Reconstruction of the Scattering Matrix of a Two-Nucleon System

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The types of experiments needed for determination of all elements of the scattering matrix are investigated. It is shown that because of the unitarity condition the required number of experiments equals the number of complex functions entering into the scattering matrix. For nucleon-nucleon scattering, the inelastic scattering matrix can be determined on the basis of five experiments measuring the cross section, polarization, normal components of the polarization correlation tensor and of the triple scattering tensors (for the scattered and recoil particles). It is shown that experiments involving spin rotation by a magnetic field are not necessary for a phase shift analysis.

1. INTRODUCTION

THE LITERATURE contains a large amount of experimental information concerning nucleon-nucleon scattering in a very wide range of energies.*

A phase shift analysis of the experimental findings has also appeared in many articles. But it is seen in all of these articles that the phase shift analysis is not unique and the authors usually present a few sets of phases which provide equally good descriptions of the result. The phase-shift analysis begins with a preliminary limitation of the number of states which participate in the scattering. Then the algebraic equations for a finite number of phases are written out. But for particles with spin, it is not clear which experiments provide the data for these equations.

The clearest example is proton-proton scattering at low energies. The two principal states involved in this case are $^1S_0$ and $^3P_0$. These states cannot be distinguished by either cross section or polarization measurements. Yet they can easily be separated by polarization correlation. Thus more complicated experiments are obviously needed. In other instances, where the scattering is anisotropic and the determination of a finite number of phases may appear to be formally possible, it is not clear to what extent these phases can also describe more complex scattering such as multiple scattering.

It therefore seemed important to attempt an analysis of all possible experiments in order to show which of these are independent in the sense that a complete reconstruction of the scattering matrix is possible when their results are known. For the purpose of clarity, we shall begin with two very simple examples: the scattering of spin-zero particles in a central force field and of particles with spin ½ by spin-zero nuclei. Subsequently we shall consider nucleon-nucleon scattering.

The case of arbitrary spin, which thus far possesses only theoretical significance and the case of photon scattering will be examined in future communications. In the present article we shall also limit ourselves to the study of the scattering matrix for a given energy. The energy dependence of the matrix elements requires further study.

2. SCATTERING OF SPIN-ZERO PARTICLES

Measurement of the differential cross section of a spin-zero particle determines a function $\sigma(\vartheta)$ which is the square of the absolute value of the scattering amplitude:

$$\sigma(\vartheta) = |f(\vartheta)|^2. \qquad (2.1)$$

It is clear that when this quantity is measured for only one value of the energy and of the angle ϑ the phase of the complex function $f(\vartheta)$ is not completely determined.

However, when the scattering is measured at all angles and is known to be elastic, the phase of $f(\vartheta)$ is also determined. Indeed, the unitarity condition for the amplitude $f(\vartheta)$ can be written as[2]

$$4\pi \operatorname{Im} f(\vartheta) = k \int d\omega f^*(\vartheta'') f(\vartheta'). \qquad (2.2)$$

Here ϑ' is the angle between **k** (the initial wave

*Wolfenstein[1] has given us a good review of recent data. We take this opportunity to thank Professor Wolfenstein for sending us his manuscript before publication.

vector) and the variable vector k″ over whose directions $d\omega$ the integration is carried out. ϑ'' is the angle between the final wave vector k′ and k″. If we denote

$$f(\vartheta) = \sqrt{\sigma(\vartheta)} \exp[i\alpha(\vartheta)], \tag{2.3}$$

we obtain from (2.2) an integral equation for $\alpha(\vartheta)$ when $\sigma(\vartheta)$ is known:

$$4\pi \sin\alpha(\vartheta) = k \int \left[\frac{\sigma(\vartheta')\sigma(\vartheta'')}{\sigma(\vartheta)}\right]^{1/2} \cos[\alpha(\vartheta') - \alpha(\vartheta'')]\, d\omega. \tag{2.4}$$

The solution of Eqs. (2.1) and (2.2) or of (2.4) is equivalent to a phase-shift analysis. The representation of $f(\vartheta)$ as a finite sum of Legendre polynomials is clearly only one method of solution, which is useful when it is known that only a few phases are involved in the solution. For the general case the direct solution of (2.4) may be more convenient.

Equation (2.4) is seen to be invariant with respect to the substitution $f(\vartheta) \to -f^*(\vartheta)$ or, equivalently,

$$\alpha(\vartheta) \to \pi - \alpha(\vartheta). \tag{2.5}$$

This transformation represents a change of sign of all scattering phases. It indicates the ambiguity in the reconstruction of the scattering amplitude from its absolute value. We know that this ambiguity is eliminated by consideration of the interference with Coulomb scattering (for charged particles) or by study of the energy dependence of scattering at low energies. Therefore, the measurement of the scattering cross section at a given energy and at all angles is a complete experiment in the sense that it provides a possibility of completely reconstructing the scattering amplitude (except possibly for the indicated ambiguity).

3. SCATTERING OF A PARTICLE WITH SPIN ½

The scattering amplitude for a particle with spin ½ is described in the usual manner as

$$M = a(\vartheta) + b(\vartheta)(\boldsymbol{\sigma}\mathbf{n}), \tag{3.1}$$

where $\boldsymbol{\sigma}$ is the Pauli matrix and $\mathbf{n} = \mathbf{kk}'/|\mathbf{kk}'|$ is the unit normal to the scattering plane.

The cross section and polarization (double scattering) are measured experimentally, thus enabling us to determine two functions:

$$\begin{aligned} \sigma(\vartheta) &= |a|^2 + |b|^2 = {}^1/_2 (|f_+|^2 + |f_-|^2), \\ \sigma(\vartheta) P(\vartheta) &= 2\,\mathrm{Re}\, ab^* = {}^1/_2 (|f_+|^2 - |f_-|^2). \end{aligned} \tag{3.2}$$

Equation (3.2) clearly determines two moduli*:

$$|f_+|^2 = |a + b|^2, \quad |f_-|^2 = |a - b|^2. \tag{3.3}$$

If the measurements are carried out over all angles and the scattering is elastic the phases of the complex functions $f_+(\vartheta)$ and $f_-(\vartheta)$ are determined again by the unitarity conditions:

$$\begin{aligned} 4\pi\,\mathrm{Im}\, a(\vartheta) &= k \int d\omega \left[a^*(\vartheta'') a(\vartheta') + b^*(\vartheta'') b(\vartheta') \frac{\cos\vartheta - \cos\vartheta' \cos\vartheta''}{\sin\vartheta' \sin\vartheta''}\right], \\ 4\pi\,\mathrm{Re}\, b(\vartheta) &= k \int d\omega \left[2\,\mathrm{Im}\, a^*(\vartheta'') b(\vartheta') \frac{\cos\vartheta \cos\vartheta' - \cos\vartheta''}{\sin\vartheta \sin\vartheta'} + \right. \\ &\left. + b^*(\vartheta'') b(\vartheta') \frac{1 + 2\cos\vartheta \cos\vartheta' \cos\vartheta'' - \cos^2\vartheta - \cos^2\vartheta' - \cos^2\vartheta''}{\sin\vartheta \sin\vartheta' \sin\vartheta''}\right]. \end{aligned} \tag{3.4}$$

Equations (3.2) and (3.4) also permit a certain invariant transformation. In order to determine this we note that the transformation (reversal of the signs of all scattering phases)

$$a(\vartheta) \to -a^*(\vartheta), \quad b(\vartheta) \to b^*(\vartheta) \tag{3.5}$$

does not affect the cross section and unitarity conditions but that it does reverse the polarization. This This is also true for the Minami[3,4] transformation

$$\begin{aligned} a(\vartheta) &\to a(\vartheta)\cos\vartheta + ib(\vartheta)\sin\vartheta, \\ b(\vartheta) &\to -ia(\vartheta)\sin\theta - b(\vartheta)\cos\vartheta. \end{aligned} \tag{3.6}$$

Therefore the product of the two transformations:

*We note that (3.3) represents the scattering cross sections of a polarized particle whose spin is directed "up" or "down" with respect to **n**.

$$a(\vartheta) \to -a^*(\vartheta)\cos\vartheta + ib^*(\vartheta)\sin\vartheta, \quad b(\vartheta) \to + ia^*(\vartheta)\sin\vartheta - b^*(\vartheta)\cos\vartheta \tag{3.7}$$

leaves all quantities invariant.*

Thus for a particle with spin ½, the measurement of the cross section and of the polarization is a complete experiment. Non-uniqueness of the phases resulting from the existence of the transformation (3.7) can be eliminated by investigating the energy dependence of the cross section at low energies. We also add that, as was shown by Wolfenstein[5], this non-uniqueness can be eliminated by the study of triple scattering (in which the scattering planes of interest are mutually perpendicular).

4. NUCLEON-NUCLEON SCATTERING

Nucleon-nucleon scattering can be described by the matrix

$$M = \alpha(\vartheta) + \beta(\vartheta)(\boldsymbol{\sigma}_1\mathbf{n})(\boldsymbol{\sigma}_2\mathbf{n}) + \gamma(\vartheta)(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\mathbf{n} + \delta(\vartheta)(\boldsymbol{\sigma}_1\mathbf{m})(\boldsymbol{\sigma}_2\mathbf{m}) + \varepsilon(\vartheta)(\boldsymbol{\sigma}_1\mathbf{l})(\boldsymbol{\sigma}_2\mathbf{l}). \tag{4.1}$$

Here $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the Pauli matrices of the two particles and **m**, **l** and **n** are the unit orthogonal vectors of the Cartesian coordinates, which are parallel to $\mathbf{k}-\mathbf{k}'$, $\mathbf{k}+\mathbf{k}'$ and $[\mathbf{kk}']$, respectively (in the center-of-mass system)[6]. This system is obviously suitable when we note that in the laboratory system the unit vectors are parallel to the wave vectors of the two particles after scattering and to the normal to the scattering plane.

Equation (4.1) differs from the general scattering matrix for two particles with spin ½ by the absence of a term proportional to $\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2$. This results from identity of the particles for a $p-p$ system and charge invariance for a $n-p$ system.

For identical particles the matrix coefficients in (4.1) are symmetrical with respect to the substitution $\vartheta \to \pi - \vartheta$. These can be formulated conveniently by introducing new functions which will also be useful subsequently[7]:

$$a = \alpha + \beta, \quad b = \alpha - \beta, \quad c = \delta + \varepsilon, \quad d = \delta - \varepsilon, \quad e = 2\gamma. \tag{4.2}$$

It can then be shown that for identical particles

$$a(\pi - \vartheta) = -a(\vartheta), \quad b = (\pi - \vartheta) = -c(\vartheta), \quad c(\pi - \vartheta) = -b(\vartheta), \quad d(\pi - \vartheta) = d(\vartheta), \quad e(\pi - \vartheta) = e(\vartheta). \tag{4.3}$$

*In terms of the phase shifts $\delta(j, l)$ this transformation represents $\delta(j, j - 1/2) \rightleftarrows -\delta(j, j + 1/2)$.

Therefore for identical particles the measurement is performed only for the angles $0 \leftrightarrow \pi/2$. For the scattering of neutrons by protons the interval of measurement is doubled to $0 \leftrightarrow \pi$, which corresponds to doubling of the number of states in this system.

Before making use of the unitarity condition, we shall briefly describe the experiments needed in a two-nucleon system.

If we have an unpolarized beam of nucleons the first scattering determines the differential cross section, which is associated with the elements of the scattering matrix by the formula

$$\sigma(\vartheta) = 1/2\,(|a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2). \tag{4.4}$$

The first scattering gave rise to a beam polarized along the **n** direction. In the second scattering the polarization in hydrogen is measured:

$$\sigma(\vartheta)P(\vartheta) = \mathrm{Re}\,ae^*. \tag{4.5}$$

It is clear that in order to obtain polarization the first scattering can be produced by any target with certain properties (the polarizer). The polarization of the recoil particle agrees with $P(\vartheta)$ ($\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2$ is absent in M) and need not be measured.

A third target can be added in two ways. Measurment of the P_{nn} component of the correlation tensor gives

$$\sigma(\vartheta)P_{nn}(\vartheta) = 1/2\,(|a|^2 - |b|^2 - |c|^2 + |d|^2 + |e|^2). \tag{4.6}$$

In these measurements all three scatterings take place in a single plane. It is possible in addition to measure the polarization of the particle after the second scattering. When it is measured by the third target (which can be any analyzer) we obtain two quantities which correspond to the two particles participating in the second scattering[5]. If all three scatterings occur in the same plane we shall have for these quantities

$$\sigma(\vartheta)D_{nn}(\vartheta) = 1/2\,\{|a|^2 + |b|^2 - |c|^2 - |d|^2 + |e|^2\} \tag{4.7}$$

(for the scattered particle) and

$$\sigma(\vartheta)K_{nn}(\vartheta) = 1/2\,\{|a|^2 - |b|^2 + |c|^2 - |d|^2 + |e|^2\} \tag{4.8}$$

(for the recoil particle). For neutron scattering by protons, the difference between these two quantities is evident; for identical particles, the first quantity corresponds to measurement in the angular interval

0 to $\pi/2$ and the second to the angular interval $\pi/2$ to π.

The methods which have been described give us five equations from which we easily determine the absolute values of the five functions $a+e$, $a-e$, b, c, d. Further consideration follows the path indicated earlier. In order to determine the phases of these complex functions measured over all angles we must make use of the unitarity condition. It can be shown (see Ref. 8) that five such conditions exist. [These conditions (see Appendix 2) are invariant with respect to

$$\alpha(\vartheta) \to -\alpha^*(\vartheta),\quad \beta(\vartheta) \to -\beta^*(\vartheta),\quad \gamma(\vartheta) \to \gamma^*(\vartheta),$$
$$\delta(\vartheta) \to -\delta^*(\vartheta),\ \varepsilon(\vartheta) \to -\varepsilon^*(\vartheta),$$

which is equivalent to simultaneous reversal of the signs of all phases (as in the preceding case this substitution reverses the polarization); a substitution which corresponds to the Minami transformation does not exist for nucleon-nucleon scattering[9].] Hence it follows that the indicated set of experiments is a complete set.

The preceding considerations are characterized by the important conclusion that determination of the scattering matrix does not require measurement of quadruple scattering or the introduction of a magnetic field but can be confined to experiments with three targets and parallel scattering planes. Up to the present time, however, such a complete set of experiments has not been performed for any single energy; this is the cause of the ambiguities in the analysis of these experiments.

In connection with the determination of the scattering matrix there arises the question of its relation to the potential. It is noteworthy that the number of independent functions in the scattering amplitude coincides with the number of scalar functions that appear in the interaction Hamiltonian. Indeed, it is easily shown that the general form of the interaction of two protons, for example, will be

$$V = V_1(r) + V_2(r)(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2) + V_3(r)(\boldsymbol{\sigma}_1\mathbf{r})(\boldsymbol{\sigma}_2\mathbf{r})$$
$$+ V_4(r)(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\mathbf{L} + V_5(r)(\boldsymbol{\sigma}_1\mathbf{L})(\boldsymbol{\sigma}_2\mathbf{L})$$

where $\mathbf{L}$ is the orbital angular momentum. It would be very interesting to determine how the scattering matrix is affected when one or more V_i vanishes. In other words, can anything be known qualitatively regarding the form of V from the characteristics of the scattering matrix?

In conclusion we wish to thank Professors E. Segre and O. Chamberlain for interesting discussions which we had with them during the summer of 1956 concerning nucleon scattering experiments.

APPENDIX 1. FORMULAS FOR POSSIBLE NUCLEON-NUCLEON SCATTERING EXPERIMENTS

Since the set of five experiments indicated above for the determination of the nucleon-nucleon scattering matrix is not the only one possible, we shall now briefly review all conceivable experiments. These experiments can differ first in the polarization of the primary beam and secondly in the nature of the quantities to be measured (cross section, polarization of the scattered particle, polarization of the recoil particle, polarization correlation). They can be summarized in the following table:

Measured quantity	Initial spin state			
	A	B	C	D
	Unpolarized beam—unpolarized target	Polarized beam—unpolarized target	Unpolarized beam—polarized target	Polarized beam—polarized target
1. Cross section	σ^*	$\sigma_i^{(1)}$	$\sigma_k^{(2)}$	σ_{ik}
2. Polarization of scattered particle	$P_p^{(1)*}$	$D_{ip}^{(1)*}$	K'_{kp}	$T_{ikp}^{(1)}$
3. Polarization of recoil particle	$P_q^{(2)}$	K_{iq}^*	$D_{kq}^{(2)}$	$T_{ikq}^{(2)}$
4. Polarization correlation	P_{qp}^*	$P_{ipq}^{(1)*}$	$P_{kpq}^{(2)}$	T_{ikpq}^*

Here each column represents a definite initial spin state of the two-nucleon system and each line gives a characteristic of the scattering process which can be measured. The subscript i denotes the direction of initial polarization of the incident particle; k denotes the initial polarization of the target; p denotes the measured polarization component of the scattered particle; q denotes the measured polarization component of the recoil particle. Each experiment will hereinafter be denoted by a letter (for the column) and numeral (for the line) indicating the initial spin state of the system and the characteristic of the scattering process which is to be measured. For example, B2 means measurement of the set of quantities $D_{ip}^{(1)}$ which determine the influence of the i-component of incident particle polarization on the p-component of scattered particle polarization, *etc.*

Not all of the experiments in this table are different. Some of them are actually identical because of symmetry. Thus in the absence of singlet-triplet transitions (the absence of the σ_1—σ_2 component) there is essential identity of experiments A2 and A3, B1 and C1, B4 and C4, D2 and D3, B2 and C3. For identical particles (see the symmetry properties in (4.3)) experiments B2 and B3 determine the quantities of B2 for supplementary angles (ϑ and $\pi-\vartheta$). Experiments C3 and C2 are related in the same way.

Because of the time-reversal symmetry of the matrix M the experiments which are situated symmetrically with respect to the main diagonal of the table are equivalent (A2 and B1, A4 and D1 etc). Thus the only distinct experiments are denoted by asterisks in the table. It appears immediately that a polarized target is needed in principle only for the most complicated experiments. But, as we have seen, for the actual determination of the scattering matrix even these experiments are unnecessary.

We now proceed to review the characterstics which are determined in the different experiments.

A1. The cross section is measured:

$$\sigma(\vartheta) = {}^1/_4 \operatorname{Sp} MM^+$$

$$= {}^1/_2(|a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2).$$

A2. The polarization is measured:

$$\sigma(\vartheta) P(\vartheta) = {}^1/_4 \operatorname{Sp} (\boldsymbol{\sigma}_1 \mathbf{n}) MM^+ = \operatorname{Re} ae^*.$$

B2. The tensor is determined:

$$\sigma(\vartheta) D_{ip}(\vartheta) = {}^1/_4 \operatorname{Sp} M\sigma_{1i} M^+ \sigma_{1p}$$
$$= {}^1/_2 (|a|^2 + |b|^2 - |c|^2 - |d|^2 + |e|^2) n_i n_p$$
$$+ \operatorname{Re}(a^* b + c^* d) m_i m_p$$
$$+ \operatorname{Re}(a^* b - c^* d) l_i l_p - \operatorname{Im} b^* e (m_i l_p - l_i m_p).$$

From triple-scattering experiments with parallel and perpendicular planes these components are determined:

$$\sigma(\vartheta) D_{nn}(\vartheta) = {}^1/_2 (|a|^2 + |b|^2 - |c|^2 - |d|^2 + |e|^2),$$

$$\sigma(\vartheta) D_{xm}(\vartheta) = -\cos\frac{\vartheta}{2} \operatorname{Re}(a^* b + c^* d) + \operatorname{Im} b^* e \sin\frac{\vartheta}{2}.$$

The determination of the components

$$\sigma(\vartheta) D_{zm}(\vartheta) = \sin\frac{\vartheta}{2} \operatorname{Re}(a^* b + c^* d) + \cos\frac{\vartheta}{2} \operatorname{Im} b^* e,$$

$$\sigma(\vartheta) D_{xl}(\vartheta) = \sin\frac{\vartheta}{2} \operatorname{Re}(a^* b - c^* d) + \cos\frac{\vartheta}{2} \operatorname{Im} b^* e$$

requires either quadruple scattering or triple scattering with a magnetic field between the targets. The unit vectors **z** and **x** are along the directions **k** and [**nk**], respectively.

B3. The tensor is measured:

$$\sigma(\vartheta) K_{iq}(\vartheta) = {}^1/_4 \operatorname{Sp} M\sigma_{1i} M^+ \sigma_{2q} = {}^1/_2 (|a|^2 - |b|^2 + |c|^2 - |d|^2 + |e|^2) n_i n_q + \operatorname{Re}(a^* c + b^* d) m_i m_q$$
$$+ \operatorname{Re}(a^* c - b^* d) l_i l_q - \operatorname{Im} c^* e (m_i l_q - l_i m_q).$$

The components K_{nn} and $K_{x,l}$ are determined from triple scattering and K_{xm} and $K_{z,l}$ from quadruple scattering.

For identical particles this is equivalent to the measurement of corresponding components of D_{ip} for supplementary angles ($\vartheta \rightarrow \pi - \vartheta$) giving $D_{nn}(\pi-\vartheta) = K_{nn}(\vartheta)$, $D_{xm}(\pi-\vartheta) = K_{x,-l}(\vartheta)$, $D_{zm}(\pi-\vartheta) = -K_{z,-l}(\vartheta)$, $D_{xl}(\pi-\vartheta) = -K_{x,m}(\vartheta)$.

A4. The polarization correlation tensor is measured:

$$\sigma(\vartheta) P_{pq}(\vartheta) = {}^1/_4 \operatorname{Sp} \sigma_{1p}\sigma_{2q} MM^+$$
$$= \frac{1}{2}(|a|^2 - |b|^2 - |c|^2 + |d|^2 + |e|^2) n_p n_q$$
$$+ \operatorname{Re}(a^* d + b^* c) m_p m_q + \operatorname{Re}(b^* c - a^* d) l_p l_q$$
$$+ \operatorname{Im} de^* (m_p l_q + m_q l_p).$$

The component

$$\sigma(\vartheta) P_{n\,n}(\vartheta) = {}^1/_2 (|a|^2 - |b|^2 - |c|^2 + |d|^2 + |e|^2)$$

is determined by an experiment in which the planes of the analyzing scatterings coincide with the basic scattering plane.

The determination of the component

$$\sigma(\vartheta) P_{ml}(\vartheta) = \mathrm{Im}\, de^* = -\sigma(\vartheta) P_{m-l}(\vartheta)$$

requires an experiment in which the planes of the analyzing scatterings are perpendicular to the basic scattering plane.

Experiments in which the planes of the analyzing scatterings are perpendicular to the basic plane, with a magnetic field perpendicular to the primary scattering plane placed in front of one of the analyzers, determine the components

$$\sigma(\vartheta) P_{mm}(\vartheta) = \mathrm{Re}\,(a^*d + b^*c),$$

$$\sigma(\vartheta) P_{ll}(\vartheta) = \mathrm{Re}\,(b^*c - a^*d).$$

B4. The tensor is measured:

$$\begin{aligned}\sigma(\vartheta) P_{ipq}(\vartheta) &= {}^1/_4\, \mathrm{Sp}\, M\sigma_{1i} M^+ \sigma_{1p}\sigma_{2q} = \\ &= \mathrm{Re}\, ae^* n_i n_p n_q + \mathrm{Re}\, be^* [m_i m_p n_q + l_i l_p n_q] - \\ &- \mathrm{Im}\,(a^*b + c^*d)\, l_i m_p n_q + \mathrm{Im}\,(a^*b - c^*d)\, m_i l_p n_q - \\ &- \mathrm{Im}\,(a^*c + b^*d)\, l_i n_p m_q + \mathrm{Im}\,(a^*c - b^*d)\, m_i n_p l_q + \\ &+ \mathrm{Im}\,(a^*d + b^*c)\, n_i l_p m_q + \mathrm{Im}\,(a^*d - b^*c)\, n_i m_p l_q + \\ &+ \mathrm{Re}\, e^*c\, [m_i n_p m_q + l_i n_p l_q] + \mathrm{Re}\, e^*d\, [n_i m_p m_q - n_i l_p l_q].\end{aligned}$$

C4. For this experiment a polarized target is required. The tensor components are measured:

$$\begin{aligned}\sigma(\vartheta) T_{ikpq}(\vartheta) &= {}^1/_4\, \mathrm{Sp}\, M\sigma_{1i}\sigma_{2k} M^+ \sigma_{1p}\sigma_{2q} = \\ &= {}^1/_2 (|a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2)\, n_i n_k n_p n_q + \\ &+ {}^1/_2 (|a|^2 + |b|^2 + |c|^2 + |d|^2 - |e|^2)\, (m_i m_k m_p m_q + l_i l_k l_p l_q) + \\ &+ \mathrm{Re}\,(a^*b + c^*d)\,(n_i l_k n_p l_q + l_i n_k l_p n_q) + \\ &+ \mathrm{Re}\,(a^*b - c^*d)\,(n_i m_k n_p m_q + m_i n_k m_p n_q) + \\ &+ {}^1/_2 (|a|^2 + |b|^2 - |c|^2 - |d|^2 - |e|^2)\,(m_i l_k m_p l_q + l_i m_k l_p m_q) + \\ &+ {}^1/_2 (|a|^2 - |b|^2 + |c|^2 - |d|^2 - |e|^2)\,(m_i l_k l_p m_q + l_i m_k m_p l_q) + \\ &+ {}^1/_2 (-|a|^2 + |b|^2 + |c|^2 - |d|^2 + |e|^2)\,(m_i m_k l_p l_q + l_i l_k m_p m_q) + \\ &+ \mathrm{Re}\,(a^*c + b^*d)\,(l_i n_k n_p l_q + n_i l_k l_p n_q) + \\ &+ \mathrm{Re}\,(a^*c - b^*d)\,(n_i m_k m_p n_q + m_i n_k n_p m_q) - \\ &- \mathrm{Re}\,(a^*d + b^*c)\,(l_i l_k n_p n_q + n_i n_k l_p l_q) + \\ &+ \mathrm{Re}\,(a^*d - b^*c)\,(n_i n_k m_p m_q + m_i m_k n_p n_q) + \\ &+ \mathrm{Im}\, e^*c\,(m_i n_k n_p l_q + n_i m_k l_p n_q - n_i l_k m_p n_q - l_i n_k n_p m_q) + \\ &+ \mathrm{Im}\, e^*d\,(n_i n_k m_p l_q + n_i n_k l_p m_q - m_i l_k n_p n_q - l_i m_k n_p n_q) + \\ &+ \mathrm{Im}\, a^*e\,(l_i m_k m_p m_q + m_i l_k m_p m_q + l_i l_k m_p l_q + l_i l_k l_p m_q - \\ &- m_i l_k l_p l_q - l_i m_k l_p l_q - m_i m_k m_p l_q - m_i m_k l_p m_q) + \\ &+ \mathrm{Im}\, b^*e\,(l_i n_k m_p n_q + n_i l_k n_p m_q - n_i m_k n_p l_q - m_i n_k l_p n_q).\end{aligned}$$

APPENDIX 2 UNITARITY CONDITIONS

These are the unitarity conditions for nucleon-nucleon scattering:

$$4\pi\, \mathrm{Im}\, \alpha(\vartheta) = \frac{k}{4} \int \mathrm{Sp}\, [M^+(\mathbf{k}', \mathbf{k}'')\, M(\mathbf{k}, \mathbf{k}'')]\, d\omega_{\mathbf{k}''},$$

$$4\pi\, \mathrm{Im}\, \beta(\vartheta) = \frac{k}{4} \int \mathrm{Sp}\, [M^+(\mathbf{k}', \mathbf{k}'')\, M(\mathbf{k}, \mathbf{k}'')\, (\boldsymbol{\sigma}_1 \mathbf{n})\, (\boldsymbol{\sigma}_2 \mathbf{n})]\, d\omega_{\mathbf{k}''},$$

$$4\pi\, \mathrm{Re}\, \gamma(\vartheta) = \frac{ik}{8} \int \mathrm{Sp}\, [M^+(\mathbf{k}', \mathbf{k}'')\, M(\mathbf{k}, \mathbf{k}'')\, (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\, \mathbf{n}]\, d\omega_{\mathbf{k}''},$$

$$4\pi\, \mathrm{Im}\, \delta(\vartheta) = \frac{k}{4} \int \mathrm{Sp}\, [M^+(\mathbf{k}', \mathbf{k}'')\, M(\mathbf{k}, \mathbf{k}'')\, (\boldsymbol{\sigma}_1 \mathbf{m})\, (\boldsymbol{\sigma}_2 \mathbf{m})]\, d\omega_{\mathbf{k}''},$$

$$4\pi\, \mathrm{Im}\, \varepsilon(\vartheta) = \frac{k}{4} \int \mathrm{Sp}\, [M^+(\mathbf{k}', \mathbf{k}'')\, M(\mathbf{k}, \mathbf{k}'')\, (\boldsymbol{\sigma}_1 \mathbf{l})\, (\boldsymbol{\sigma}_2 \mathbf{l})]\, d\omega_{\mathbf{k}''}.$$

The calculation of the traces in the unitarity conditions is elementary but results in complicated expressions which we shall not reproduce here. It should be noted that when Wolfenstein's[5] form of M is used the unitarity condition for $B(\vartheta)$ (singlet scattering) is of the same form as the unitarity condition for the scattering of spin-zero particles.

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132