# Skin Effect in Thin Films and Wires

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Equations are obtained for the skin effect in thin films and wires by using kinetic theory. A method is developed for the approximate solution of these equations which gives expressions for the impedance of thin films and wires.

**1.** A KINETIC theory of skin effect for bulk conductors was developed by Reuter and Sondheimer[1]. If $l \ll \delta_0$, where $l$ is the electron free path in a metal and $\delta_0$ is the classic skin layer depth, then the usual or normal skin effect occurs. If $l \gtrsim \delta_0$ or $l \gg \delta_0$, the skin effect is anomalous. In the case of samples of small size (wires of radius $R \lesssim l$ or films of thickness $h \lesssim l$), just as in the $\delta_0 \lesssim l$ case, it is impossible to the use the normal skin effect theory. Hence, the kinetic approach for thin films and wires is necessary for direct current[2-6]. Also the question arises of the behavior of small-size samples in an alternating field. Dingle[7] considered the question for the optical frequency case, taking only diffusion reflection of the electrons from the metal surface into account. The work of Azbel' and Kaganov[8] is also devoted to this question, where, as in Ref. 7, the passage of a plane electromagnetic wave through a film is analyzed. Hence, the impedance for this plane wave passage is calculated in these works. It is also interesting to determine the impedance of thin films and wires with respect to the high-frequency current flowing therein.

Below, we consider the conclusions to which kinetic theory leads for the case of skin-effect in thin conductors, *i.e.*, in such conductors as are much less thick than the free path length of the electrons in an unlimited metal. To be specific, let us visualize cylindrical wires of radius $R$ and plane films of thickness $h$.

**2.** Let us direct the $z$ axis parallel to the wire axis (or parallel to the film surface). Let us impose an electric field $E = E_z e^{i\omega t}$, directed along the $z$ axis, on the sample. The quantity $E_z$ is a function of the points in the wire cross section. If this function is known, then the energy losses in the conductor are characterized by the impedance*

$$Z = R + iX = (4\pi i\omega / c\alpha)\, E(P) / E'(P), \qquad (1)$$

where $P$ is a point on the conductor surface, $E'(P)$ is the normal derivative relative to the conductor surface, and

$$\alpha = \begin{cases} 1 & \text{for a conducting half-space} \\ 2 & \text{for films} \\ 2\pi R & \text{for wires.} \end{cases} \qquad (2)$$

The real part of the impedance $R$ gives the ohmic resistance of the conductor and the imaginary part $X$ gives the inductance.

The classical skin effect theory, valid for $l \ll h$, $l \ll \delta_0$, leads to the following expression for films of thickness $h$:

$$Z = -\frac{2\pi i\omega}{c^2 k_0} \operatorname{ctg} k_0 \frac{h}{2} = -\frac{4\pi i\omega}{c^2 k_0^2 h}\left[1 - \frac{1}{12}(k_0 h)^2 - \cdots\right], \qquad (3)$$

$$k_0^2 = -2i / \delta_0^2 = -4\pi i\omega\sigma_0 / c^2, \qquad (4)$$

where $\sigma_0$ is the conductivity of the unlimited sample.

**3.** As shown in Refs. 2-6, if the lateral dimensions of the conductor become comparable to the free path length $l$, the direct current sample conductivity ($\omega = 0$) starts to depend on the lateral dimensions. For thin films with $h \ll l$, we have[2]:

$$\sigma = \sigma_0 \frac{3}{4} \frac{1+\varepsilon}{1-\varepsilon} \frac{h}{l} \ln \frac{l}{h} \quad (\text{if } \varepsilon \lessdot 1); \qquad (5)$$

$$\sigma = \sigma_0 \quad (\text{if } \varepsilon = 1),$$

where $\varepsilon$ is the portion of the electrons reflected specularly from the sample surface. For thin wires with $R \ll l$, we have[4]:

$$\sigma = \sigma_0 \frac{1+\varepsilon}{1-\varepsilon} \frac{2R}{l} \quad (\text{if } \varepsilon \lessdot 1); \quad \sigma = \sigma_0 \quad (\text{if } \varepsilon = 1). \qquad (6)$$

**4.** In deriving the skin effect equations, we used the Chambers method[5] which, being equivalent to the kinetic equations method, permits a whole set of intermediate calculations to be avoided. The gen-

*We use the impedance definition $Z = E(P)/I$, which leads to (1) and (2) under the assumption of field symmetry over the wire cross-section. Consequently, we will always select the solutions corresponding to field symmetry subsequently.

eralization of the Chambers method to the case of a field dependent on the coordinates is made without difficulty. This method yields for the average velocity increment (in the $z$ direction) taken by electrons arriving at a point $M$ from $P$ in the volume of the conductor (Fig. 1):

$$\Delta v_z = \frac{e}{mv}\left[\int_a E(s)\, e^{-s/l}\, ds + \frac{\varepsilon e^{-a/l}}{1-\varepsilon e^{-b/l}} \int_b E(s)\, e^{-s/l}\, ds\right], \tag{7}$$

where $a = PM$ and the same notation is used as in Ref. 5. If $E$ is independent of $s$, the result of Ref. 5 is obtained from (7).

The expression (7) permits the current at the point $M$ to be expressed through $E_z$:

$$j_z(M) = \int v^2 dv\, d\Omega\, ev_z \frac{\partial f_0}{\partial v_z} \Delta v_z, \tag{8}$$

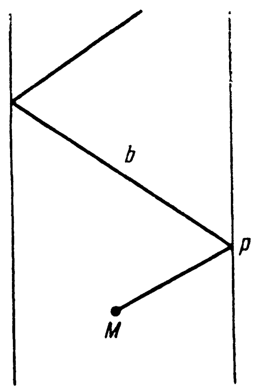


FIG. 1

where $f_0$ is the equilibrium distribution of the electrons in the metal. Substituting (8) in the equation for $E_z$

$$\Delta E_z + (\omega/c)^2 E_z = (4\pi i\omega/c^2)\, j_z,$$

we obtain an integro-differential equation for $E_z$. Let us give the final form of the equation. For films of thickness $h$:

$$\frac{d^2E}{\partial x^2} + \frac{\omega^2}{c^2}E = \frac{3i}{2\delta_0^2 l}\int_0^h E(x')\,[\varphi_1(x-x') + \varepsilon\varphi_2(x-x')]\, dx', \tag{9}$$

where $\varphi_1(x) = \int_1^\infty e^{-t|x|/l}\left(\frac{1}{t} - \frac{1}{t^3}\right) dt,$

$$\varphi_2^{(x)} = 2\int_1^\infty \frac{e^{-ht/l}}{1-\varepsilon e^{-ht/l}} \operatorname{ch}\frac{xt}{l}\left(\frac{1}{t} - \frac{1}{t^3}\right) dt. \tag{10}$$

For wires of radius $R$

$$\frac{d^2E}{dr^2} + \frac{1}{r}\frac{dE}{dr} + \frac{\omega^2}{c^2}E = \frac{3i}{2\pi\delta_0^2 l}\int_0^{2\pi} d\varphi \int_0^R E(r') \times [\varphi_1(|\mathbf{r}-\mathbf{r}'|) + \varepsilon\varphi_2(|\mathbf{r}-\mathbf{r}'|)]\, r'\, dr', \tag{11}$$

$$\varphi_1(|\mathbf{r}|) = \frac{2}{r}\int_1^\infty e^{-rt/l}\sqrt{t^2-1}\,\frac{dt}{t^3} = \frac{2}{r} S_3\left(\frac{r}{l}\right);$$

$$\varphi_2(|\mathbf{r}|) = \frac{2}{r}\int_1^\infty \frac{e^{-at/l} + e^{-bt/l}}{1-\varepsilon e^{-bt/l}} e^{-rt/l}\sqrt{t^2-1}\,\frac{dt}{t^3}. \tag{12}$$

In the expressions (12), $S_3$ is a function introduced in Ref. 4, $\mathbf{r}$ and $\mathbf{r}'$ are points in the wire cross section, $b$ is the length of the chord drawn through $\mathbf{r}$ and $\mathbf{r}'$, $b-a$ and $a$ are segments into which the point $\mathbf{r}$

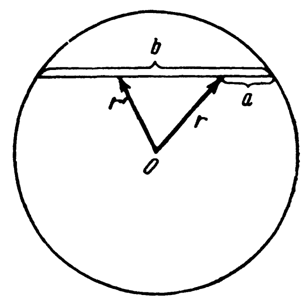


FIG. 2

divides the chord (Fig. 2). We neglected the relaxation time in deriving these equations. The relaxation time can be taken into account if $l/(1-i\omega\tau)$ is written instead of $l$ in (10) and (12) for $\varphi_1$ and $\varphi_2$. We will neglect the quantity $\omega\tau$ in comparison with unity below.

**5.** Let us turn to the solution of the equations obtained. Let us first analyze Eq. (9) for the thin films. This equation is easily reduced to the integral equation

$$E(x) = \cos\frac{\omega}{c}\left(x - \frac{h}{2}\right) + \frac{3i}{2\delta_0^2 l}\int_0^h K(x, x')\, E(x')\, dx'; \tag{13}$$

$$K(x, x') = \frac{c}{\omega}\int_{h/2}^x \sin\frac{\omega}{c}(x-t)\,[\varphi_1(t-x') + \varepsilon\varphi_2(t-x')]\, dt. \tag{14}$$

It is easy to be convinced of the equivalence of (9) and (13) by calculating the quantity $E'' + (\omega/c)^2 E$ using (13). This quantity gives the right side of (9).

While the kernel of the original equation (9) would have a logarithmic singularity at $x=x'$, the kernel of Eq. (13) is bounded. If

$$\left| \frac{3ih}{2\delta_0^2 l} \max K(x, x') \right| < 1, \qquad (15)$$

Eq. (13) can be solved by an iteration method. Putting $\cos \frac{\omega}{c}(x - \frac{h}{2}) = 1$ in the $h \ll c/\omega$ case, and taking $E_0 = 1$ for the zero approximation, we obtain

$$E(x) = 1 + \frac{3i}{2\delta_0^2 l} \int_0^h K(x,x')dx' \qquad (16)$$

$$+ \left( \frac{3i}{2\delta_0^2 l} \right)^2 \int_0^h \int_0^h K(x,x') K(x',x'')\, dx'\, dx'' + \ldots$$

Substituting (16) in (1) for the impedance, we find the latter as the ratio of two power series in the parameter (15). In its turn, this ratio can be expanded in a power series in the same parameter. The first term of the expansion is independent of the field frequency $\omega$ and gives the ohmic resistance of the film to direct current:

$$Z_0 = \frac{1}{\sigma_0 l} \left[ \frac{h}{l} \right. \qquad (17)$$

$$\left. - \frac{3}{2}(1 - \varepsilon) \int_1^\infty \left( \frac{1}{t^3} - \frac{1}{t^5} \right) \frac{1 - e^{-ht/l}}{1 - \varepsilon e^{-ht/l}}\, dt \right]^{-1} = \frac{1}{\sigma h}.$$

This expression is equivalent to that obtained earlier by Sondheimer[6]. It can be used as a definition of the film conductivity $\sigma$ in a direct current. The expression (5) for $\sigma$ is obtained from (17) for $h \ll l$.

The evaluation of the next terms of the expansion of $Z$ in powers of the parameter (15) appears to be somewhat complex. Let us give the results for the case of purely diffuse ($\varepsilon = 0$) and purely specular ($\varepsilon = 1$) reflection for $h \ll l$. In both cases,

$$Z = Z_0 [1 - (kh)^2 / 12 + \cdots], \qquad (18)$$

where $Z$ is defined by (17) and $k$ by (4) if $\sigma_0$ is replaced by $\sigma(\varepsilon)$ according to (5). It is easy to see that the parameter (15) is the square of the ratio of the film thickness to the depth of field penetration into an unlimited sample with conductivity equal to the film conductivity.

As is seen from (3) and (18), the first terms of the impedance expansion in the normal skin effect case ($h$, $\sigma_0 \gg l$) agree with the terms of the impedance expansion for thin films ($h \ll l$, $\sigma \gg l$) with $k$ replaced by $k_0$. The region of such "quasi-normal" skin effect in thin films appears to be broader than the region of normal skin effect in bulk conductors. Higher terms of the impedance expansion in powers of $kh$ were not calculated. It should not be expected that the coefficients of the expansions (3) and (18) will agree exactly. As will be clear from the sequel, it can be imagined that they will be close in order of magnitude.

**6.** In order to determine the impedance at higher frequencies, it is convenient to use a less exact approximate method of solution which we will illustrate first by an example of a metallic half-space (a problem analyzed in Ref. 1). In this case, the skin effect equation is (for simplicity, we consider the reflection to be diffuse):

$$E'' + \frac{\omega^2}{c^2} E = \frac{3i}{2\delta_0^2 l} \int_0^\infty \varphi_1(x - y)\, E(y)\, dy. \qquad (19)$$

As is seen from (19), the current intensity at the given point $x$ is determined by the whole region near this point. The effective distance characterizing the size of this region equals (in order of magnitude):

$$\lambda = \int_0^\infty \varphi_1(x)\, x\, dx \Big/ \int_0^\infty \varphi_1(x)\, dx = \frac{3}{8} l \qquad (20)$$

[see Eq. (10)]. In order to determine the impedance, we must know how the field itself behaves in the depth of the metal, even if values of the field and its derivative on the metal surface are essential. In the region close to the metal surface ($x = 0$), $E$ satisfies the equation

$$E'' + \frac{\omega^2}{c^2} E = \frac{3i}{2\delta_0^2 l} \int_0^\infty \varphi_1(y)\, E(y)\, dy. \qquad (21)$$

Let us replace the kernel $\varphi_1(y)$ by a function which equals a certain constant $\beta$ in the limits from 0 to $\lambda$[Eq. (20)] and which becomes zero outside this range. Then (21) becomes:

$$E'' + \frac{\omega^2}{c^2} E = \frac{3i\beta}{2\delta_0^2 l} \int_0^\lambda E(y)\, dy. \qquad (22)$$

This equation admits of a solution $E(x) = e^{-kx}$. Substitution in (22) gives an equation for $k$

$$k^2 + \omega^2/c^2 = (3i\beta/2\delta_0^2 lk)(1 - e^{-3kl/8}). \qquad (23)$$

If $|kl| \ll 1$, *i.e.*, if the field attenuates at distances much larger than the free path length $l$, then (23) becomes

$$k^2 + (\omega/c)^2 = 9\,i\beta/16\delta_0^2. \tag{24}$$

Inasmuch as the classical skin-effect theory is valid in this case[1], the right side must be equal to $2i/\delta_0^2$. Hence, we find:

$$\beta = 32/9. \tag{25}$$

In the other extreme case, we obtain for $|kl| \gg 1$, neglecting displacement currents:

$$k = (16i/3\delta_0^2 l)^{1/3} = (\sqrt{3} + i)\,(2/3\delta_0^2 l)^{1/3}, \tag{26}$$

from which the impedance is

$$Z = 4\pi i\omega/c^2 k = (1 + i\sqrt{3})\,(3\pi^2 l\omega^2/4\sigma_0 c^4)^{1/3}. \tag{27}$$

According to exact theory[1],

$$Z = (1 + i\sqrt{3})\,(\sqrt{3}\,\pi l\omega^2/\sigma_0 c^4)^{1/3}. \tag{27a}$$

These values are very close. If both $\lambda$ and $\beta$ are considered as adjustable parameters, then exact agreement is obtained with the Reuter-Sondheimer theory for

$$\lambda = \sqrt{3}\,l/2\pi, \quad \beta = 8\pi/3\sqrt{3}.$$

It is understood that the agreement with exact theory can hold in the two extreme cases: $|kl| \gg 1$ and $|kl| \ll 1$ since the exact solution is slightly different from the exponential $e^{-kx}$ only in these cases (see Ref. 1, Appendix III).

**7.** Let us use this rough theory in the thin film case ($h \ll l$). In this case, we can again write (22), where the film thickness $h$ should be substituted as the upper limit of integration instead of $\lambda$ [Eq. (20)]:

$$E''(h) + \frac{\omega^2}{c^2} E(h) = \frac{3i\beta}{2\delta_0^2 l} \int_0^h E(y)\,dy. \tag{28}$$

Let us look for the solution as $E = \cos k(x - h/2)$. Substituting this function into (28), again neglecting the term $\omega^2/c^2$ in comparison with $k^2$, we obtain

$$k^3 \cos(kh/2) = (3i\beta/\delta_0^2 l)\sin(kh/2). \tag{29}$$

Hence, for $|kh| \ll 1$, we find

$$k^2 = 3i\beta h/2\delta_0^2 l. \tag{30}$$

As in Sec. 6, the constant $\beta$ can be determined by requiring that the impedance agree with (17), obtained in Sec. 5 by the iteration method, $\omega \to 0$. This yields

$$\beta = \frac{4l\sigma}{3h\sigma_0} = \frac{4}{3}\frac{l}{h} - 2\frac{l^2}{h^2}(1 - \varepsilon)\int_1^\infty \left(\frac{1}{t^3} - \frac{1}{t^5}\right)\frac{1 - e^{-ht/l}}{1 - \varepsilon e^{-ht/l}}\,dt. \tag{31}$$

In the $|kh| \gg 1$ case, we find

$$k = {}^1\!/_2\,(\sqrt{3} + i)\,(3\beta/\delta_0^2 l)^{1/3}. \tag{32}$$

In this case, we find for the impedance

$$Z = (1 + i\sqrt{3})\,(\pi^2\omega^2 h/4\sigma c^4)^{1/3}, \tag{33}$$

where $\sigma$ is the static conductivity of the film of thickness $h$.

An approximate expression can be obtained from (27a) by replacing $l$ by $h$ and $\sigma_0$ by $\sigma$ therein.

**8.** Simplifying (11) for thin wires ($R \ll l$) by the same method, we obtain:

$$\frac{d^2E}{dr^2} + \frac{1}{r}\frac{dE}{dr} + \frac{\omega^2}{c^2}E = \frac{3i\beta}{2\pi\delta_0^2 l}\int_0^R E(r')\,dr'. \tag{34}$$

We look for the solution of this equation as $E(r) = J_0(kr)$. Substituting $J_0(kr)$ in (34), we find:

$$\left(\frac{\omega^2}{c^2} - k^2\right) J_0(kr) = \frac{3i\beta}{2\pi\delta_0^2 l}\int_0^R J_0(kr')\,dr'. \tag{35}$$

Hence, again neglecting the quantity $\omega^2/c^2$, we obtain for $|kR| \ll 1$:

$$k^2 = -3i\beta R/2\pi\delta_0^2 l. \tag{36}$$

Just as in the thin film case, we determine the constant $\beta$ from the condition that the impedance at zero frequency should give the known expression[4] for the resistance of a thin wire. This yields

$$\beta = \frac{4\pi l\sigma}{3R\sigma_0} = \frac{8\pi}{3}\frac{1 + \varepsilon}{1 - \varepsilon}\ (\varepsilon \leqslant 1). \tag{37}$$

Hence, we obtain for $|kR| \gg 1$

$$Z = \frac{1 + i\sqrt{3}}{2\pi R}\left(\frac{2\pi^2\omega^2 R}{c^4\sigma}\right)^{1/3}. \tag{38}$$

**9.** Let us discuss the results obtained. The parameter characterizing the skin effect in the conducting half-space case is the ratio $l/\delta_0$, where $\delta_0$ is the classical skin layer depth. As follows from our computations, the critical parameter in the thin conductors case is the ratio of the size of the conductor cross-section (film thickness $h$ or wire radius $R$) to the depth of field penetration in a bulk conductor with the same static conductivity as in the thin conductor. Namely, if the depth of field penetration is large in comparison with the film thickness or with the wire radius [see inequality (15)], the analog of the classic skin effect holds with the difference, however, that the conductivity is determined by (5) and (6) in the thin conductor case. In this case, the field and the impedance can be found by a power series expansion in the parameter (15). As is easy to see, the real part of the impedance (the ohmic resistance) is expanded in even powers of the frequency $\omega$ in this case, just as in the classic skin effect case, and the imaginary part of the impedance, in odd powers of $\omega$. The expansions for thick and thin films agree to the accuracy of terms of order $(kh)^2$ (with the difference that $k_0$ replaces $k$ in the thick films case). The order of magnitude of the next terms of the expansion can be determined by using an approximate method explained in Secs. 6, 7 and 8. According to this method, the impedance of a thin sample can be obtained from the expansion (3) if we speak of a thin film and from a similar expansion for a wire, by replacing the conductivity of the unlimited sample $\sigma_0$ by the conductivity of a thin conductor $\sigma$. Hence, the normal skin effect theory can actually be used to estimate the impedance of thin samples in the frequency band under consideration. Let us note that the frequency band where such a quasi-classical skin effect occurs appears to be much broader than the region of classical skin effect in bulk conductors. Actually, condition (15) for a thin conductor or the corresponding condition $|kh| \ll 1$ ($|kR| \ll 1$) in the approximate method, is satisfied, as is easy to see, for a much larger frequency band than in the bulk conductor. A case is even possible when anamalous skin effect conditions are not generally realized in thin conductors. This occurs when condition (15) is violated at such high frequencies for which it is impossible to neglect the relaxation time and to take the relaxation time $\tau$ into account yields a condition analogous to the inequality (9) of Ref. 1, *i.e.*, again the classical skin effect condition.

When there is an anomalous skin effect region (as always occurs if the conductor is not too thin), the impedance of the thin conductors is expressed by the approximate formulas (33) and (38). In this case, the impedance depends on the thickness of the film wire. The impedances of thin wires and thin films are, respectively, proportional to

$$\frac{1}{R}\left[\frac{1-\varepsilon}{1+\varepsilon}\right]^{1/3} \text{and} \left[\frac{1-\varepsilon}{1+\varepsilon}\left(\ln\frac{l}{h}\right)^{-1}\right]^{1/3}.$$

The impedance of a thin film depends on $h$ as $h^{1/3}$ and the impedance of a thin wire is proportional to $R^{-2/3}$ for $\omega=1$, $\sigma=\sigma_0$. The skin effect in a thin film was analyzed earlier by Ginzburg[9], where $\sigma_{eff}=\sigma_0\gamma h/l$ with $\gamma\approx 1$ was assumed for the thin film. As is clear from the above, $\gamma=\ln(l/h)$.

It is impossible to compare the conclusions obtained with the results of Dingle[7] and Azbel' and Kaganov[8] since they determined the impedance for the passage of a plane wave and not for a current as in the present work.

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