# SOVIET PHYSICS

# JETP

*A translation of the Journal of Experimental and Theoretical Physics of the USSR.*

SOVIET PHYSICS–JETP Vol. 5, No. 4, pp 527-778 November, 1957


## Bremsstrahlung and Pair Production at High Energies in Condensed Media


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(Submitted to JETP editor January 17, 1957)

J. Exptl. Theoret. Phys. (U.S.S.R.) 32, 633-646 (April 1957)



The effect of multiple scattering on bremsstrahlung and pair production is considered. The probability of these processes decreases appreciably at energies $\sim 10^{13}$e. The computation is carried out with the aid of the density matrix. Equations are derived for the probability of pair production and bremsstrahlung per unit path for electrons and quanta of arbitrary energy.


### 1. INTRODUCTION

IN BREMSSTRAHLUNG and pair production at high energies the participating particles travel in the same direction and the characteristic length associated with these processes will be considerable. Thus, the characteristic length associated with the emission of a quantum of wavelength $\lambdabar$ by bremsstrahlung is given by $l \sim \lambdabar/(1-v/c)$ where $v$ is the velocity of the electron. Landau and Pomeranchuk[1,2] have shown that the multiple scattering occurring in a length of that order will considerably reduce the probability of these processes. The cross-sections for pair production and bremsstrahlung have been given in Ref. 2 for the case of extremely high energies ($E \gg 10^{13}\,e$).

In an earlier paper[3] the author calculated the intensity of soft quanta emitted by an electron of arbitrary energy by determining the radiation emitted classically by an electron moving along a given trajectory and then averaging over all possible trajectories. This was accomplished by means of a distribution function averaged over the different arrangements of the atoms of the scattering medium and obeying the usual transport equation.

The present paper is devoted to the derivation of expressions for the probability of bremsstrahlung [Eq. (67)] and of pair production [Eq.(69)] per unit path length in a condensed medium without restrictions on the electron and photon energies. A connection is first established between the transition probabilities and the density matrix. Certain expressions for the density matrix averaged over the coordinates of the scattering atoms, which were derived in earlier papers[4,5], are then employed.

At low energies Eqs. (67) and (69) become the Bethe-Heitler expressions[6], and at very high energies they confirm the estimate of Ref. 2. For photon energies much smaller than the electron energy, (67) becomes identical with the expressions of Ref. 3.

Finally, for very soft quanta for which the dielectric constant is appreciably different from 1 Eq. (62), of the present paper gives in the limiting cases the results obtained by Ter-Mikaelian[7].

Eqs. (67) and (69) can be used as starting points for the formulation of a shower theory in condensed media for energies greater than $10^{13}$e.

## 2. CONNECTION BETWEEN TRANSITION PROBABILITIES AND THE DENSITY MATRIX

To solve our problem we must average the probability of bremsstrahlung and pair production over all possible arrangements of the atoms of the scattering medium. We shall express the number of radiating transitions per unit time in terms of the density matrix, and then utilize the expressions of Ref. 4 in averaging the density matrix.

Denoting the eigenfunctions of the electron in the scattering medium by $\psi_s$ and the initial wave function by $\psi_0$ we have to first order in the interaction between electron and the radiation field

$$i\dot{C}_s^{(1)} = \sum_{s'} (\psi_s | e^{iHt} A e^{ikt} e^{-iHt} | \psi_{s'}) C_{s'}^0$$

$$= (\psi_s | e^{iHt} A e^{ikt} e^{-iHt} | \psi_0), \quad A = \varepsilon_\nu \alpha e^{-i\mathbf{kr}} e \sqrt{2\pi/k},$$

Here $A$ is the photon emission operator, $\epsilon_\nu$ the polarization vector, $k$ the wave vector, and $H$ is the Hamiltonian for the electron, including the potential from all scattering centers:

$$H = H_0 + \sum_m V(\mathbf{r} - \mathbf{r}_m); \quad H\psi_s = E_s\psi_s.$$

We use units such that $m = \hbar = c = 1$. The $\psi$-functions of the electrons and quanta are normalized per unit volume.

For the number of radiating transitions per unit time we obtain

$$Q_s = \frac{d}{dt} |C_s|^2 = 2\,\mathrm{Re}\, \dot{C}_s^* C_s$$

$$= 2\mathrm{Re} \int_0^t (\psi_0 | e^{iHt_1} A^+ e^{-iHt_1} | \psi_s)(\psi_s | e^{iHt} A e^{-iHt} | \psi_0) \times e^{ik(t-t_1)}\, dt_1. \quad (1)$$

We now shall sum over the final electron states. First we consider the case of bremsstrahlung. We introduce the operator of the sign of the energy

$$K = (H + |E(\mathbf{p})|)/2|E(\mathbf{p})|,$$

where $\mathbf{p}$ is the momentum operator for the electron. With

$$\sum_s \psi_s^*(x)\psi_s(x') = \delta(x - x'),$$

we obtain

$$Q = \sum_{E_s > 0} Q_s = 2\,\mathrm{Re} \int_0^t (\psi_0 | e^{iHt_1} A^+ K e^{iH(t-t_1)} A e^{-iHt} | \psi_0) \times e^{ik(t-t_1)}\, dt_1. \quad (2)$$

In (2) the operator $K$ at large energies can be replaced with negligible error by

$$K_0 = (H_0 + |E_p^0|)/2|E_p^0|$$

which is the equivalent operator for a free electron. Then the coordinates of the scattering centers enter in (2) only through the factor $e^{\pm iHt}$.

The averaging over the coordinates of the scatterers can be performed independently in the factors $e^{\pm iHt_1}$ and $e^{\pm iH(t-t_1)}$. Indeed, the coordinates in the former term correspond to collisions occurring during the time interval $0 - t_1$ while the latter term contains the coordinates of scatterers participating in collisions during the later time interval $t_1 - t$. If after the time $t_1$ there are many collisions, then after averaging over the first collisions the factors of the form $e^{\pm iHt_1}$ will become practically independent of the collisions which took place close to $t_1$.

We now shall write the integrand of (2) as a matrix element of a product of operators in the representation of the wave functions.

$$\varphi_p^\lambda = u_p^\lambda e^{i\mathbf{pr}}$$

of the free electron. Putting $\psi_0 = \varphi_{p_0}^{\lambda_0}$ ($p_0$ - initial momentum of the electron) and denoting average by $\langle\,\rangle$ we obtain from (2)

$$\langle Q\rangle = 2\,\mathrm{Re} \int_0^t d\tau e^{ik\tau} J,$$

$$J = \langle (\psi_0 | e^{iHt_1} A^+ K_0 e^{iH\tau} A e^{-iH\tau} e^{-iHt_1} | \psi_0)\rangle \quad (3)$$

$$= \sum_{\substack{p_1p_2\\ \lambda_1\lambda_2}} \langle (e^{iHt_1})_{p_0p_1}^{\lambda_0\lambda_1} (A^+K_0 e^{iH\tau} A e^{-iH\tau})_{p_1p_2}^{\lambda_1\lambda_2} (e^{-iHt_1})_{p_2p_0}^{\lambda_2\lambda_0}\rangle;$$

here $\tau = t - t_1$. Because of the statistical independence of the factors $e^{\pm iHt_1}$ and $e^{\pm iHt}$ we have

$$J = \sum_{\substack{p_1p_2\\ \lambda_1\lambda_2}} \langle (e^{-iHt_1})_{p_2p_0}^{\lambda_2\lambda_0} (e^{iHt_1})_{p_0p_1}^{\lambda_0\lambda_1}\rangle \langle (A^+K_0 e^{iH\tau} A e^{-iH\tau})_{p_1p_2}^{\lambda_1\lambda_2}\rangle.$$

At high energy the momentum changes of the electron are essentially small, and the spin does not change in the scattering process, i.e.

$$(e^{\pm iHt})_{\mathbf{pp}'}^{\lambda\lambda'} = \delta_{\lambda\lambda'} (e^{\pm iHt})_{\mathbf{pp}'}^{\lambda\lambda'}$$

with an error on the order $|\mathbf{p}' - \mathbf{p}|/p$. In this approximation

$$J = \sum_{p_1p_2} \langle (e^{-iHt_1})_{p_2p_0}^{\lambda_0\lambda_0} (e^{iHt_1})_{p_0p_1}^{\lambda_0\lambda_0}\rangle \langle (A^+K_0 e^{iH\tau} A e^{-iH\tau})_{p_1p_2}^{\lambda_0\lambda_0}\rangle. \quad (4)$$

The first factor in (4) satisfies as a function of $p_2$, $p_1$ and $t_1$ the same equation as the averaged density matrix

$$\langle \rho^{\lambda_0\lambda_0}_{p_2 p_1} \rangle = \langle (e^{-iHt_1}\rho_0 e^{iHt_1})^{\lambda_0\lambda_0}_{p_2 p_1} \rangle.$$

It follows from this (see Ref. 4 and 5) that the difference $p_2 - p_1$ does not change in the scattering (this is is due to the homogeneity of the scattering medium). Since the first factor in (4) for $t_1 = 0$ is $\delta_{p,p_0}\delta_{p_2 p_0}$ one can write it in the form

$$\langle (e^{-iHt_1})^{\lambda_0\lambda_0}_{p_2 p_0} (e^{iHt_1})^{\lambda_0\lambda_0}_{p_0 p_1} \rangle = f_0^{\lambda_0\lambda_0}(p_1, t_1)\,\delta_{p_2 p_1}. \quad (5)$$

We shall now write the second factor of (4) as a sum of products of operators in the momentum representation. Using (5) and the expression for the operator A we obtain

$$\langle (A^+ K_0 e^{iH\tau} A e^{-iH\tau})^{\lambda_0\lambda_0}_{p_1 p_1} \rangle$$
$$= \frac{2\pi e^2}{k} \Big\langle \sum_{\substack{p_0\,\lambda_1 \\ E^{\lambda_1}>0}} (\alpha e_\nu)^{\lambda_0\lambda_1}_{p_1,\, p_1-k} (e^{iH\tau})^{\lambda_1\lambda_1}_{p_1-k,\, p-k/2} (\alpha e_\nu)^{\lambda_1\lambda_0}_{p-k/2,\, p+k/2} (e^{-iH\tau})^{\lambda_0\lambda_0}_{p+k/2,\, p_1} \Big\rangle.$$

Here

$$(\alpha e_\nu)^{\mu_1\mu_2}_{g_1 g_2} = (u^{\mu_1}_{g_1} \alpha e_\nu u^{\mu_2}_{g_2}),$$

where $u^\mu_g$ are spin functions.

We introduce

$$\langle (e^{-iH\tau})^{\lambda_0\lambda_0}_{p+k/2,\, p_1} (e^{iH\tau})^{\lambda_1\lambda_1}_{p_1-k,\, p-k/2} \rangle = f_k^{\lambda_0\lambda_1}(p, \tau). \quad (6)$$

Both the coefficients in the equations for $f_k^{\lambda_0\lambda_1}(p, \tau)$ and $f^{\lambda_0\lambda_1}(p_1 t_1)$ and their boundary conditions [see below, Eq. (10) and (12)] do not depend on the spin orientation. At a fixed sign of the energy, one can therefore omit these indices in the summation with respect to $\lambda_0$ and $\lambda_1$.

Inserting (5) and (6) into (4), summing over the photon polarization, and averaging over the spin of the initial state we obtain

$$J_1 = \frac{1}{2} \sum_{\substack{\lambda_0,\, \nu \\ E^{\lambda_0}>0}} J$$
$$= \frac{\pi e^2}{k} \int \mathscr{L}(p_1, p)\, f_0(p_1, t_1)\, f_k(p, \tau)\, \frac{d p_1}{(2\pi)^3} \frac{d p}{(2\pi)^3}; \quad (7)$$

$$\mathscr{L}(p_1, p) = \sum_{\substack{\lambda_0\lambda_1\nu \\ E^{\lambda_0},\, E^{\lambda_1}>0}} (\alpha e_\nu)^{\lambda_0\lambda_1}_{p_1,\, p-k} (\alpha e_\nu)^{\lambda_1\lambda_0}_{p-k/2,\, p+k/2}. \quad (8)$$

We note that

$$\rho^{\lambda_0\lambda_1}_{p+k/2,\, p-k/2} = (e^{-iH\tau})^{\lambda_0\lambda_0}_{p+k/2,\, p_1} (e^{iH\tau})^{\lambda_1\lambda_1}_{p_1-k,\, p-k/2}$$
$$= (e^{-iH\tau}\rho_0 e^{iH\tau})^{\lambda_0\lambda_1}_{p+k/2,\, p-k/2}$$

satisfies the equation

$$\partial\rho / \partial\tau = -i\,[H, \rho].$$

Furthermore

$$\mathrm{Sp}\,\rho = \sum_{g_1\mu_1} \rho^{\mu_1\mu_1}_{g_1 g_1} = \sum_{p\lambda_1} (e^{iH\tau})^{\lambda_0\lambda_1}_{p_1,\, p} (e^{-iH\tau})^{\lambda_1\lambda_0}_{p,\, p_1} = 1,$$

i.e. $\rho$ is an element of the density matrix in the momentum representation. We shall call $f_k(p, \tau)$ the averaged density matrix.

Thus the problem of averaging the number of transitions per unit time has been reduced to finding the averaged density matrix and doing the summation (8) and the integration (7).

## 3. EQUATIONS FOR THE AVERAGED DENSITY MATRIX

As shown in Ref. 4 the averaged density matrix satisfies the equation

$$\partial f_k^{\lambda_0\lambda_1}(p, \tau) / \partial\tau + i\,(E^{\lambda_0}_{p+k/2} - E^{\lambda_1}_{p-k/2})\, f_k^{\lambda_0\lambda_1}(p, \tau)$$
$$= n\pi \int \frac{d p'}{(2\pi)^3} |V_{p'-p}|^2 \{\delta(E^{\lambda_0}_{p'+k/2} - E^{\lambda_0}_{p+k/2}) \quad (9)$$
$$+ \delta(E^{\lambda_1}_{p'-k/2} - E^{\lambda_1}_{p'-k/2})\}\,[f_k^{\lambda_0\lambda_1}(p', \tau) - f_k^{\lambda_0\lambda_1}(p, \tau)]$$

with the initial conditions following from the definition of $f_k(p, \tau)$:

$$f_k^{\lambda_0\lambda_1}(p, \tau)\,|_{\tau=0} = \delta_{p,\, p_1-k/2}. \quad (10)$$

These equations differ from the classical transport equations in these points. First, $k\,\delta E / \delta p$ has been replaced by the difference $E^{\lambda_0}_{p+k/2} - E^{\lambda_1}_{p-k/2}$. Further, the term describing the collisions has been replaced by one half the sum of a term for the momentum $p + k/2$ and energy $E^{\lambda_0}_{p+k/2}$ and a term for $p - k/2$ and energy $E^{\lambda_0}_{p-k/2}$. For $k \ll p$ and $\lambda_0 = \lambda_1$, Eq. (9) goes over into the classical equation for the $k$ th Fourier component of the distribution function.

The equation for $f_0(p_1, t_1)$ is

$$\frac{\partial f_0^{\lambda_0\lambda_0}(p_1, t_1)}{\partial t_1} = 2\pi n \int \frac{d p'}{(2\pi)^3} |V_{p'-p_1}|^2 \delta(E^{\lambda_0}_{p_1} - E^{\lambda_0}_{p'})$$
$$\times [f_0^{\lambda_0\lambda_0}(p', t_1) - f_0^{\lambda_0\lambda_0}(p_1, t_1)] \quad (11)$$

with an initial condition

$$f_0^{\lambda_0\lambda_0}(\mathbf{p}_1, t_1)|_{t_1=0} = \delta_{\mathbf{p}_1, \mathbf{p}_0}. \quad (12)$$

From (11) and (12) we have

$$\int (2\pi)^{-3} f_0^{\lambda_0\lambda_0}(\mathbf{p}_1, t_1)\, d\mathbf{p}_1 = 1. \quad (13)$$

From (11) and (12) follows that $f_0^{\lambda_0\lambda_0}$ is different from zero only for $p_1 = p_0$. One can therefore introduce a function $v_0(\theta, t_1)$ where $\theta$ is the vector associated with the angle between $\mathbf{p}_0$ and $\mathbf{p}_1$:

$$f_0^{\lambda_0\lambda_0}(\mathbf{p}_1, t_1)(2\pi)^{-3} d\mathbf{p}_1 = \delta(p_1 - p_0)\, v_0(\theta, t_1)\, dp_1\, d\theta,$$
$$v_0(\theta, 0) = \delta(\theta). \quad (14)$$

From (13) we obtain

$$\int v_0(\theta, t_1)\, d\theta = 1. \quad (14')$$

It is easy to see that $f_{\mathbf{k}}(p, \tau)$ differs from zero only for $p$ close to $g = p_0 - k/2$. For $\tau = 0$ this follows from (10):

$$p^2 = p_0^2 + \frac{k^2}{4} - p_0 k + p_0 k(1 - \cos\widehat{\mathbf{p}_1 \mathbf{k}}) = g^2\left(1 - \frac{p_0 k}{g}\eta_1^2\right),$$

Here $\eta_1$ is the acute angle between $\mathbf{p}_1$ and $\mathbf{k}$.

We introduce the vectors for the angles between $\mathbf{p}$ and $\mathbf{k}$, and $\mathbf{p}'$ and $\mathbf{k}$.

$$\boldsymbol{\eta} = \mathbf{p}_\perp/g; \quad \boldsymbol{\eta}' = \mathbf{p}'_\perp/g; \quad g = p_0 - k/2. \quad (15)$$

Here $\mathbf{p}_\perp$ and $\mathbf{p}'_\perp$ denote the projections of $\mathbf{p}$ and $\mathbf{p}'$ respectively on a plane normal to $\mathbf{k}$. The $\delta$-functions on the right hand side of (9) can be rewritten as

$$\delta(E_{\mathbf{p}'\pm\mathbf{k}/2} - E_{\mathbf{p}\pm\mathbf{k}/2}) \approx \delta\left(p' - p \pm \frac{gk(\eta'^2 - \eta^2)}{2(g \pm k/2)}\right) \approx \delta(p' - p).$$

Hence the absolute value of $\mathbf{p}$ is approximately conserved in a collision.

The values of $p$ for which the function $f_{\mathbf{k}}(p, \tau)$ appreciably differs from zero are given by $(p - g)/g \sim \eta^2$ An estimate of the magnitude of $\eta^2$ will be given below.

Taking into account the approximate constancy of $p$, the function $v(\eta, \tau)$ will be given by

$$f_{\mathbf{k}}(\mathbf{p}, \tau)(2\pi)^{-3} d\mathbf{p} = \delta(p - g)\, v(\boldsymbol{\eta}, \tau)\, dp\, d\boldsymbol{\eta}. \quad (16)$$

From (10) we have

$$v(\boldsymbol{\eta}, \tau)|_{\tau=0} = \delta(\boldsymbol{\eta} - \boldsymbol{\eta}_0);$$
$$\boldsymbol{\eta}_0 = (\mathbf{p}_1 - \mathbf{k}/2)_\perp/g = \frac{\mathbf{p}_{1\perp}}{g}, \quad (17)$$

where $\eta_0$ is the vector of the angle between $\mathbf{p}_\perp - \mathbf{k}/2$ and $\mathbf{k}$. The vector $\eta_0$ is connected with the previously introduced vector $\theta$ by the relation

$$\theta = \mathbf{p}_1/p_0 - \mathbf{p}_0/p_0 = \mathbf{p}_1/p_0 - \mathbf{n} + \mathbf{n} - \mathbf{p}_0/p_0$$
$$= \mathbf{p}_{1\perp}/p_0 + \boldsymbol{\vartheta} = g\boldsymbol{\eta}_0/p_0 + \boldsymbol{\vartheta}. \quad (18)$$

Here $\boldsymbol{\vartheta} = \mathbf{n} - \mathbf{p}_0/p_0$ denotes the vector of the angle between $\mathbf{n} = \mathbf{k}/k$ and the initial direction of the electron. From (15) we obtain

$$\mathbf{p} = (\mathbf{pn})\mathbf{n} + \mathbf{p}_\perp \approx g\mathbf{n} + g\boldsymbol{\eta}; \quad \mathbf{p} + \mathbf{k}/2 \approx p_0\mathbf{n} + g\boldsymbol{\eta};$$
$$\mathbf{p} - \mathbf{k}/2 \approx (p_0 - k)\,\mathbf{n} + g\boldsymbol{\eta}. \quad (19)$$

The energy difference entering as an argument in (9) can be written as

$$E^{\lambda_0}_{\mathbf{p}+\mathbf{k}/2} - E^{\lambda_1}_{\mathbf{p}-\mathbf{k}/2}$$
$$= \sqrt{1 + (p_0\mathbf{n} + g\boldsymbol{\eta})^2} - \sqrt{1 + [(p_0 - k)\mathbf{n} + g\boldsymbol{\eta}]^2}$$
$$= k[1 - 1/2p_0(p_0 - k) - g^2\eta^2/2p_0(p_0 - k)]. \quad (20)$$

Utilizing (18) in integrating (9) with respect to $p$ we find

$$\frac{\partial v(\boldsymbol{\eta}, \tau)}{\partial \tau} + i\left(a - b\frac{\eta^2}{2}\right)v(\boldsymbol{\eta}, \tau)$$
$$= \frac{ng^2}{(2\pi)^2}\int |V_{g(\boldsymbol{\eta}'-\boldsymbol{\eta})}|^2 \{v(\boldsymbol{\eta}', \tau) - v(\boldsymbol{\eta}, \tau)\}\, d\boldsymbol{\eta}', \quad (21)$$
$$a = k\left(1 - \frac{1}{2p_0(p_0 - k)}\right); \quad b = \frac{g^2 k}{p_0(p_0 - k)}; \quad \boldsymbol{\eta}' = \frac{\mathbf{p}'_\perp}{g}.$$

For $V_{\mathbf{q}}$ we choose the expression

$$V_{\mathbf{q}} = 4\pi Z e^2/(q^2 + \varkappa^2), \quad \varkappa \sim 1/a, \quad (22)$$

where $a$ is the Thomas-Fermi radius $a \sim 137 Z^{-1/3}$.

Inserting (22) into (21) gives

$$\frac{\partial v}{\partial \tau} + i\left(a - b\frac{\eta^2}{2}\right)v$$
$$= \frac{4nZ^2e^4}{g^2}\int \frac{d\boldsymbol{\eta}'}{[(\boldsymbol{\eta}' - \boldsymbol{\eta})^2 + \theta_\perp^2]^2}[v(\boldsymbol{\eta}', \tau) - v(\boldsymbol{\eta}, \tau)],$$
$$\theta_\perp = \varkappa/g. \quad (23)$$

By expanding $v(\eta', \tau)$ into a power series in $\boldsymbol{\eta}' - \eta$ we obtain from (23) the Fokker-Planck differential equation

$$\frac{\partial v}{\partial \tau} + i\left(a - b\frac{\eta^2}{2}\right)v = q\Delta_\eta v,$$
$$q = \frac{2\pi n Z^2 e^4}{g^2}\ln\frac{\theta_2}{\theta_1} = \frac{B}{g^2}. \quad (24)$$

We will determine the quantity $\theta_2$ from the conditions of validity of the Fokker-Planck equation. The first term of the expansion is

$$\int_{\theta_1}^{\theta_2} \frac{\theta d\theta}{\theta^4}\,\theta^2 \frac{1}{4}\,\Delta_\eta\, v \approx \frac{1}{4}\ln\frac{\theta_2}{\theta_1}\Delta_\eta v.$$

The next term of the expansion is of the order

$$\int_{\theta_1}^{\theta_2} \frac{\theta d\theta}{\theta^4}\,\theta^4\,\frac{\partial^4 v}{\partial\eta^4} \sim \theta^2{}_1 \frac{1}{\eta^2}\Delta_\eta v.$$

We furthermore need

$$\int_0^\infty v e^{-ik\tau}\,d\tau = \int_0^\infty v' d\tau, \qquad v' = e^{-ik\tau}\,v;$$

$v'$ satisfies an equation which one obtains when replacing in (24) the quantity $a$ by the quantity $a' = a - k$.

The essential values of $\eta^2$ are given by the relations

$$(a' - b\eta^2/2)\,v \sim b\eta^2 v \sim q\Delta_\eta v;\ \ \eta^2 \sim \sqrt{q/b}. \quad (25)$$

This estimate will be confirmed below. The condition for the possibility of the series expansion of $v$ therefore has the form

$$\theta_2^2\sqrt{\frac{b}{q}} \sim \ln\frac{\theta_2}{\theta_1};\ \ \theta_2 \sim \left(\frac{q}{b}\right)^{1/4} L^{1/2};\quad L = \ln\frac{\theta_2}{\theta_1}\ (26)$$

At sufficiently large energies $\theta_2$ becomes equal to the diffraction angle of the nucleus, which is $1/gR$. Then the upper limit of integration of (23) over $|\boldsymbol{\eta}' - \boldsymbol{\eta}|$ is given by $1/gR$ and the value of $L$ becomes identical with the well-known expression from the theory of multiple scattering[6]. Putting $R \approx 0.5\, r_0 Z^{1/3}$, we obtain for $\theta_2 > 1/gR$.

$$L = \ln(137^2/0{,}5\, Z^{2/3}) = 2\ln(190\, Z^{-1/3}). \quad (27)$$

## 4. SUMMATION OVER ELECTRON SPIN AND PHOTON POLARIZATION

The summation over $\lambda_0$ and $\lambda_1$ cannot be performed in the usual manner since the momenta are different for the different spin functions.

It is however possible to reduce the sum (8) to an evaluation of the trace of two-by-two matrices. The spin functions can be written in the form

$$u_g^\mu = \begin{Bmatrix} v_\mu \\ \dfrac{\sigma \mathbf{g}}{E_g + 1} v_\mu \end{Bmatrix} N_g;\qquad v_1 = \begin{Bmatrix}1\\0\end{Bmatrix},\ \ v_2 = \begin{Bmatrix}0\\1\end{Bmatrix};$$

$$N_g^2 = \frac{1}{1 + g^2(E_g+1)^{-2}} \approx \frac{1}{2}. \quad (28)$$

Here $\sigma_{1,2,3}$ are the Pauli spin matrices, and the $z$ axis is parallel to $\mathbf{n}$. Substitution in (8) yields

$$\mathscr{L}(\mathbf{p}_1\mathbf{p}) = \frac{1}{4}\sum_{i=1,2}\mathrm{Sp}\left\{\left[\frac{\sigma_i\sigma(\mathbf{p}_1-\mathbf{k})}{E_{\mathbf{p}_1-\mathbf{k}}+1} + \frac{\sigma\mathbf{p}_1\sigma_i}{E_{\mathbf{p}_1}+1}\right]\times\left[\frac{\sigma_i\sigma(\mathbf{p}+\mathbf{k}/2)}{E_{\mathbf{p}+\mathbf{k}/2}+1} + \frac{\sigma(\mathbf{p}-\mathbf{k}/2)\sigma_i}{E_{\mathbf{p}-\mathbf{k}/2}+1}\right]\right\}. \quad (29)$$

We introduce the abbreviations

$$\mathbf{A} = \frac{\mathbf{p}_1}{E_{\mathbf{p}_1}+1};\quad \mathbf{B} = \frac{\mathbf{p}_1-\mathbf{k}}{E_{\mathbf{p}_1-\mathbf{k}}+1};\quad \mathbf{C} = \frac{\mathbf{p}+\mathbf{k}/2}{E_{\mathbf{p}+\mathbf{k}/2}+1};$$

$$\mathbf{D} = \frac{\mathbf{p}-\mathbf{k}/2}{E_{\mathbf{p}-\mathbf{k}/2}+1}. \quad (30)$$

We then have from (29)

$$\mathscr{L}(\mathbf{p}_1,\mathbf{p}) = \frac{1}{4}\sum_{i=1,2}\mathrm{Sp}\,(\sigma_i\mathbf{B}\sigma + \mathbf{A}\sigma\sigma_i)(\sigma_i\sigma\mathbf{C} + \mathbf{D}\sigma\sigma_i)$$
$$= \mathbf{BD} + \mathbf{AC} - (\mathbf{Bn})(\mathbf{Cn}) - (\mathbf{An})(\mathbf{Dn}). \quad (31)$$

Each of the scalar products of (31) has a value close to unity. However, as will become clear later, the complete expression is of order $\sim\eta^2$. We therefore rewrite (31) as a sum of small terms so that it will be possible to keep only the main terms of expansions in powers of $1/p$.

We express each of the vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ as a sum of two terms, one is parallel and the other perpendicular to $\mathbf{n}$: $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$; $\mathbf{A}_1 \parallel \mathbf{n}$, $\mathbf{A}_2 \perp \mathbf{n}$ and similarly for $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$. Then (31) becomes

$$\mathscr{L} = (\mathbf{D}_1 - \mathbf{C}_1)(\mathbf{B}_1 - \mathbf{A}_1) + \mathbf{B}_2\mathbf{D}_2 + \mathbf{A}_2\mathbf{C}_2. \quad (32)$$

In (32) each term is of order $\sim p^{-2}$ or $\eta^{+2}$.

From (17), (19), and (30) we obtain up to terms of required accuracy

$$A_1 = C_1 = 1 - 1/p_0;\ \ B_1 = D_1 = 1 - 1/(p_0 - k),$$
$$\mathbf{A}_2 = g\boldsymbol{\eta}_0/p_0;\quad \mathbf{B}_2 = g\boldsymbol{\eta}_0/(p_0 - k);\quad \mathbf{C}_2 = g\boldsymbol{\eta}/p_0;$$
$$\mathbf{D}_2 = g\boldsymbol{\eta}/(p_0 - k). \quad (33)$$

Inserting these expressions into (32) we finally obtain

$$\mathcal{L} = K_1 + K_2\eta\eta_0,$$

$$K_1 = \frac{k^2}{p_0^2(p_0-k)^2}; \quad K_2 = \frac{g^2[p_0^2+(p_0-k)^2]}{p_0^2(p_0-k)^2}. \quad (34)$$

## 5. PROBABILITY OF BREMSSTRAHLUNG

Let $W_r(p_0 k)dk$ denote the probability of emission of a photon of energy between $k$ and $k+dk$ per unit path. Since the initial $\psi$-function of the electron was normalized to unit volume (or to unit flux for $c = 1$) we have from (3) and (7)

$$W_r = \frac{1}{2}\sum_{\substack{\lambda_0\nu \\ E^{\lambda_0}>0}} \int <Q> d\vartheta \frac{k^2}{(2\pi)^3} = \frac{k^2}{(2\pi)^3} 2\,\mathrm{Re}\int_0^t d\tau e^{ik\tau}\int J_1 d\vartheta. \quad (35)$$

Inserting into (7) the expressions for $v_0$, $v$, and $\mathcal{L}$, as given by (14), (16), and (33) respectively, we find

$$\int J_1 d\vartheta = \frac{\pi e^2}{k}\int v_0(\theta, t_1)\, v(\eta,\tau)\,[K_1 + K_2\eta\eta_0]\, d\theta\, d\eta\, d\vartheta.$$

We now express $\theta$ in terms of $\eta_0$ and $\vartheta$ according to (18) and so obtain

$$\int J_1 d\vartheta = \frac{\pi e^2}{k}\int v_0\left(\frac{g}{p_0}\eta_0 + \vartheta,\ t_1\right) \times v(\eta,\ \tau)\,(K_1 + K_2\eta\eta_0)\,\frac{g^2}{p_0^2}\, d\eta_0\, d\eta\, d\vartheta.$$

Utilizing the normalization condition (14′) for $v_0$ we have

$$W_r = \frac{e^2 g^2 k}{(2\pi)^2 p_0^2} \times \mathrm{Re}\int_0^t d\tau e^{ik\tau}\int (K_1 + K_2\eta\eta_0)\, v(\eta,\tau)\, d\eta\, d\eta_0. \quad (36)$$

We denote by $\tau_0$ the values of $\tau$ that are significant in the integral (36). From (24) and (25) we obtain the estimate

$$\eta^4 \sim q/b,\ \ \tau_0 \sim 1/b\eta^2 \sim (bq)^{-1/2},\ \ \eta^2 \sim q\tau_0. \quad (37)$$

If the time $\tau$ the electron spends in the medium is much larger than $\tau_0$ the upper limit of the integral over $\tau$ in (35) can be replaced by infinity and then $W$ will not depend on $\tau$. We shall treat below the case $\tau \gg \tau_0$ or $1 \gg l \gg l_k$, where here $l$ is the thickness of the scattering medium, and $l_k = c\tau_0$.

From (37) one obtains for a condensed medium ($n \approx 3\times 10^{23}$)

$$l_k \sim (p_0/\sqrt{k}\, Z)\, 10^{-6}\ \text{cm}. \quad (38)$$

For $Z = 10$, $E_0 = 10^{18}$ eV ($p_0 = \frac{10^{18}}{5\cdot 10^5} = 2\cdot 10^{12}$), $k = p_0/2$, this gives $l_k \sim 0.2$ cm.

For $t \gg \tau_0$ one easily obtains the angular distribution of the photons. The width of the angular distribution is given by the function $v_0(\theta, t_1)$ and has the order $\bar{\theta}^2 \sim qt_1$. Furthermore, $\eta_0^2 \sim \eta^2 \sim q\tau$ and therefore one can replace the function $v_0(g^2\eta_0/p_0^2 + \vartheta,\ t_1)$ by $v_0(\vartheta,\ t)$ and the photon angular distribution is given by

$$W'_r(p_0,\ k,\ \vartheta)\, d\vartheta = v_0(\vartheta, t)\, W_r(p_0, k)\, d\vartheta, \quad (39)$$

*i.e.*, the angular distribution of the photons is the same as the angular distribution of multiply scattered electrons of energy $p_0$.

We now define

$$\int v(\eta,\ \tau;\ \eta_0)\, d\eta_0 = h_1(\eta,\ \tau);$$

$$\int v(\eta,\ \tau;\ \eta_0)\, \eta_0 d\eta_0 = \mathbf{R}(\eta,\ \tau).$$

Since the coefficients of Eq. (24) do not depend on $\eta_0$, the equations for $h_1$ and $\mathbf{R}$ are identical to (24). The boundary conditions are $h_1(\eta, 0) = 1$; $\mathbf{R}(\eta, 0) = \eta$ which can be obtained immediately from (17).

The coefficients of Eq. (24) depend only on $\eta^2$. One can therefore write the solutions in the form

$$h_1(\eta,\ \tau) = h(z,\ \tau);$$

$$\mathbf{R}(\eta,\ \tau) = \eta g(z,\ \tau), \quad z = \eta^2/2.$$

Here $h$ and $g$ satisfy the equations

$$\frac{\partial h}{\partial \tau} + i(a - bz)\, h = 2zq\left(\frac{\partial^2 h}{\partial z^2} + \frac{1}{z}\frac{\partial h}{\partial z}\right),$$

$$\frac{\partial g}{\partial \tau} + i(a - bz)\, g = 2zq\left(\frac{\partial^2 g}{\partial z^2} + \frac{2}{z}\frac{\partial g}{\partial z}\right). \quad (40)$$

We introduce the functions

$$\varphi_1(z) = \int_0^\infty e^{ik\tau} h(z,\ \tau)\, d\tau;\ \ \varphi_2(z) = z\int_0^\infty e^{ik\tau} g(z,\ \tau)\, d\tau. \quad (41)$$

Then (36) becomes

$$W_r = \frac{e^2 k g^2}{2\pi p_0^2}\,\mathrm{Re}\left\{K_1\int_0^\infty \varphi_1 dz + 2K_2\int_0^\infty \varphi_2 dz\right\}. \quad (42)$$

The equations for $\varphi_1$ and $\varphi_2$ are obtained by integrating (40) with respect to $\tau$ taking into account the initial conditions for $h$ and $g$:

$$z\varphi_1'' + \varphi_1' + i(\alpha + \beta z)\varphi_1 = -1/2q; \quad (43)$$

$$z\varphi_2'' + i(\alpha + \beta z)\varphi_2 = -z/2q; \quad (44)$$

$$\alpha = \frac{k-a}{2g} = \frac{k}{4p_0(p_0-k)q} > 0,$$
$$\beta = \frac{kg^2}{2p_0(p_0-k)q} > 0. \quad (45)$$

Eq. (43) and (44) can be solved by means of the Laplace transformation. Putting

$$u(\lambda) = \int_0^\infty \varphi_1(z)e^{-\lambda z}dz,$$

we have from (43)

$$u' + \frac{\lambda - i\alpha}{\lambda^2 + i\beta}u = \frac{1}{2q\lambda(\lambda^2 + i\beta)}. \quad (46)$$

This gives

$$u(\lambda) = $$
$$= \frac{1}{2q(\lambda_1^2-\lambda^2)^{1/2}}\left(\frac{\lambda_1+\lambda}{\lambda_1-\lambda}\right)^\mu \int^{\xi_1} \frac{d\xi}{\xi(\lambda_1^2-\xi^2)^{1/2}}\left(\frac{\lambda_1-\xi}{\lambda_1+\xi}\right)^\mu,$$
$$\lambda_1 = \sqrt{\beta}\,e^{-i\pi/4}, \quad \mu = (\alpha/2\sqrt{\beta})\,e^{-i\pi/4}. \quad (47)$$

The arbitrary constant $\xi_1$ is determined by the condition of boundedness of $\varphi_1(z)$ for $z \to \infty$. For this to hold it is necessary that the function $u(\lambda)$ has no singularities on the right half-plane, which gives $\xi_1 = \lambda_1$.

In (42) we need the expression

$$\operatorname{Re}\int_0^\infty \varphi(z)dz = \operatorname*{Re}_{\lambda\to 0} u(\lambda).$$

The function $u(\lambda)$ has a logarithmic singularity at $\lambda = 0$. The above limit therefore depends on the way $\lambda$ approaches zero. It follows from

$$\operatorname{Re} u(\lambda) = \int_0^\infty e^{-\lambda^0 z}(\operatorname{Re}\varphi_1 \cos\lambda' z - \operatorname{Im}\varphi_1 \sin\lambda' z)\,dz;$$
$$\lambda = \lambda^0 + i\lambda'$$

that in order for

$$\int_0^\infty \operatorname{Re}\varphi_1(z)\,dz = \operatorname{Re} u(0)$$

to hold $\lambda$ has to approach zero along the real axis.

We now separate in the integral (47) the part which is singular at $\xi = 0$. We thus obtain

$$\operatorname*{Re}_{\substack{\lambda\to 0\\ \arg\lambda=0}} u(\lambda) =$$
$$= \operatorname*{Re}_{\substack{\lambda\to 0,\\ \arg\lambda=0}} \frac{1}{2q}\frac{1}{\lambda_1}\left\{\int_\lambda^{\lambda_1}\frac{d\xi}{\xi}\left[\frac{1}{(\lambda_1^2-\xi^2)^{1/2}}\left(\frac{\lambda_1-\xi}{\lambda_1+\xi}\right)^\mu - \frac{1}{\lambda_1}\right] + \frac{1}{\lambda_1}\ln\frac{\lambda_1}{\lambda}\right\}.$$

Since

$$\operatorname*{Re}_{\substack{\lambda\to 0\\ \arg\lambda=0}} \frac{1}{\lambda_1^2}\ln\frac{\lambda_1}{\lambda} = \frac{\pi}{4\beta},$$

this equals

$$\operatorname*{Re}_{\substack{\lambda\to 0\\ \arg\lambda=0}} u(\lambda) = \frac{\pi}{8\beta q}$$
$$- \frac{1}{2\beta q}\operatorname{Im}\int_0^1 \frac{dx}{x}\left\{\frac{1}{(1-x^2)^{1/2}}\left(\frac{1-x}{1+x}\right)^\mu - 1\right\}.$$

Putting $x = \tanh(t/2)$ and separating the imaginary part we find

$$\int_0^\infty \operatorname{Re}\varphi_1 dz = \frac{1}{12q\alpha^2}G(s);$$

$$G(s) = 48s^2\left(\frac{\pi}{4} - \frac{1}{2}\int_0^\infty e^{-st}\frac{\sin st}{\sinh st}\,dt\right); \quad (48)$$

$$s = \frac{\alpha}{2\sqrt{2\beta}} = \frac{1}{8g}\sqrt{\frac{k}{p_0(p_0-k)q}}.$$

To solve (44) we introduce

$$f = \varphi_2 - \frac{i}{2q\beta}, \quad zf'' + i(\alpha + \beta z)f = \frac{\alpha}{2q\beta}. \quad (44')$$

We further introduce

$$v(\lambda) = \int e^{-\lambda z} f dz, \quad v' + \frac{2\lambda - i\alpha}{\lambda^2 + i\beta}v = \frac{f(0) - \alpha/2q\beta\lambda}{\lambda^2 + i\beta}. \quad (49)$$

Since $\varphi_2(0) = 0$, we have $f(0) = -i/2q\beta$. The solution of (49) is

$$v(\lambda) = -\frac{i}{2q\beta}\frac{1}{\lambda_1^2 - \lambda^2}$$
$$\times\left(\frac{\lambda_1+\lambda}{\lambda_1-\lambda}\right)^\mu \int_\lambda^{\lambda_1}\left(1 - \frac{i\alpha}{\xi}\right)\left(\frac{\lambda_1-\xi}{\lambda_1+\xi}\right)^\mu d\xi.$$

The calculation of $\operatorname{Re} v(0)$ is completely analogous to the above calculation of $\operatorname{Re} u(0)$.

Introducing once more the substitution $\xi/\lambda_1 = x$

and separating the divergent part of the integral we obtain

$$\operatorname*{Re}_{\substack{\lambda\to 0\\ \arg\lambda=0}} v(\lambda) = \frac{1}{2q\beta^2} \operatorname*{Re}_{\substack{\lambda\to 0\\ \arg\lambda=0}} \left\{\lambda_1 \int_0^1 \left(\frac{1-x}{1+x}\right)^{\mu} dx \right.$$

$$\left. - i\alpha \int_0^1 \frac{dx}{x}\left[\left(\frac{1-x}{1+x}\right)^{\mu} - 1\right] - i\alpha \ln \frac{\lambda_1}{\lambda}\right\}.$$

Introducing $x = \tanh(t/2)$ and separating the real part we find

$$\operatorname*{Re}_{\substack{\lambda\to 0\\ \arg\lambda=0}} v(\lambda) = \frac{1}{2q\beta^2}\left\{-\frac{\pi\alpha}{4} + \alpha\int_0^\infty e^{-st}\frac{\sin st}{\sinh t}\,dt\right.$$

$$\left. + \frac{1}{2}\sqrt{\frac{\beta}{2}}\int_0^\infty e^{-st}\,\frac{\cos st + \sin st}{\sinh \frac{t}{2}}\,dt\right\}.$$

Thus we obtain

$$\int_0^\infty \operatorname{Re}\varphi_2 dz = \int_0^\infty \operatorname{Re} f dz = \frac{1}{6\alpha\beta q}\,\Phi(s), \qquad (50)$$

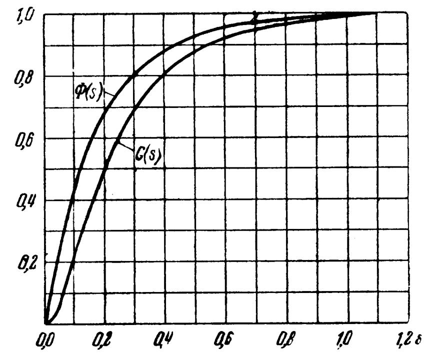
$$\Phi(s) = 3s\int_0^\infty e^{-sx}\,\frac{\cos sx + \sin sx}{\cosh^2(x/2)}\,dx$$
$$+ 24s^2\int_0^\infty e^{-sx}\frac{\sin sx}{\sinh x}\,dx - 6\pi s^2 \qquad (51)$$
$$= 12s^2\int_0^\infty \coth\frac{x}{2}\, e^{-sx}\sin sx\, dx - 6\pi s^2.$$

The function $\Phi(s)$ was introduced in Ref. 3. The functions $\Phi(s)$ and $G(s)$ can be expressed in terms of the logarithmic derivative of the Γ-function*

$$\Phi(s) = 12s^2\{-\operatorname{Im}[\Psi(s-is) + \Psi(s+1-is)] - \pi/2\}$$
$$= 6s - 6\pi s^2 + 24s^3\sum_{k=1}^{\infty}\frac{2}{(s+k)^2+s^2},$$
$$G(s) = 48s^2[\pi/4 + \operatorname{Im}\Psi(s+1/2-is)] \qquad (52)$$
$$= 12\pi s^2 - 48s^3\sum_{k=0}^{\infty}\frac{1}{(k+s+1/2)^2+s^2}.$$

These formulae are useful for tabulating the functions..

* Expression (52) has been derived by S. A. Kheifets.




Some values of the functions Φ and $G$ are given in the table, and the functions are plotted in the graph.

| $s$ | $\Phi(s)$ | $G(s)$ |
|---|---|---|
| 0 | 0 | 0 |
| 0,05 | 0,258 | 0,094 |
| 0,1 | 0,446 | 0,206 |
| 0,2 | 0,686 | 0,475 |
| 0,3 | 0,805 | 0,695 |
| 0,4 | 0,880 | 0,800 |
| 0,5 | 0,93 | 0,875 |
| 0,6 | 0,95 | 0,917 |
| 0,7 | 0,965 | 0,945 |
| 0,8 | 0,975 | 0,963 |
| 0,9 | 0,985 | 0,975 |
| 1,0 | 0,990 | 0,985 |
| 1,5 | 0,998 | 0,994 |
| 2 | 0,999 | 0,998 |

The asymptotic behavior of Φ and $G$ is: for $s \to 0$

$$\Phi \to 6s;\quad G \to 12\pi s^2;$$

and for $s \to \infty$

$$\Phi \to 1 - 0{,}012/s^4;\quad G \to 1 - 0{,}029/s^4. \qquad (53)$$

For $s \lesssim 1$ we have

$$\eta_0^2 \sim \eta^2 \sim \frac{\operatorname{Re}\int\varphi_2 dz}{\operatorname{Re}\int\varphi_1 dz} \sim \frac{\alpha}{\beta s} \sim \frac{1}{\sqrt{\beta}} = \sqrt{\frac{2q}{b}},$$

This confirms the previously given estimate of the significant values of $\eta^2$.

Inserting (48) and (50) into (42) we obtain

$$W_r = \frac{ekg^2}{2\pi p_0^2}\left\{K_1\,\frac{G}{12q\alpha^2} + 2K_2\,\frac{\Phi}{6\alpha\beta q}\right\},$$

or, utilizing (24), (34), and (45) this becomes

$$W_r = \frac{2e^2}{3\pi p_0^2 k}\,B\,\{k^2 G(s) + 2\,[p_0^2 + (p_0 - k)^2]\,\Phi(s)\};$$
$$B = 2\pi Z^2 e^4 n \ln\frac{\theta_2}{\theta_1}. \qquad (54)$$

The estimate (26) for $\theta_2$ can be rewritten in the more convenient form

$$\theta_2 \sim (q/b)^{1/4} L^{1/2} = (2\beta)^{-1/4} L^{1/2} \sim L^{1/2}/gs^{1/2}. \quad (55)$$

For $s \gtrsim 1$

$$W_r = (2e^2/3\pi p_0^2 k) B \{k^2 + 2[p_0^2 + (p_0 - k)^2]\}. \quad (56)$$

This expression differs from known formulae (*e.g*, Ref. 6) only by a factor of the order of magnitude unity inside the logarithm. The inaccuracy connected with the appearance of this factor is associated with the inaccuracy introduced by the use of the Fokker-Planck method. More accurate formulae can be obtained by solving the integral equation (23). This however, is a rather difficult task.

Since for $s = 1$ both $\Phi$ and $G$ have values close to 1, one can obtain a simple formula. For the logarithm one has

$$L = \ln(\theta_2/\theta_1) = \ln(190/Z^{1/3}s^{1/2}). \quad (57)$$

Then $W_r$ is determined for $s \leq 1$; for $s = 1$ the expression becomes the Bethe-Heitler formula.

For $s \ll 1$ we obtain from (53) and (55)

$$W_r = \frac{8e^2}{\pi p_0^2 k} Bs[p_0^2 + (p_0 - k)^2] = \frac{e^2}{\pi p_0^2}\sqrt{\frac{B}{kp_0(p_0 - k)}}[p_0^2 + (p_0 - k)^2]. \quad (58)$$

In this case the probability is proportional to the square root of the density.

For $k \ll p_0$, (55) yields (see Ref. 3)

$$W_r = (8e^2/3\pi k) B\Phi(s). \quad (59)$$

For extremely soft quanta one has to take into account the fact that the dielectric constant differs from unity. The dielectric constant $\epsilon$ enters the expressions via the normalization of the operator $A$; further in the integral over $\tau$ in (3) one has to replace $e^{ik\tau}$ by $e^{i\omega\tau}$ where $\omega = k/\sqrt{\epsilon}$. Considering frequencies $\omega \gg \omega_0 = \sqrt{4\pi n Z e^2}$ we have

$$\varepsilon \approx 1 - \omega_0^2/\omega^2; \quad \omega = k/\sqrt{\varepsilon} \approx k(1 + \omega_0^2/2\omega^2). \quad (60)$$

The change of normalization of $A$ due to $\epsilon$ results in the multiplication of $W_r$ by a factor which has a numerical value close to unity; it can therefore be discarded. Thus one takes into account the influence of the dielectric constant by replacing in (45) the quantity $\alpha$ by $\alpha'$.

$$\alpha' = \frac{\omega - a}{2q} \approx \alpha + \frac{\omega - k}{2q} \approx \alpha\left(1 + p_0^2\frac{\omega_0^2}{\omega^2}\right). \quad (61)$$

Using (54) we obtain for small $k$ the more general formula

$$W_r = (8e^2/3\pi k) B'\Phi(s\gamma)/\gamma; \quad \gamma = 1 + p_0^2\omega_0^2/\omega^2, \quad (62)$$

where $B'$ differs from $B$ in that in the logarithm $s$ is replaced by $s\gamma$. For $s\gamma > 1$; $\gamma \gg 1$ we have

$$W_r = (4/3\pi) Ze^4 Lk/p_0^2 \quad (63)$$

in accordance with the result obtained by Ter-Mikaelian[7] for this limiting case. Thus, the difficulties associated with the infrared catastrophe do not appear for the case of radiation inside a medium.

We now will give an expression for the radiation length. For that purpose we define a function $\xi(s)$ which describes the change of $L$ with energy:

$$\xi(s) = 1 + \frac{\ln(1/s)}{2\ln(190 Z^{-1/3})}; \quad 1 \geqslant s \geqslant s_1; \qquad \xi(s) = 1; \quad s > 1;$$
$$\xi(s) = 2; \quad s_1^{1/2} = Z^{1/3}/190; \quad s < s_1. \quad (64)$$

Here $s_1$ is the value of $s$ for which $L = 2\ln(190 Z^{-1/3})$.

From (55), (57), and (64) we have

$$B = 2\pi n e^4 Z^2 \xi(s) \ln\frac{190}{Z^{1/3}} = \frac{\pi\, 137}{2t_0'}\xi(s),$$
$$1/t_0' = (4ne^4Z^2/137)\ln(190/Z^{1/3}) \quad t_0' = t_0 mc/\hbar, \quad (65)$$

Here $t_0$ is the radiation length in cm.

From (48) we have for $s$

$$s = 1{,}4 \cdot 10^3 [kt_0/p_0(p_0 - k)\xi(s)]^{1/2}. \quad (66)$$

The probability for radiation in a path length equal equal to one radiation length is

$$W_r t_0' = (\xi(s)/3kp_0^2)\{k^2 G(s) + 2[p_0^2 + (p_0 - k)^2]\Phi(s)\}. \quad (67)$$

In lead ($t_0 \approx 0.5$ cm) for $k = p_0/2$ and at $s = 1$ we have $p_0 = 2 \times 10^6 t_0 = 5 \times 10^{11}$ ev; at $s = 0.2$, which corresponds to a deviation of (62) from the Bethe-Heitler formula by 30%, we have $p_0 \approx 1.2 \times 10^{13}$ ev; $s = s_1$ corresponds to an energy $p_0 \approx 10^{18}$ ev.

## 6. PROBABILITY OF PAIR PRODUCTION

We denote by $W_p(k, p_0)$ the probability of pair production per unit path length for an electron energy between $p_0$ and $p_0 + dp_0$; $W_p$ is summed over all possible positron states.

The probability of the inverse reaction to pair production, $\widetilde{W}_r$, can be found in a similar manner as $W_r$; one just has to sum (8) over the negative energy states, and one has to invert the sign of $E^{\lambda_1}$ in (9). The final formulae can be obtained from the ones written down by replacing $p_0 - k$ by $k - p_0$. So, for example, the quantity $g = (p_0 + p_0 - k)/2$ becomes $\widetilde{g} = (p_0 - p_0 + k)/2$.

Thus for the probability $\widetilde{W}_r$, which differs from $W_p$ only by a statistical weight factor, one obtains from (55)

$$W_p = \frac{p_0^2}{k^2}\widetilde{W}_r = \frac{2e^2}{3\pi k}$$

$$B\left\{G(\widetilde{s}) + 2\left[\frac{p_0^2}{k^2} + \left(1 - \frac{p_0}{k}\right)^2\right]\Phi(\widetilde{s})\right\}, \quad (68)$$

$$\widetilde{s} = \frac{1}{8}\sqrt{\frac{k}{p_0(k-p_0)B}} = 1{,}4\cdot 10^3\sqrt{\frac{kt_0}{p_0(k-p_0)\,\xi(\widetilde{s})}}.$$

Here $\widetilde{s}$ differs from $s$ only by the replacement of $p_0 - k$ by $k - p_0$. The probability for pair creation in a path length equal to one radiation length is

$$W_p t_0' = \frac{\xi(\widetilde{s})}{3k}\left\{G(\widetilde{s}) + 2\left[\frac{p_0^2}{k^2} + \left(1 - \frac{p_0}{k}\right)^2\right]\Phi(\widetilde{s})\right\}. \quad (69)$$

For $\widetilde{s} = 1$ this expression becomes the well-known formula for pair creation[6]. For $\widetilde{s} \ll 1$ we have

$$W_p t_0' = \frac{4\xi(\widetilde{s})}{k}\left\{\frac{p_0^2}{k^2} + \left(1 - \frac{p_0}{k^2}\right)^2\right\}\widetilde{s}. \quad (70)$$

The equations (67) and (69) represent the solution to the problem of bremsstrahlung and pair creation at high energies.

The author is grateful to V. M. Galitski for his critical discussion of the results and to S. A. Kheifets for numerical computations.

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Translated by M. Danos
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