(i)} = U^{(i)} = 0. \tag{3.12}$$

Thus [see Eq. (3.1)]

$$\sin 2\beta^{(i)} = \mp 1, \tag{3.13}$$

i.e., as might have been expected, longitudinal thermal emission from a point source is circularly polarized. The total flux is only partly polarized, naturally, though again circularly and with the same direction of rotation as for an extraordinary plane wave. The degree of polarization for this particular case is,

$$p(0) = \frac{|n_1(1-h)^2(n_2^2+1-v)^2 - n_2(1+h)^2(n_1^2+1-v)^2|}{n_1(1-h)^2(n_2^2+1-v)^2 + n_2(1+h)^2(n_1^2+1-v)^2}, \tag{3.14}$$

$$n_i^2 = 1 - v/(1 \pm h). \tag{3.15}$$

4. THE EQUATION FOR TRANSFER OF THERMAL RADIATION IN A MAGNETOACTIVE MEDIUM

The problem of thermal radiation in inhomogeneous media reduces, as is known, to solving the transfer equation; to write this equation one must know the emission and absorption coefficients of the medium. Since we have obtained these values for a magnetoactive medium, it is now easy for us to write the transfer equation. If $J_\omega^{(i)}$ is the intensity of the i-type thermal radiation ($i = 1, 2$) in the direction θ (θ is the angle between the direction of interest and the magnetic field at the given point), then

$$(dJ_\omega^{(i)}/d\sigma) + \alpha_\omega^{(i)}(\theta) J_\omega^{(i)} = \eta_\omega^{(i)}(\theta), \tag{4.1}$$

where $d\sigma$ is the element of ray length, and $\alpha_\omega^{(i)}(\theta)$ and $\eta_\omega^{(i)}(\theta)$ are respectively the absorption and emission coefficients of the medium.

The intensity of radiation emitted from the medium in a given direction is obtained from Eq. (4.1), thus

$$J_{\omega}^{(i)} = \int_{0}^{\infty} \eta_{\omega}^{(i)}(\theta, \sigma)\, e^{-\tau_i(\sigma)}\, d\sigma, \tag{4.2}$$

where $\tau_i(\sigma)$ is the optical thickness,

$$\tau_i(\sigma) = \int_{0}^{\sigma} \alpha_{\omega}^{(i)}(\theta, \sigma)\, d\sigma. \tag{4.3}$$

The integration in Eq. (4.2) and Eq. (4.3) is to be performed along the ray under consideration.

Thus the problem reduces to (as in the case of an isotropic medium) a computation of the trajectory of the ray. However, in contrast to the case of an isotropic medium, where it is sufficient to use the refraction law (Snell's law) to determine the trajectory of the ray, the corresponding computations in the present case require consideration of both the refraction law (whose form is considerably more complicated) and the relationship between the directions of the wave normal and the energy flow. The result is that even for a comparatively simple case of anisotropy (*e.g.*, a plane inhomogeneous ionosphere stratum in a homogeneous magnetic field[19-20]) the computation of the trajectory of the ray necessitates cumbersome calculations and can be completed only by combining analytic and graphic methods. In the case of the sun, which is of practical interest, the ray trajectories have not, to our knowledge, been computed with allowance for the magnetic field (*i.e.*, for the "anisotropic" sun).

It is not out purpose here to compute the ray trajectories for any concrete problems, but to examine the particular, but practically important, case of weak gyrotropy, *i.e.*, effects due to anisotropy which can be treated more or less as *corrections* to the solution for a corresponding isotropic medium.

5. THE CASE OF WEAK GYROTROPY

The condition for weak gyrotropy $\omega_H/\omega \equiv h \ll 1$ is realized in the earth's ionosphere (where the magnetic field $H_0 \approx 0.5$ oersted) for wavelengths of a meter or less and on the sun—for the general magnetic field ($H_0 \approx 50$ oersted) and the field of small sun spots ($H_0 \approx 10^2$ oersted)—for wavelengths from a decimeter to a centimeter (here $h \lesssim 0.1$).

We shall find approximate expressions for the quantities that are of interest which are accurate to terms of the order of h. The general expression for the index of refraction for an ionized gas in a magnetic field[11,19] reduces in this approximation to a simple equation,

$$n_i^2 = 1 - v \pm hv\,|\cos\xi|, \tag{5.1}$$

where ξ is the angle formed by the wave normal and the magnetic field. The quadratic term in h in the expansion of n_i^2 is one order of magnitude smaller than the preceeding term only when the angle satisfies the following condition,

$$\sin\xi \cdot \tan\xi \nless 2(1 - v). \tag{5.2}$$

When this condition is not fulfilled the anisotropy of the medium proves to be of order h^2, *i.e.*, the medium is isotropic in the approximation discussed here ($\sim h$) and consequently an analysis of the phenomena that interest us becomes superfluous. Eq. (5.2) obviously means that the directions of propagation must not be too close to the transverse direction.* Thus, when $v = 0.4$, the angle ξ should not exceed about 56°. Henceforth we shall invariably assume Eq. (5.2) to be fulfilled.

When Eq. (5.1) holds, it is easy to produce the corresponding approximate expression for the angle $\chi_i(\xi_{0i}) = \xi_{0i} - \theta$, as well as for the absorption $[\alpha_{\omega}^{(i)}(\theta)]$ and emission $[\eta_{\omega}^{(i)}(\theta)]$ coefficients [the exact expressions for these quantities are given by Eq. (2.8), (2.9) and (2.10) respectively]. Thus,

$$\chi_i[\xi_{0i}(\theta)] = \mp \frac{v \sin\theta}{2(1-v)} h, \tag{5.3}$$

$$\alpha_{\omega}^{(i)}(\theta) = \alpha_{\omega}^{(0)}\left[1 \mp 2\left(1 + \frac{1}{4}\frac{v}{1-v}\right) h\,|\cos\theta\,|\right], \tag{5.4}$$

$$\eta_{\omega}^{(i)}(\theta) = \alpha_{\omega}^{(i)}(\theta)\, I_{\omega}^{(i)}(\theta) = {}^1\!/_2\, \eta_{\omega}^{(0)}\left[1 \mp 2\left(1 + {}^1\!/_4 \frac{v}{1-v}\right) h\,|\cos\theta\,|\right]. \tag{5.5}$$

Here

$$\alpha_{\omega}^{(0)} = qkv / \sqrt{1 - v},$$

$$\eta_{\omega}^{(0)} = \alpha_{\omega}^{(0)} I_{\omega}^{(0)} = \alpha_{\omega}^{(0)} I_{0\omega}(1 - v)$$

are the absorption and emission coefficients in the corresponding isotropic medium ($h = 0$). We note that the equilibrium intensities $I_{\omega}^{(i)}(\theta)$ in the dis-

*It should be noted that Eq. (5.2) is not a condition of "quasilongitudinality" (Ref. 11).

cussed approximation are equal to just half of the equilibrium intensity in the corresponding isotropic medium,

$$I_\omega^{(i)}(\theta) = {}^1/_2 I_\omega^{(0)} + O(h^2) \\ = {}^1/_2 (1-v) I_{0\omega} + O(h^2), \tag{5.6}$$

where $I_{0\omega}$ is the equilibrium intensity in a vacuum.

Let us now compute by Eq. (4.2) and (4.3), accurate to terms of the order of h, the radiation intensities $J_\omega^{(i)}$ from an inhomogeneous magnetoactive medium.

Let the ray equation be given in parametric form,

$$x = x(\sigma, h), \quad y = y(\sigma, h), \quad z = z(\sigma, h), \tag{5.7}$$

i.e., the arc length σ $(0 < \sigma < \infty)$ is treated as a parameter. Since the angle $\chi_i(\xi_{i0})$ between the wave normal and the energy flow vector is of order h, according to Eq. (5.3), the perturbation of the ray trajectory by the magnetic field is also of order h. This means that the expansion of x, y and z in powers of h is linear,

$$x(\sigma, h) = x_0(\sigma) + x_1(\sigma) h + \ldots, \tag{5.8}$$

where $x_1(\sigma)$, in general, differs from zero.

In our computations we shall allow for continuous changes in the properties of the medium by treating all the subsequent quantities as dependent on the coordinates only through

$$\xi = \mu x, \quad \eta = \mu y, \quad \zeta = \mu z, \tag{5.9}$$

where μ is a small parameter. The magnitude of μ is determined by consideration of the fact that the relative changes in all the quantities over a wave length must be on the order of μ.* We shall consider henceforth that

$$\nu \lesssim h. \tag{5.10}$$

Substituting Eq. (5.4) in Eq. (4.3) and utilizing Eq. (5.8) to (5.10) we obtain the following expression for the optical thickness

$$\tau_i(\sigma) = \tau^{(0)}(\sigma) \mp \Delta\tau(\sigma), \tag{5.11}$$

$$\tau^{(0)}(\sigma) = \int_0^\sigma \alpha_\omega^{(0)} \, d\sigma, \tag{5.12}$$

$$\Delta\tau(\sigma) = 2\int_0^\sigma \alpha_\omega^{(0)} \left(1 + {}^1/_4 \frac{v}{1-v}\right) h \, |\cos\theta| \, d\sigma, \tag{5.13}$$

where the integration is along the unperturbed ray $x = x_0(\sigma)$, $y = y_0(\sigma)$, $z = z_0(\sigma)$.

An approximate expression for intensity is obtained by substituting Eq. (5.5) and (5.11) in Eq. (4.2) and employing Eq. (5.8) – (5.10). This gives

$$J_\omega^{(i)} = {}^1/_2 (J_\omega^{(0)} \mp \Delta J_\omega), \tag{5.14}$$

$$J_\omega^{(0)} = \int_0^\infty \eta_\omega^{(0)} e^{-\tau^{(0)}(\sigma)} \, d\sigma, \tag{5.15}$$

$$\Delta J_\omega = \int_0^\infty \eta_\omega^{(0)} \left\{2\left(1 + {}^1/_4 \frac{v}{1-v}\right) \times h\,|\cos\theta| - \Delta\tau(\sigma)\right\} e^{-\tau^{(0)}(\sigma)} \, d\sigma, \tag{5.16}$$

again, as in Eq. (5.12) and (5.13), the integrals are taken along the unperturbed ray.

On the basis of Eq. (5.14), the degree of polarization of thermal radiation from a magnetoactive medium in this approximation is,

$$p = |J_\omega^{(1)} - J_\omega^{(2)}| / (J_\omega^{(1)} + J_\omega^{(2)}) = |\Delta J_\omega| / J_\omega^{(0)}. \tag{5.17}$$

The question of thermal radiation from the "anisotropic" sun was analyzed by Smerd[3] who calculated its total magnetic field assuming $h \ll 1$. He made use of earlier unpublished theoretical equations that in some respects do not coincide with the results obtained here. Thus, our expression for the correction to the optical depth $\Delta\tau(\sigma)$ [Eq. (5.13)] differs from the corresponding expression in Smerd's paper by having the factor $\left(1 + {}^1/_4 \frac{v}{1-v}\right)$. Coincidence is obtained only if terms of the order of v^2 $(\alpha_\omega^{(0)} \sim v)$ are neglected, a procedure that is far from being always justified.

The advantage of the approximate method outlined here for solving the transfer equation is that it permits one to deal with only the unperturbed form of the ray. This is due to the assumption that $\mu \lesssim h$. It can be shown that this advantage remains valid also for calculation up to terms on the order of h^k, if it is assumed that $\mu \lesssim h^k$.

In conclusion the author wishes to avail himself of the opportunity to express his deep gratitude to Prof. S. M. Rytov for suggesting this subject and for his constant help in the preparation of this paper.

*The region in which the transfer equation is valid coincides, as is known, with the region to which geometric optics apply.

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On the Mechanism of Fission of Heavy Nuclei

V. V. VLADIMIRSKII

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The effect of the state of individual nucleons on the shape of the nucleus prior to fission is studied. It is shown that the presence of excess nucleons with large values of the angular momentum projection on the symmetry axis of the nucleus may lead to loss of stability of the nucleus with respect to asymmetric deformations in the saddle point. This facilitates the explanation of some of the experimental facts.

OUR PRESENT IDEAS about the fission of heavy nuclei at low excitation, based on the liquid drop model[1], are connected with the fact that, for a sufficient elongation of an incompressible drop, the sum of the Coulomb and of the surface energies attains a maximum equal to the fission threshold, further elongation of the drop being energetically favorable. It was shown by various authors[2] that, at the critical elongation, the nucleus retains its stability with respect to asymmetric deformations. The energy of the nucleus expressed in terms of the deformation parameters possesses therefore a saddle point at the critical elongation, the loss of stability depending only on the one deformation parameters that characterizes the elongation. The shape of the nucleus in the saddle point remains symmetric.

The quantitative comparison of calculations based on the liquid drop model with experimental data encounters a number of difficulties. The theoretically predicted strong dependence of the fission threshold $U \sim (1-x)^3$ on the parameter $x \sim Z^2/A$ has not been confirmed experimentally[3,4]. In fact, the threshold was found to be almost identical for a number of elements. Difficulties are also encountered in attempts to explain the observed asymmetry in the mass distribution of fission fragments. It has been shown in recent works[5,6] that it possible to explain this asymmetry on the basis of the liquid drop model. The authors indicate that upon further elongation of the nucleus, after the saddle point has been passed, the stability with respect to asymmetric deformations is lost and there may be a fast increase in the asymmetry of the nucleus. It seems very probable that their estimate of the mean ratio of the masses of the fission fragments is correct. The calculations pertaining to the dynamics of