The Application of the Microcanonical Distribution to the Statistical Theory of Multiple Production of Particles

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The general statistical method of the microcanonical distribution is applied to the problem of the computation of statistical weights for the reactions of production of π-mesons in collisions of nucleons with nucleons. A general formula is derived for computation of the statistical weight of a state with an arbitray set of particles, considering not only the laws of conservation of energy and momentum, but also the different type of statistics for fermions and bosons. In allowing all particles to obey Boltzmann statistics, this general formula reduces to the formula proposed in Ref. 2. In particular, corrections are found which are due to consideration of the type of statistics for all possible collision processes of nucleons with nucleons, in which not more than three mesons are created.

THE statistical theory of multiple production of mesons proposed by Fermi[1], was made more exact by a series of authors[2-4]. The most general formula for computation of the statistical weights of processes of different multiplicity was put forth by Lepore and Stuart[2].

In contrast to Ref. 1, the authors of Ref. 2 took into account exactly not only the law of conservation of energy, but also the law of conservation of momentum for particles of arbitrary mass; however, both in Ref. 1 and in Ref. 2, the different type of statistics for fermions and bosons was not considered. For calculation of the statistical weights in Refs. 1 and 2, a method was used which is suitable only for a set of particles obeying Boltzmann statistics.

For calculating the probabilities of the states of a system with different numbers of particles, taking into account both the conservation laws and the type of statistics, it is simpler to use the general statistical method of Gibbs. In the given case, for a system with exactly given total energy and total momentum, it is obviously necessary to use the formula for the microcanonical distribution. We shall use this distribution to derive a more general formula than that of Lepore and Stuart. We shall consider the general case of a system in which there may exist and be created particles of several kinds, both bosons and fermions, having arbitrary masses. Finally, after an analysis of the general properties of the expressions for the probabilities of different states with given numbers or particles, several applications of the derived formulas to the processes of multiple production of π-mesons will be considered.

1. THE MICROCANONICAL DISTRIBUTION

Let us consider a system consisting of ν kinds of noninteracting particles having masses $m_1, m_2, \ldots$ The total energy E of the system and the total momentum $\mathbf{P}$ is given, but the total number of particles of arbitrary kind is in no way limited. Let the discrete set of vectors $\mathbf{p}_s$ (*i.e.*, $\mathbf{p}_s'$, $\mathbf{p}_s''$, ...) represent all possible momenta of one particle of type s. Inasmuch as the particles do not interact with each other, the set of vectors $\mathbf{p}_s$ also represents the possible states of the whole system, while the set of occupation numbers $n_s(\mathbf{p}_s)$ completely determines the state of the system under consideration.* The probability of a given set of occupation numbers can be represented in the form

$$w\{n_1(\mathbf{p}_1), n_2(\mathbf{p}_2), \ldots, n_\nu(\mathbf{p}_\nu)\}$$
$$=A\delta\left\{E-\sum_{s=1}^{\nu}\sum_{\mathbf{p}_s}\varepsilon_s(\mathbf{p}_s)\,n_s(\mathbf{p}_s)\right\}\delta\left\{\mathbf{P}-\sum_{s=1}^{\nu}\sum_{\mathbf{p}_s}\mathbf{p}_s n_s(\mathbf{p}_s)\right\}\times\prod_{s=1}^{\nu}\Omega_s\{n_s(\mathbf{p}_s)\}, \quad (1)$$

where δ is the Dirac δ-function; A a normalization factor; $\Omega_s\ \{n_s(\mathbf{p}_s)\}$ is the multiplicity of a state

*Particles having the same momentum $\mathbf{p}_s$ can be found in different states of polarization; therefore it is necessary to keep in mind that to each vector $\mathbf{p}_s$ correspond g_s independent states.

with given p_s; $\varepsilon_s(p_s)=(p_s^2+m_s^2)^{1/2}$ is the energy of this state*; the summation over p_s is taken over all permitted values of this vector and over all states of polarization.

Obviously, both in the case of Bose statistics and in the case of Fermi statistics $\Omega_s=1$ on account of the indistinguishability of the particles. But in the case of Boltzmann statistics,

$$\Omega_s\{n_s(\mathbf{p}_s)\}=\Big[\sum_{\mathbf{p}_s} n_s(\mathbf{p}_s)\Big]!\prod_{\mathbf{p}_s}[n_s(\mathbf{p}_s)!]^{-1}, \quad (2)$$

where the product is taken over all p_s.

The probability of a state in which there are N_1 particles of the first kind, N_2 particles of the second kind, *etc.*, is clearly equal to

$$W_{N_1N_2\ldots N_\nu}=\sum_{n_1,n_2,\ldots} w\{n_1(\mathbf{p}_1),\ldots,n_\nu(\mathbf{p}_\nu)\}\prod_{s=1}^{\nu}\delta\Big\{N_s-\sum_{\mathbf{p}_s} n_s(\mathbf{p}_s)\Big\}, \quad (3)$$

where $\delta\{N\}=\delta_{N0}$ is the Kronecker symbol; the summation is carried out over all occupation numbers, *i.e.*,

$$\sum_{n(\mathbf{p})} f\{n(\mathbf{p})\}=\sum_{n'}\sum_{n''}\ldots f(n',n'',\ldots),$$

whereby in the cases of Boltzmann and Bose statistics all $n^{(k)}$ run through the values 0, 1, 2, 3, ...; in the case of Fermi statistics, however, only the two values $n^{(k)}=0,1$ are allowed. Obviously from the normalization conditions

$$\sum_{N_1,N_2,\ldots=0}^{\infty} W_{N_1N_2\ldots N_\nu}=\sum_{n_1,n_2,\ldots} w\{n_1(\mathbf{p}_1),\ldots,n_\nu(\mathbf{p}_\nu)\}=1, \quad (4)$$

and, consequently, according to Eqs. (1) and (4),

$$W_{N_1N_2\ldots N_\nu}=\sigma(E,\mathbf{P};N_1,\ldots,N_\nu)\Big/\sum_{N_1,N_2,\ldots=0}^{\infty}\sigma(E,\mathbf{P};N_1,\ldots,N_\nu), \quad (5)$$

where σ is the statistical weight determined by

$$\sigma(E,\mathbf{P};N_1,\ldots,N_\nu)=\sum_{n_1,n_2,\ldots}\delta\Big\{E-\sum_{s=1}^{\nu}\sum_{\mathbf{p}_s}\varepsilon_s n_s(\mathbf{p}_s)\Big\}\times\delta\Big\{\mathbf{P}-\sum_{s=1}^{\nu}\sum_{\mathbf{p}_s}p_s n_s(\mathbf{p}_s)\Big\}\prod_{s=1}^{\nu}\delta\Big\{N_s-\sum_{\mathbf{p}_s}n_s(\mathbf{p}_s)\Big\}\times\Omega_s\{n_s(\mathbf{p}_s)\}. \quad (6)$$

Noting that the δ-function and the δ-symbol can be represented in the form

$$\delta(x)=\frac{1}{2\pi}\int_{-\infty-i\beta}^{+\infty-i\beta}e^{i\alpha x}\,d\alpha,\quad \delta\{N\}=\frac{1}{2\pi}\int_0^{2\pi}e^{i\varphi N}\,d\varphi,$$

where β is a small, positive quantity, set equal to zero in the final equation,* we get

$$\sigma(E,\mathbf{P};N_1,\ldots,N_\nu)=\frac{1}{(2\pi)^{4+\nu}}\int_{-\infty-i\beta}^{+\infty-i\beta}\iiint_{-\infty}^{+\infty}\times\overbrace{\int_0^{2\pi}\ldots\int}\exp\Big\{i\Big(\alpha E+\mathbf{bP}+\sum_{s=1}^{\nu}\varphi_s N_s\Big)+\sum_{s=1}^{\nu}\Phi_s(\alpha,\mathbf{b},\varphi_s)\Big\}\,d\alpha\,d\mathbf{b}\,d\varphi_1\ldots d\varphi_\nu, \quad (7)$$

$$e^{\Phi_s(\alpha,\mathbf{b},\varphi_s)}=\sum_{n_s}\prod_{\mathbf{p}_s}\exp\{-in_s(\mathbf{p}_s)[\alpha\varepsilon_s(\mathbf{p}_s)+\mathbf{bp}_s+\varphi_s]\}\,\Omega_s\{n_s(\mathbf{p}_s)\}. \quad (8)$$

The last sums can be easily calculated both in the case of Fermi and Bose statistics and in the case of Boltzmann statistics. For Fermi and Bose statistics, $\Omega_s=1$, but it is necessary to sum over all $n_s^{(k)}=0,1$ in the former case and over all $n_s^{(k)}=0,1,2,3,\ldots$ in the latter. Carrying out these summations we get

$$\Phi_s(\alpha,\mathbf{b},\varphi_s)=\pm\sum_{\mathbf{p}_s}\ln\{1\pm\exp(-i[\alpha\varepsilon_s(\mathbf{p}_s)+\mathbf{bp}_s+\varphi_s])\}. \quad (9)$$

Here as in what follows, the upper sign refers to the case of Fermi statistics and the lower sign to the

*Here a system of units has been chosen in which the velocity of light $c=1$.

* The introduction of a small parameter β enables one, in integrations to come, to avoid a pole, analogously to the way this was done in Ref. 4.

case of Bose statistics. But in the case of Boltzmann statistics, according to Eqs. (2) and (8), we get

$$\Phi_s(\alpha, \mathbf{b}, \varphi_s) = \sum_{\mathbf{p}_s} \exp\{-i[\alpha\varepsilon_s(\mathbf{p}_s) + \mathbf{b}\mathbf{p}_s + \varphi_s]\} + \ln N_s!\,. \tag{10}$$

Expanding the logarithm after the summation sign in Eq. (9) into a series and interchanging the order of summation, we obtain for the Fermi and Bose cases*

$$\Phi_s(\alpha, \mathbf{b}, \varphi_s) = \sum_{k=1}^{\infty} k^{-1}(\mp 1)_s^{k+1} \sum_{\mathbf{p}_s} \exp\{-i[\alpha\varepsilon_s(\mathbf{p}_s) + \mathbf{b}\mathbf{p}_s + \varphi_s]k\}, \tag{11}$$

from which it is seen that the case of Boltzmann statistics corresponds to a limitation of the series (11) to only the first term, with addition of a term $\ln N_s!$

If the linear dimensions L of the volume of space $V = L^3$ occupied by the gas are sufficiently large, or if, in a finite volume, the energy is sufficiently large that the de Broglie wavelengths of the different particles are small compared with L, then the summation over p in Eq. (11) can be replaced by integration and instead of (11) one can take the approximate expression

$$\Phi_s(\alpha, \mathbf{b}, \varphi_s) = \sum_{k=1}^{\infty} k^{-1}(\mp 1)_s^{k+1} \frac{g_s}{\omega} \int \exp\{-i[\alpha\varepsilon_s(\mathbf{p}) + \mathbf{b}\mathbf{p}_s + \varphi_s]k\}\, d\mathbf{p}_s, \tag{12}$$

where $\omega = (2\pi\hbar/L)^3$ is the volume of the elementary cell in momentum space. Introducing the notation

$$B_s(km_s, \alpha, \mathbf{b}) = \frac{g_s k^3}{\omega} \int \exp\{-i[\alpha(\mathbf{p}_s^2 + m_s^2)^{1/2} + \mathbf{b}\mathbf{p}_s]k\}\, d\mathbf{p}_s, \tag{13}$$

we have in place of Eq. (12)

$$\Phi_s(\alpha, \mathbf{b}, \varphi_s) = \sum_{k=1}^{\infty} k^{-4}(\pm 1)_s^{k+1} e^{-ik\varphi_s} B_s(km_s, \alpha, \mathbf{b}). \tag{14}$$

It is possible to carry out the integration over the angular variables in the space of the vector $\mathbf{p}_s$, as a result of which we obtain

$$B_s = \frac{g_s k^3}{\omega} \frac{2\pi}{ikb} \int_{-\infty}^{+\infty} \exp\{ik[bp - \alpha\sqrt{p^2 + m_s^2}]\}\, p\,dp = -\frac{2\pi^2 g_s (km_s)^2 \alpha}{\alpha^2 - b^2} H_2^{(2)}[km_s(\alpha^2 - b^2)^{1/2}], \tag{15}$$

where $H_2^{(2)}[x]$ is Hankel function. (For an analogous transformation, see Ref. 4).

To compute σ, according to Eq. (7) it is necessary to calculate the intermediate integrals

$$I_s(\alpha, \mathbf{b}) = \frac{1}{2\pi} \int_0^{2\pi} \exp\{iN_s\varphi_s + \Phi_s(\alpha, \mathbf{b}, \varphi_s)\}\, d\varphi_s, \tag{16}$$

where Φ_s is determined by means of Eqs. (14) and (15).

Carrying out the change of variables $z = e^{-i\varphi}$, we get, according to the calculus of residues,

$$I_s = -\frac{i}{2\pi} \int_{C^+} z^{-N_s - 1} e^{\Phi_s(z)}\, dz = \frac{1}{N_s!} \left[\frac{d^{N_s}}{dz^{N_s}} e^{\Phi_s(z)}\right]_{z=0}, \tag{17}$$

where the cymbol C^+ denotes that the integration is taken counterclockwise along the contour $|z| = 1$.

Putting Eq. (17) into Eq. (7), and integrating over all directions of the vector b, in analogy with Eq. (15), we get

$$\sigma(E, \mathbf{P}; N_1, \ldots, N_\nu) = \frac{2}{(2\pi)^3} \int_0^{\infty} \int_{-\infty - i\beta}^{+\infty - i\beta} \frac{\sin bP}{bP} e^{i\alpha E} \times \prod_{s=1}^{\nu} \left[\frac{1}{N_s!} \frac{d^{N_s}}{dz^{N_s}} e^{\Phi_s(z)}\right]_{z=0} b^2\, db\, d\alpha. \tag{18}$$

On account of Eqs. (14) and (15), the last formula can be written in the following final form:

$$\sigma(E, \mathbf{P}; N_1, \ldots, N_\nu) = \frac{2}{(2\pi)^3} \int_0^{\infty} \int_{-\infty - i\beta}^{+\infty - i\beta} \frac{\sin bP}{bP} e^{i\alpha E} \times \prod_{s=1}^{\nu} \left[\frac{1}{N_s!} \frac{d^{N_s}}{dz^{N_s}} \times \exp\left\{\frac{2\pi^2 g_s \alpha}{\omega(\alpha^2 - b^2)} \sum_{k=1}^{\infty} k^{-4}(\mp 1)_s^{k+1} \times (km_s)^2 H_2^{(2)}[km_s(\alpha^2 - b^2)^{1/2}] z^k\right\}\right]_{z=0} b^2\, db\, d\alpha. \tag{19}$$

Inasmuch as it is necessary in the case of Boltzmann statistics to limit oneself, according to Eq. (10), to only the first term in the sum over k in the exponent of the integrand, and to add $\ln N_s!$ to it in this case

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*Here $(\pm 1)_s = -1$ for all s designating particles obeying Fermi statistics, and $(\pm 1)_s = +1$ correspondingly for the case of Bose statistics.

$$\sigma_0(E, \mathbf{P}; N_1, \ldots, N_\nu) = \frac{2}{(2\pi)^3}\frac{(2\pi^2)^N}{\omega^N}$$
$$\times \int_0^\infty \int_{-\infty-i\beta}^{+\infty-i\beta} \frac{\sin bP}{bP}\frac{\alpha^N e^{i\alpha E}}{(\alpha^2-b^2)^N} \tag{20}$$
$$\times \prod_{s=1}^{\nu} \{g_s m_s^2 H_2^{(2)}[m_s(\alpha^2-b^2)^{1/2}]\}^{N_s} b^2 db d\alpha,$$

where $N=\sum_{s=1}^{\nu} N_s$ is the total number of all particles in the system. For $P=0$ this formula agrees with the corresponding formula in Ref. 4.

In this way Eq. (19) is a more exact formula, correctly taking into account the type of statistics for computation of the statistical weights. But the formula of Lepore and Stuart is correct only for the case when all particles obey Boltzmann statistics.

2. STATISTICAL WEIGHTS FOR REACTIONS PRODUCING MESONS IN NUCLEON-NUCLEON COLLISIONS

We shall apply the general formula (18) to the determination of the statistical weight of the reactions of meson production in collisions of a nucleon with a nucleon. Since the greatest interest in connection with experiments on the cosmotron is provided by the cases in which 1, 2, or 3 mesons are created, we shall limit outselves to particular expressions obtained from Eq. (18) when at the end of the reaction there are formed not more than three identical particles of each kind.

The factors entering into the integrand of Eq. (18) in the cases $N_s=0$, 1, 2, and 3 are, according to Eq. (14), respectively equal to

$$\left[\frac{d^0}{dz^0}e^{\Phi_s(z)}\right]_{z=0} = 1; \quad \left[\frac{d^1}{dz^1}e^{\Phi_s(z)}\right]_{z=0} = B_s(m_s),$$
$$\left[\frac{d^2}{dz^2}e^{\Phi_s(z)}\right]_{z=0} = [B_s(m_s)]^2 \mp \frac{1}{2^3}B_s(2m_s), \tag{21}$$
$$\left[\frac{d^3}{dz^3}e^{\Phi_s(z)}\right]_{z=0} = [B_s(m_s)]^3 \mp \frac{3}{2^3}B_s(2m_s)B_s(m_s) + \frac{2}{3^3}B_s(3m_s),$$

where as before the upper sign refers to particles obeying Fermi statistics and the lower sign to particles obeying Bose statistics.

From Eqs. (18) and (21), it is seen that in the case in which all $N_s \leqslant 1$, Eq. (18) does not differ from Eq. (20), obtained for Boltzmann statistics, which in the notations of Eqs. (18) and (14) takes the form

$$\sigma_0(E, \mathbf{P}; N_1, m_1; \ldots; N_\nu, m_\nu)$$
$$= \frac{2}{(2\pi)^3}\int_0^\infty \int_{-\infty-i\beta}^{+\infty-i\beta} \frac{\sin bP}{bP} e^{i\alpha E}\prod_{s=1}^{\nu}[B_s(m_s)]^{N_s} b^2 db d\alpha. \tag{22}$$

But in the case when any of the N_s are equal to 2 or 3, it is necessary to add to the statistical weight, calculated according to Eq. (22) and multiplied by $(\prod_{s=1} N_s!)^{-1}$, additional terms which also can be calculated according to Eq. (22), but for particles of double or triple mass.

We shall compute σ for all possible cases in which at the end of the reaction there are 2 Fermi particles of mass M (*i.e.* nucleons) and 1, 2, or 3 Bose particles of mass μ (*i.e.* π-mesons). We concisely designate the statistical weights (18) and (22) by means of bracket expressions, in which the figures in the upper line denote the numbers of particles of each kind, while in the lower line below each figure is correspondingly indicated the mass of the particle, *i.e.*,

$$\sigma \equiv \begin{pmatrix} N_1, & N_2, & \ldots, & N_\nu \\ m_1, & m_2, & \ldots, & m_\nu \end{pmatrix}, \sigma_0 \equiv \begin{pmatrix} N_1, & N_2, & \ldots, & N_\nu \\ m_1, & m_2, & \ldots, & m_\nu \end{pmatrix}_0.$$

According to Eqs. (18), (21), and (22), we obtain for the statistical weights σ, computed by the strict formula (18) their expressions in terms of the statistical weights computed according to the formula [Eq. (22)] of Lepore and Stuart. In the table presented below, expressions are given for σ in terms of σ_0, and the corresponding concrete reactions are written down.

From the table it is seen that corrections for the type of statistics have a twofold character. Firstly, there are the corrections which do not depend on the difference between Fermi and Bose statistics and are caused only by the indistinguishability of the particles. These corrections lead to a decrease in the statistical weight by a factor of $N_1! \ldots N_\nu!$ Secondly three occur specific corrections, the sign of which depends on the difference between Fermi and Bose statistics. We shall call these corrections "corrections of the second kind" in distinction to the corrections caused only by indistinguishability, which we call "corrections of the first kind". From the expressions presented in the table it is seen that the corrections of the second kind decrease the statistical weights in the case of the presence of identical particles obeying Fermi statistics, and they increase them in the case of the presence of identical particles obeying Bose statistics.

Therefore, the calculation of exact statistical weights, with consideration of the type of statistics, for all reactions at the end of which no more than three mesons are formed and two nucleons are left, reduces to calculation of statistical weights of the same reactions by Boltzmann statistics and of statistical weights of states with fewer numbers of particles, but with double or triple masses, also by Boltzmann statistics, *i.e.*, by Eq. (22).

Statistical weight	Type of reaction: *np*	*pp*	*nn*
$\begin{pmatrix}1&1&1\\M&M&\mu\end{pmatrix}=\begin{pmatrix}1&1&1\\M&M&\mu\end{pmatrix}_0$	*np* 0	*np* +	*np* —
$\begin{pmatrix}2&1\\M&\mu\end{pmatrix}=\frac{1}{2}\begin{pmatrix}2&1\\M&\mu\end{pmatrix}_0-\frac{1}{2^4}\begin{pmatrix}1&1\\2M&\mu\end{pmatrix}_0$	*nn* + pp —	*pp* 0	*nn* 0
$\begin{pmatrix}1&1&1&1\\M&M&\mu&\mu\end{pmatrix}=\begin{pmatrix}1&1&1&1\\M&M&\mu&\mu\end{pmatrix}_0$	*np* +—	*np* + 0	*np* — 0
$\begin{pmatrix}1&1&2\\M&M&\mu\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&1&2\\M&M&\mu\end{pmatrix}_0+\frac{1}{2^4}\begin{pmatrix}1&1&1\\M&M&2\mu\end{pmatrix}_0$	*np* 00		
$\begin{pmatrix}2&1&1\\M&\mu&\mu\end{pmatrix}=\frac{1}{2}\begin{pmatrix}2&1&1\\M&\mu&\mu\end{pmatrix}_0-\frac{1}{2^4}\begin{pmatrix}1&1&1\\2M&\mu&\mu\end{pmatrix}_0$	*nn* + 0 *pp* — 0	*pp* +—	*nn* +—
$\begin{pmatrix}2&2\\M&\mu\end{pmatrix}=\frac{1}{4}\begin{pmatrix}2&2\\M&\mu\end{pmatrix}_0-\frac{1}{2^5}\begin{pmatrix}1&2\\2M&\mu\end{pmatrix}_0+\frac{1}{2^5}\begin{pmatrix}2&1\\M&2\mu\end{pmatrix}_0-\frac{1}{2^8}\begin{pmatrix}1&1\\2M&2\mu\end{pmatrix}_0$		*pp* 00 *nn* ++	*pp* — — *nn* 00
$\begin{pmatrix}1&1&1&1&1\\M&M&\mu&\mu&\mu\end{pmatrix}=\begin{pmatrix}1&1&1&1&1\\M&M&\mu&\mu&\mu\end{pmatrix}_0$	*np* +— 0		
$\begin{pmatrix}1&1&2&1\\M&M&\mu&\mu\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&1&2&1\\M&M&\mu&\mu\end{pmatrix}_0+\frac{1}{2^4}\begin{pmatrix}1&1&1&1\\M&M&2\mu&\mu\end{pmatrix}_0$		*np* + 00 *np* ++—	
$\begin{pmatrix}1&1&3\\M&M&\mu\end{pmatrix}=\frac{1}{6}\begin{pmatrix}1&1&3\\M&M&\mu\end{pmatrix}_0+\frac{1}{2^4}\begin{pmatrix}1&1&1&1\\M&M&2\mu&\mu\end{pmatrix}_0+\frac{1}{3^4}\begin{pmatrix}1&1&1\\M&M&3\mu\end{pmatrix}_0$	*np* 000		
$\begin{pmatrix}2&1&1&1\\M&\mu&\mu&\mu\end{pmatrix}=\frac{1}{2}\begin{pmatrix}2&1&1&1\\M&\mu&\mu&\mu\end{pmatrix}_0-\frac{1}{2^4}\begin{pmatrix}1&1&1&1\\2M&\mu&\mu&\mu\end{pmatrix}_0$		*pp* +— 0	*nn* +— 0
$\begin{pmatrix}2&2&1\\M&\mu&\mu\end{pmatrix}=\frac{1}{4}\begin{pmatrix}2&2&1\\M&\mu&\mu\end{pmatrix}_0-\frac{1}{2^5}\begin{pmatrix}1&2&1\\2M&\mu&\mu\end{pmatrix}_0+\frac{1}{2^5}\begin{pmatrix}2&1&1\\M&2\mu&\mu\end{pmatrix}_0-\frac{1}{2^8}\begin{pmatrix}1&1&1\\2M&2\mu&\mu\end{pmatrix}_0$	*pp* — 00 *pp* — — + *nn* + 00 *nn* ++—	*nn* ++ 0	*pp* — — 0
$\begin{pmatrix}2&3\\M&\mu\end{pmatrix}=\frac{1}{12}\begin{pmatrix}2&3\\M&\mu\end{pmatrix}_0+\frac{1}{2^5}\begin{pmatrix}2&1&1\\M&2\mu&\mu\end{pmatrix}+\frac{1}{2\cdot 3^4}\begin{pmatrix}2&1\\M&3\mu\end{pmatrix}_0-\frac{1}{3\cdot 2^5}\begin{pmatrix}1&3\\2M&\mu\end{pmatrix}_0-\frac{1}{2^8}\begin{pmatrix}1&1&1\\2M&2\mu&\mu\end{pmatrix}_0-\frac{1}{6^4}\begin{pmatrix}1&1\\2M&3\mu\end{pmatrix}_0$		*pp* 000	*nn* 000

3. STATISTICAL WEIGHTS FOR CREATION OF SINGLE MESONS.

As the simplest example of the application of the relations derived, we shall consider the statistical weights of the reactions

$$p+p\rightarrow n+p+\pi^+ \quad \text{and} \quad p+p\rightarrow p+p+\pi^0.$$

Inasmuch as direct calculations by Eq. (22) are tedious, we limit ourselves to only the simplified particular cases when the π-mesons can be considered ultrarelativistic, while the nucleons are nonrelativistic. In this case, according to Ref. 4, one can approximately compute

$$\sigma_0=\frac{2^s}{\omega^N}\left[\frac{M^{3s/2}(2\pi)^{3(s-1)/2}}{(sM)^{3/2}}\right]\frac{2^{3n}\pi^n T^{3s/2+3n-5/2}}{\Gamma\left[{}^3/_2(s-1)+3n\right]}, \tag{23}$$

where s is the number of nucleons, n the number of π-mesons, $T=E-\sum_{s=1}^{\nu} m_s N_s$ the kinetic energy. According to the table and Eq. (23) we obtain

$$\frac{\sigma(pp \to pp\,0)}{\sigma(pp \to pn+)} = \frac{1}{2}\left[1 - \frac{35\,\omega}{208\sqrt{2}\pi\,(MT)^{3/2}}\right]. \quad (24)$$

The last equation is valid for $\mu < T < M$. In this region, the change from summation to integration in Eqs. (11) and (12) can be considered justified.

Assuming, as in Ref. 1, $L^3 = (4\pi/3)(\hbar/\mu)^3$ and, consequently, $\omega = (2\pi\hbar/L)^3 = 6\pi^2\mu^3$, we get

$$\frac{\sigma(pp \to pp\,0)}{\sigma(pp \to np+)} = \frac{1}{2}\left[1 - 2{,}24\left(\frac{\mu}{M}\right)^{3/2}\left(\frac{\mu}{T}\right)^{3/2}\right]. \quad (25)$$

From the last formula it is seen that the correction caused by the difference between Fermi and Bose statistics, in the region of its applicability, amounts approximately to from 1 to 10%.

4. SOME REMARKS ABOUT THE METHOD

The method of the microcanonical distribution applied in this article is most exact for computation of statistical weights for a system of noninteracting particles with precisely given energy and momentum, but with arbitrary, unspecified total angular momentum. In a system of colliding nucleons, the angular momentum is not fixed in advance, but certain values of the total angular momentum of the final system may be forbidden, which must lead to a change in the statistical weights, as was remarked by Fermi[1,5]. For evaluation of this circumstance it is obviously necessary to change at the very beginning the assumptions about the allowed states of the system of noninteracting particles and to consider not the states with given momenta, but states of a system of particles with given angular momenta. However, this requires special investigations.

Consideration of the conservation of charge and of the relations between the cross-sections resulting from conservation of isotopic spin can be introduced into the present method even in the final expressions (as was done in making up the table, where the laws of conservation of charge were taken into account).

In case of sufficiently large energy, one can use the mathematically simpler canonical distribution instead of the microcanonical distribution. Obviously the "statistical" weight, calculated according to the formulas of the canonical distribution (σ_θ) is expressed in terms of the statistical weight by the equations of the microcanonical distribution, as

$$\sigma_\Theta(N_1, \ldots, N_\nu) = \int\limits_{E_0}^{\infty}\iiint\limits_{-\infty}^{+\infty} e^{-E/\Theta}\sigma(E, \mathbf{P}; N_1, \ldots, N_\nu) \times dE\, d\mathbf{P}. \quad (26)$$

The general relations for the statistics with conserved charge, found in Ref. 6, are a particular case of formulas which can be derived from Eq. (26). The so-called thermodynamical method, applied by Fermi[1,5] for computing the average number of particles formed in collisions of nucleons with extremely high energies, is also a special case of the method of the canonical distribution.

It is well known that in the limit of sufficiently large energies, and correspondingly for large average numbers of particles, the results given by the microcanonical and the canonical distributions practically coincide. Thus the method of the canonical distribution and, in particular, the thermodynamical method, applied in the region of small average numbers of particles, are approximate methods of computation of statistical weights, which are more exactly computed by the method of the microcanonical distribution.

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131