Concerning Some Possibilities of Formulation of a Relativistically Invariant Theory of Extended Particles

V. S. BARASHENKOV
Academy of Sciences, USSR

(Submitted to JETP editor, February 12, 1956)
J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 566-569 (March, 1957)

Various problems pertaining to the theory of extended particles are examined in the case of the classical (non-quantum) theory of interacting dynamically indeformable extended particles. Although the structure of the theory is not of the Hamiltonian type, a formulation in Lagrangian variables $\mathbf{x}_k$; $\dot{\mathbf{x}}_k$; A_μ; $\dot{A}_\mu$, as well as Hamiltonian variables $\mathbf{x}_k$; $\mathbf{p}_k$; A_μ; π_μ can be given. The possibility of a many time formalism is examined and the latter is found to be non-unique. However, in none of the cases was a transition to a single-time theory found and the system of equations of motion was found to be inconsistent.

1. LAGRANGIAN AND HAMILTONIAN VARIABLES.

NUMEROUS attempts to formulate a theory of extended particles are described in literature[1-8]. The search of possible ways leading in this direction is carried on up to the present time[2,8]. In the published works, however, the conditions of relativistic invariance are not sufficiently well formulated. The present work has for its aim the investigation of possible covariant formulations.

It is easy to obtain the equations of motion of the particles and of the field from the action principle[3,4]:

$$m_i du_{i\alpha}/ds_i - u_{i\beta}\int_{-\infty}^{+\infty} F^{\alpha\beta}(x)\,G(x-x_i)\,d^4x = 0; \tag{1}$$

$$\partial^2 A_\alpha(x)/\partial x_\mu \partial x^\mu + \sum_i e_i \int_{-\infty}^{+\infty} G(x-x_i)\,u_{i\alpha}ds_i = 0. \tag{2}$$

In contrast with the local interaction theory, where the equations of motion of the particles and of the field can be obtained from tha action principle

$$\delta S = \int \delta L(t)\,dt = 0, \tag{3}$$

with the function $L(t)$ defined in a unique way (with the exception of an irrelevant divergence), in a nonlocal theory the expression under the sign of integral in Eq. (3), that is, $L(t)$, can be represented in two basically different forms

$$S = \int L_i(t)\,dt = \int (\mathscr{L}_0(t) + \mathscr{L}'_i(t)\,dt; \tag{4}$$

$$\mathscr{L}'_1(t) = \sum_i e_i v^\mu(t)\int A_\mu(x)\,G(x-x_i)\,d^4x; \tag{5}$$

$$\mathscr{L}'_2(t) = \sum_i e_i \int A_\mu(x)\,d^3x \int v_i^\mu(\tau)\,G(x-x_i(\tau))\,d\tau, \tag{6}$$

where $\mathfrak{L}_0(t)$ is the Lagrangian of the non-interacting particles and the field.

Making use of $L_1(t)$ we can obtain the equations of motion of the particles, but it is impossible to obtain the equations of motion of the field, since $\delta\mathfrak{L}'_1(t)/\delta A_\nu(x)=0$. Using $L_2(t)$, on the other hand, we can obtain the equations of motions of the field but, since $\partial\mathfrak{L}'_2(t)/\partial \mathbf{v}_i=0$, it is impossible to obtain the equations of motion of the particles. If we formally assume that the terms $\delta\mathfrak{L}'_1(t)/\delta A_\nu$ and $\partial\mathfrak{L}'_2(t)/\partial\mathbf{v}_i$ in the equations of motion vanish, then these equations can be written by means of one function $\mathfrak{L}(t)=\mathfrak{L}_0(t)+\mathfrak{L}'_1(t)+\mathfrak{L}'_2(t)$. In this case, however, $\mathfrak{L}$ does not approach the Lagrange function of the local theory $\mathscr{H}(t)$:

$$\mathscr{H}(t) = \sum_i e_i \int A_0(x)\,G(x-x_i)\,d^4x + \sum_i \sqrt{\mathbf{P}_i^2 + m_i^2} - \sum_i e_i \int A_\mu(x)\,v_i^\mu(\tau)\,G(x-x_i)\,d^3x\,d\tau + \mathscr{H}_0(t); \tag{7}$$

$$\mathbf{v}_i(\tau) = \mathbf{P}_i(\tau)/\sqrt{\mathbf{P}_i^2 + m_i^2}; \quad v_i^0 = 1; \qquad \mathbf{P}_i = \mathbf{p}_i - e_i\int \mathbf{A}(x)\,G(x-x_i)\,d^4x, \tag{8}$$

where $\mathscr{H}_0(t)$ is the Hamiltonian of the free electromagnetic field.

We shall further determine the Hamilton system of equations by means of a well-known procedure. In these equations, the dynamical variables $\mathbf{x}_i$; $\mathbf{P}_i$; A_μ; π_μ tend to the corresponding Hamiltonian variables of the local theory as $\lambda \to 0$; they can therefore be regarded as their generalization. Both in the "Lagrange" and in the "Hamilton" equations, however, the dynamical variables are taken for different moments of time, and the state of a physical system at the time t does not determine, in a unique way, the state of the system at the time $t+\delta t$.[10] The functions $\mathfrak{L}(t)$ and $\mathcal{H}(t)$ are only auxiliary functions and are useful for the study of the relativistically invariant generalizations of the theory.

2. MANY-TIME FORMULATION OF THE EQUATIONS OF MOTION.

A relativistically invariant generalization of the theory can be obtained within the framework of the many-time formalism of Dirac, Fock and Podolsk[4]. For this purpose, we shall generalize the Hamilton system of equations known from the local theory, making use of Eq. (7) in the following way:

$$\partial \mathbf{p}_i/\partial t_j = -\partial \mathcal{H}'_j(t_j)/\partial \mathbf{x}_i; \quad \partial \mathbf{x}_i/\partial t_j = \partial \mathcal{H}'_j(t_j)/\partial \mathbf{p}_i; \tag{9}$$

$$\partial P^\mu_k/\partial t_j = -\partial \mathcal{H}''_j(t_j)/\partial Q^\mu_k; \quad \partial Q^\mu_k/\partial t_j = \partial \mathcal{H}''_j(t_j)/\partial P^\mu_k; \tag{10}$$

$$\mathcal{H}'_j = e_j \int A_0(x)\, G(x-x_j)\, d^4x + \sqrt{\mathbf{P}_i^2 + m_j^2}; \tag{11}$$

$$\mathcal{H}''_j = -e_j \int A_\mu(x)\, d^3x \int v^\mu_j(\tau_j)\, G(x - x_j(\tau_j))\, d\tau_j; \tag{12}$$

$$A^\mu(x) = \sum_k \left\{ \frac{1}{k} P^\mu_k \cos(\mathbf{kx} - kt) + Q^\mu_k \sin(\mathbf{kx} - kt) \right\}, \tag{13}$$

v^μ_j being determined by Eq. (8).

Clearly, we have

$$\partial^2 A^\mu(X)/\partial X_\nu \partial X^\nu = 0, \tag{14}$$

where $X = \{X, T, Z, T\}$ are the space-time coordinates of the field.

Equations (9) and (10) can be written in the form

$$m_i du^\mu_i/\partial s_j = e_i \delta_{ij} u_{i\nu} \int F^{\mu\nu}(x)\, G(x-x_i)\, d^4x - e_i e_j \int u^\mu_j(s_j)\, G(x-x_j)\, D \times (x-x_i)\, d^3x ds_j/\sqrt{1-v_j^2}, \text{ where } x_0 \equiv t_j; \tag{15}$$

$$dA^\mu(x)/dt_j = e_i \int u^\mu_j(s_j)\, D(x'-x) \times G(x'-x_j)\, d^3x'\, ds_j, \text{ where } x'_0 \equiv t_j. \tag{16}$$

Since for any relativistically invariant function we have $G(x) = \int g(k^2) e^{i\mathbf{kx}} d^4k$ and for any function $B(x)$ fulfilling Eq. (14) the following identity is true

$$\int B(x')\, G(x-x')\, d^4x' = B(x),$$

then the expression (11) does not differ from the Hamiltonian of the local theory, and the first term of the right-hand side of Eq. (15) can be written in the form*

$$e_i \delta_{ij} u_{i\nu} F^{\mu\nu}(X_i;\, t_1;\, t_2;\, \ldots)_\lambda.$$

It is important to note that the system of equations (9) and (10) [or (15) and (16)] is not the only possible many-time generalization of the single-time equations. For example, the equation

$$dA^\mu(x)/dt_j = e_i \int u^\mu_j(s_j)\, D(x_i - x') \times G(x'-x)\, d^3x' ds_j \quad \text{etc.}$$

may be chosen instead of Eq. (16).
In all cases, however, the conclusions of Sec. 3 and 4 remain in remain in force.

3. IMPOSSIBILITY OF TRANSITION TO SINGLE-TIME EQUATIONS.

In transition to a single-time theory, it is necessary to take into account the fact that, for an arbitrary function $R(x;\, t_1;\, t_2;\, \ldots)$, we have

$$dR(x)/dt = \left\{ \left(d/dT + \sum_i d/dt_i \right) R(x;\, t_1;\, t_2;\, \ldots) \right\}_{t=T=t_i}.$$

Putting $R(x;\, t_1;\, \ldots) = dA^\mu(x;\, t_1;\, \ldots)/dT$ we obtain making use of Eqs. (14) and (16),

$$\partial^2 A^\mu(x)/\partial x_\nu \partial x^\nu + \sum_i e_i \int u^\mu_i(t_i;\, t)\, G(x - x_i(t_i))\, ds_i = 0, \tag{17}$$

where $u^\mu_i(t_i;\, t) = u^\mu_i(t_1;\, t_2;\, \ldots)|_{t=t_1=\ldots=t_{i-1}=t_{i+1}=\ldots;\, t_i \neq t}$.

The resulting equation does not coincide with Eq.

---

*This cannot be done if the form function depends on any four-dimensional vectors characterizing the system of interacting particles or their internal structure.

(2), since it is not permissible to equate t_i with t under the sign of integral. In an analogous way, for the Eq. (15), we obtain

$$m_i du_i^\mu/dt = e_i u_{i\nu} \int F^{\mu\nu}(x_i;\, t)\, G(x' - x_i)\, d^4x' \qquad (18)$$
$$= e_i u_{i\nu} F^{\mu\nu}(x_i),$$

which is different from Eq. (1). If we determine the first term on the right-hand side of Eq. (15) as follows:

$$e_i \delta_{ij} \int u_{i\nu}(s_i)\, G(x - x_i(s_i))\, F^{\mu\nu}(x')\, ds_i\, d^3x';\quad x_0' = t_i,$$

then, in transition to the single time theory, Eq. (15) will be different from Eq. (18) as well as from Eq. (1).

4. CONDITIONS OF CONSISTENCY OF THE MANY-TIME EQUATIONS.

Similarly to the case of the theory dealing with point particles, the system of equations of motion can be integrated only if the Bloch conditions are fulfilled. From Eq. (10) and (15), we obtain, for $i \neq j$,

$$m_j \partial^2 P_k^\mu/\partial t_i \partial t_j = \qquad (19)$$
$$- e_j^2 e_i \int \sin(\mathbf{kx} - kt_j)\, v_i^\mu(\tau') \sqrt{(1 - \mathbf{v}_j^2(\tau))^3}$$
$$\times G(x - x_j(\tau))\, D(x' - x_j(\tau))\, G(x' - x_i(\tau'))$$
$$\times d^3(xx')\, d(\tau\tau'),$$

where

$$x_0 = t_j \neq \tau;\quad x_0' = t_i \neq \tau'.$$

The expression for $\partial^2 P_k^\mu/\partial t_j \partial t_i$ we can obtain from Eq. (19) by exchanging the indices i and j. It can be easily seen that, for $G(x) \neq \delta(x)$, the Bloch condition is not satisfied:

$$\partial^2 P_k^\mu/\partial t_i dt_j - \partial^2 P_k^\mu/\partial t_j \partial t_i \neq 0.$$

The actual choice of the generalization of Eqs. (1) and (2) in the form (9) and (10) does not impair the generality of discussion. Repeating the procedure analogous to that which leads to relation (19), it can easily be seen that the presence of a form-function, even if only in one of the equations of motion, which does not vanish in accordance with Eq. (17), always leads to an inconsistent system of equations.*

In conclusion I wish to express my gratitude to Prof. D. I. Blokhintsev for his interest and valuable advice, and to Prof. M. A. Markov for interesting comments concerning the various problems of the theory with the form-factor.

[1] G. Wataghin, Z. Physik **88**, 92 (1934)

[2] G. F. Chew, Phys. Rev. **95**, 1669 (1954)

[3] D. I. Blokhintsev, Vestn. Moscow State Univ. (phys) **77**, 3 (1945); J. Phys. (U.S.S.R.) **10**, 162 (1946); J. Exptl. Theoret. Phys (U.S.S.R.) **16**, 480 (1946); **18**, 566 (1948)

[4] R. E. Peierls, 8th Conseil de Physique, Inst. Solvay p. 291 (1948); R. E. Peierls and H. McManus, Proc. Roy. Soc. (London) **195A**, 323 (1948); I. Irving, Proc. Phys. Soc. (London) **62**, 780 (1949)

[5] Iu. M. Shirokov, J. Exptl. Theor. Phys. (U.S.S.R.) **24**, 47 (1953)

[6] M. A. Markov, J. Exptl. Theor. Phys. (U.S.S.R.) **10**, 1311 (1940); **16**, 790 (1946); **21**, 11 (1951)

[7] M. A. Markov, J. Exptl. Theor. Phys. (U.S.S.R.) **25**, 527 (1953)

[8] M. A. Markov, Dokl. Akad. Nauk. SSSR **101**, 51, 449 (1955)

[9] M. A. Markov, J. Phys. **7**, 42 (1944)

[10] V. S. Barashenkov, J. Exptl. Theor. Phys. (U.S.S.R.) **31**, 694 (1956); Soviet Phys. JETP **4**, 599 (1957)

[11] W. Pauli, Nuovo Cimento **10**, 648 (1953)

[12] Y. Takahashi and H. Umezawa, Progr. Theor. Phys. **9**, 14, 501 (1953); C. Hayashi, Progr. Theor. Phys. **10**, 533 (1953); Y. Katayama, Progr. Theor. Phys. **10**, 31 (1953)

Translated by H. Kasha
128

*The attempts to obtain a covariant Lagrange or Hamilton system of equations by the method of Pauli[11] or by the generalized Young-Feldman method[12] have also been futile[10].