Excitation of Betatron Oscillations by Synchrotron Momentum Oscillations in a Strong Focusing Accelerator

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The occurrence of resonances between the synchrotron oscillations of the momentum p and the amplitude beats near resonance on magnetic field perturbations is demonstrated. Resonances occur if the ratio between the beat frequency (for $\Delta p/p = 0$) and the frequency of synchrotron oscillations is an integer. Transitions through these resonances are examined in the linear and non-linear approximations.

1. EQUATIONS OF MOTION AND RESONANCES

LET us examine the simultaneous effect of the perturbation of the magnetic field and of the betatron frequency, due to the synchrotron oscillations of the momentum. The simultaneous effect of a parametric resonance is not important here. Let us consider, for example, the radial oscillations. The initial equations have the form

$$\frac{d^2r}{d\theta^2} - \left(\frac{l}{2\pi}\right)^2 \frac{\partial H/\partial r}{P_0} r + \left(\frac{l}{2\pi}\right)^2 \frac{\partial H/\partial r}{P_0}\frac{\Delta p}{p} r = \left(\frac{l}{2\pi}\right)^2 \frac{\Delta H}{P_0}, \quad (1)$$

where $1/p_0 = e/cp_0 = 1/\rho_0$; ρ_0 is the radius of the unperturbed closed orbit; l is the length of the periodic sector; $\theta = (2\pi/l)s$; s is the coordinate along the unperturbed closed orbit. The small synchrotron oscillations of the momentum are described by the term $\Delta p/p$. The gradient of the magnetic field $\partial H_0/\partial r$ has a period of 2π.

The general solution of the unperturbed equation ($\Delta p/p = \Delta H = 0$) has the form

$$r = a\varphi^* + a^*\varphi, \quad \varphi(\theta) = f(\theta)\exp(i\nu\theta), \quad f(\theta) = f(\theta + 2\pi) \quad (2)$$

(φ is the Floquet function and ν the known betatron quasi-frequency). As usually, we seek a solution of Eq. (1) in the form

$$r = x\varphi^* e^{-i\nu\theta} + x^*\varphi e^{i\nu\theta}, \quad r' = x\varphi'^* e^{-i\nu\theta} + x^*\varphi' e^{i\nu\theta}. \quad (3)$$

For x we have the equation

$$\frac{dx}{d\theta} + i\nu x = -i\left(\frac{l}{2\pi}\right)^2 \frac{1}{w} f(\theta) \frac{\partial H/\partial r}{P_0}(xf^* + x^*f)\frac{\Delta p}{p} - i\left(\frac{l}{2\pi}\right)^2 \frac{1}{w} f(\theta)\frac{\Delta H}{H}, \quad (4)$$
$$w = i(\varphi\varphi^{*\prime} - \varphi^*\varphi').$$

In the linear approximation,

$$\Delta p/p = (\Delta p/p)_{\max} \sin\Omega\theta, \quad (5)$$

where Ω is the synchrotron frequency in appropriate units.

Equation (4) can be solved in the usual way, by finding the general solution for $\Delta H = 0$ and then the solution of the complete equation (4). If $\epsilon_1 \ll 2\Omega$, then

$$\frac{\varepsilon_1}{2\Omega} \ll 1, \quad \varepsilon_1 = \frac{1}{2\pi}\left(\frac{\Delta p}{p}\right)_{\max} \times \left(\frac{l}{2\pi}\right)^2 \int_0^{2\pi} |\varphi|^2 \frac{\partial H/\partial r}{P_0} d\theta, \quad (6)$$

and the solution for $\Delta H = 0$ has the form

$$x_0 \approx Ae^{-i\nu\theta}\left(1 + 2\sum_k \frac{1}{k!}\left(\frac{i\varepsilon_1}{2\Omega}\right)^k \cos k\Omega\theta\right). \quad (7)$$

This solution can be obtained by expanding all the terms of the right hand side of (4) in Fourier series. In the expansion (7), we are interested in $k \sim$ 3-5. For such k's the following strong equality is true.

$$1 > \nu \gg k\Omega. \quad (8)$$

Hence the only term which contributes appreciably to the right hand side of (4) is

$$-i\left(\frac{l}{2\pi}\right)^2 \frac{1}{\omega} \left\langle |\varphi|^2 \frac{\partial H/\partial r}{P_0} \right\rangle \frac{\Delta p}{p} x = -i\varepsilon_1 (\sin\Omega\theta)\, x. \quad (9)$$

The remaining linear terms are small corrections for x_0 and have, furthermore, such frequencies which are of no interest.

For $\epsilon_1/2\Omega \gtrsim 1$ the expansion (7) is not adequate. Let us note, however, that Ω being small, we can speak of an "instantaneous" frequency

$$\nu_\theta = \nu + \varepsilon_1 \sin\Omega\theta. \quad (10)$$

Formula (10) is obtained in correspondence with (4) and (9). Instead of (7) we get

$$x_0 \approx A \exp\left\{-i\int_0^\theta \nu_\theta d\theta\right\} \tag{11}$$

$$= A \exp\left\{-i\nu\theta + i\frac{\varepsilon_1}{\Omega}(\cos\Omega\theta - 1)\right\}.$$

It is easy to see that (7) is a Fourier expansion of the function

$$\exp\left[-i\nu\theta + i(\varepsilon_1/\Omega)(\cos\Omega\theta - 1)\right]$$

for $\epsilon_1/2\Omega \ll 1$. Hence (11) is the general solution of (4) for $\Delta H = 0$.

Let us note that, according to (7) and (11), we can speak of an approximate Floquet function for Eq. (4), namely:

$$\Phi(\theta) = F(\theta)\exp(i\nu\theta), \quad F(\theta) = F(\theta + \tau) \tag{12}$$

$$= f(\theta)\exp\{i(\varepsilon_1/\Omega)(\cos\Omega\theta - 1)\},$$

$$\Phi(\theta) = \varphi(\theta)\exp\{(i\varepsilon_1/\Omega)(\cos\Omega\theta - 1)\}, \tag{13}$$

where τ is the period of the synchrotron oscillations.

We seek the solution of the complete equation (4) in the form

$$x = a\Phi^* + a^*\Phi. \tag{14}$$

For a we have

$$a = \text{const} + \frac{i}{w\rho_0}\left(\frac{l}{2\pi}\right)^2\int_0^\theta f(\theta)\frac{\Delta H}{H_0} \tag{15}$$

$$\times\exp\left[i\nu\theta + \frac{i\varepsilon_1}{\Omega}(\cos\Omega\theta - 1)\right]d\theta.$$

The frequency ν is always chosen within the limits

$$k/M - 1/2M < \nu < k/M, \tag{16}$$

where k is an integer and M is the number of periodic sectors. For $\nu = k/M$ we have two resonances: the so-called outside resonance (usual resonance on magnetic field perturbations) and the parametric one; for $\nu = k/M - 1/2M$ we have only the parametric resonance (resonance on the gradient perturbation).

Let us consider only a single resonant harmonic in the integral of Eq. (15):

$$he^{-i(k\theta/M+\gamma)} = \left[\frac{i}{w\rho_0}\left(\frac{l}{2\pi}\right)^2\frac{1}{2\pi M}\right. \tag{17}$$

$$\left.\times\int_0^{2\pi M} f(\theta)\frac{\Delta H}{H}\exp(ik\theta/M)\,d\theta\right]e^{-ik\theta/M}.$$

then

$$a = \text{const} + he^{-i\gamma_1}\int_0^\theta \exp\left[i\varepsilon_0\theta + \frac{i\varepsilon_1}{\Omega}\cos\Omega\theta\right]d\theta, \tag{18}$$

$$\gamma_1 = \gamma + \varepsilon_1/\Omega;$$

$$\varepsilon_0 = \nu - k/M. \tag{19}$$

where ϵ_0 is the distance to the outside resonance for $\Delta p/p = 0$. The integral in (18) describes, for $E_1 = 0$ the amplitude beat of an equilibrium orbit with a frequency ϵ_0*. For $\epsilon_1 \neq 0$, $\Omega \to 0$, $\epsilon_0/\Omega \neq n$. (n is an integer), Eq. (18) describes the change of the equilibrium orbit for an adiabatic change of ϵ

$$\varepsilon = \varepsilon_0 + \varepsilon_1\sin\Omega\theta, \tag{20}$$

$$a = \text{const} + (h/\varepsilon)\exp(i\varepsilon\theta + i\gamma_2).$$

Resonances occur when the equalities $\epsilon = n\Omega$ are satisfied[1]. According to (18),

$$a \approx \text{const} + \frac{h}{\varepsilon}\exp(i\varepsilon\theta + i\gamma_2) + \theta hJ_n\left(\frac{\varepsilon_1}{\Omega}\right)e^{-i\gamma_1}. \tag{21}$$

2. TRANSITION THROUGH RESONANCES IN THE LINEAR APPROXIMATION

As it is known, Ω changes in time. In a slow transition through a resonance, the main contribution to the integral

$$\int_0^{t=\Omega\theta}\frac{1}{\Omega}\exp\left(i\frac{\varepsilon_0}{\Omega}t + i\frac{\varepsilon_1}{\Omega}\cos t\right)dt \tag{22}$$

comes from the variation of the coefficient ϵ_0/Ω in the exponential. The resonance being very sharp, the variation of Ω does not practically affect the factor $(1/\Omega)\exp\{i\epsilon_1/\Omega)\cos t\}$; the variation of ϵ_0/Ω is, however, quite appreciable because the resonance occurs only if ϵ_0/Ω is an integer.

Writing ϵ_0/Ω in the form

$$\varepsilon_0/\Omega = n - \frac{n}{\Omega}\frac{d\Omega}{dt}(t - t_0), \quad t_0 \sim t \tag{23}$$

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* The sign of $\epsilon_0 = \nu - k/M$ is irrelevant for the effect under consideration. We put $\epsilon_0 > 0$.

we must make the usual substitution

$$\frac{\varepsilon_0}{\Omega} t \to \int_0^t \frac{\varepsilon_0}{\Omega} dt = \left(n + \frac{n}{\Omega}\frac{d\Omega}{dt} t_0\right) t - \frac{n}{2\Omega}\frac{d\Omega}{dt} t^2 \quad (24)$$

in the exponent of the integral (22). During a single period of synchrotron oscillations, Ω changes only slightly; therefore,

$$\int_0^t \frac{1}{\Omega} \exp\left(\frac{i\varepsilon_0}{\Omega} t + \frac{i\varepsilon_1}{\Omega} \cos t\right) dt \quad (25)$$

$$\approx J_n\left(\frac{\varepsilon_1}{\Omega}\right) \int_0^\theta \exp\left(i\beta\theta - \frac{n}{2}\frac{d\Omega}{d\theta}\theta^2\right) d\theta, \quad \beta = n\frac{d\Omega}{d\theta}\theta_0.$$

the quantity $J_n(\epsilon_1/\Omega)$ falls off rapidly as $n = \epsilon_0/\Omega$ increases. Hence only the transitions through the first resonances $n = 2, 3, 4, 5$ are appreciable. In the initial period, the acceleration changes according to the law $\Omega \approx \Omega_0 p_0/p$; hence

$$\left|\frac{n d\Omega}{d\theta}\right| \approx \left|\frac{\varepsilon_0}{2\pi M}\frac{dT/dN}{2T}\right|, \quad T = \frac{p^2}{2m}, \quad (26)$$

where dT/dN is the kinetic increase per revolution. Substituting this into (25) we get

$$a \approx \text{const} + (h/\varepsilon) \exp(i\varepsilon_0\theta + i\gamma) + 2\pi h J_n(\varepsilon_1/\Omega) e^{i\delta} (C(u) - iS(u)) \times \sqrt{MT/(\varepsilon_0 dT/dN)}; \quad (27)$$

$$u = u_0 + \theta\sqrt{\frac{n}{\pi}\frac{d\Omega}{d\theta}} = u_0 + \theta\sqrt{\frac{\varepsilon_0\, dT/dN}{4\pi^2 MT}}, \quad (28)$$

where $C(u)$ and $S(u)$ are the Fresnel integrals.

According to (14) and (17), the maximum increase of r after the transition through a resonance is equal to

$$(\Delta r)_{\max} = 4\pi |\varphi|_{\max} h J_n\left(\frac{\varepsilon_1}{\Omega}\right) \times (C^2 + S^2)^{1/2}_{\max} \left(\frac{MT}{\varepsilon_0\, dT/dN}\right)^{1/2} \approx 4\pi |\varphi|_{\max} h J_n\left(\frac{\varepsilon_1}{\varepsilon_0} n\right)\left(\frac{MT}{\varepsilon_0\, dT/dN}\right)^{1/2}. \quad (29)$$

Assuming a static independence of the perturbations in the various magnets, we have for $\sqrt{\langle h^2\rangle}$, according to (17)

$$\sqrt{\langle h^2\rangle} = \frac{1}{w\rho_0}\left(\frac{l}{2\pi}\right)^2 \frac{1}{2\pi M^{1/2}} \times \sqrt{\left\langle\left(\frac{\Delta H}{H}\right)^2\right\rangle} \left(\left|\int_0^\pi \varphi dt\right|^2 + \left|\int_\pi^{2\pi} \varphi dt\right|^2\right)^{1/2}. \quad (30)$$

The following approximate equality is usually satisfied:

$$\partial H(\theta)/\partial r \approx -\partial H(\theta + \pi)/\partial r. \quad (31)$$

Hence

$$\varepsilon_1 \approx \frac{1}{2\pi}\left(\frac{\Delta p}{p}\right)_{\max}\left(\frac{l}{2\pi}\right)^2 \frac{1}{b\rho_0} \left|\int_0^\pi |\varphi|^2 dt - \int_\pi^{2\pi} |\varphi|^2 dt\right|, \quad (32)$$

$$b = H/(\partial H(\partial r)). \quad (33)$$

The order of magnitude of ϵ_0 is $1/8\, M$ -$1/4\, M$. Equation (29) can be written in another form. Let us denote by r_0 the amplitude of the equilibrium orbit for $\Delta p/p = 0$:

$$r_0 = 2|\varphi|_{\max} h/\varepsilon_0. \quad (34)$$

Then

$$\frac{(\Delta r)_{\max}}{r_0} \approx 2\pi\varepsilon_0 J_n\left(n\frac{\varepsilon_1}{\varepsilon_0}\right)\left(\frac{MT}{\varepsilon_0\, dT/dN}\right)^{1/2}. \quad (35)$$

And we have an analogous formula for the vertical oscillations.

3. EFFECT OF NONLINEARITY IN THE TRANSITION THROUGH RESONANCES

The effect of nonlinearity can be taken into account by making the substitution

$$\varepsilon \to \varepsilon + \alpha a^2, \quad (36)$$

$$\alpha = \pm \frac{l^2}{8\pi^2 w}\frac{1}{H_0\rho_0}\frac{1}{2\pi}\int_0^{2\pi} |\varphi|^4 \frac{\partial^3 H}{\partial r^3} d\theta$$

[a from Eq. (18)]; if the plus sign is taken for r, z-oscillations, then the minus sign is taken for the z, r-oscillations. This is related to the fact that, usually,

$$\partial^3 H(\theta)/\partial r^3 \approx -\partial^3 H(\theta + \pi)/\partial r^3.$$

During the transition through resonance, a^2 contains a constant term, an oscillating term and a slowly increasing term. The constant term changes only the magnitude ϵ_0, and the oscillating term has no effect. As far as the slowly increasing term is concerned, when the inequality $\alpha(\nu - k/M) < 0$ is satisfied, the increase of a^2 lead to the situation where the ratio $(\nu - k/M + \alpha a^2)/\Omega$ remains constant as Ω decreases. This will lead to particle loss.

For this not to happen, it is obviously sufficient that the following inequality be satisfied.

$$n d\Omega/d\theta \gg \alpha (da^2/d\theta)_{\max},$$

where $(da^2/d\theta)_{\max}$ has to be taken in a region of monotonic increase of a, according to (27). In this region,

$$\frac{dC}{d\theta} \approx \frac{dS}{d\theta} = \frac{dS}{du}\frac{du}{d\theta} \approx 0.7 \frac{1}{2\pi}\left(\frac{\varepsilon_0\, dT/dN}{MT}\right)^{1/2}.$$

Therefore

$$\alpha\left(\frac{da^2}{d\theta}\right)_{\max} \approx 3.6\pi\alpha h^2 J_n^2\left(n\frac{\varepsilon_1}{\varepsilon_2}\right)\left(\frac{MT}{\varepsilon_0\, dT/dN}\right)^{1/2}.$$

And we finally get the safety factor condition

$$150\alpha h^2 J_n^2\left(n\frac{\varepsilon_1}{\varepsilon_0}\right)\left(\frac{MT}{\varepsilon_0\, dT/dN}\right)^{3/2} \ll 1 \qquad (37)$$

or

$$\frac{\alpha(\Delta r)^2_{\max}}{|\varphi|^2_{\max}}\left(\frac{MT}{\varepsilon_0\, dT/dN}\right)^{1/2} \ll 1. \qquad (38)$$

This condition is not difficult to satisfy. It is automatically satisfied for the usual specifications on $\partial^3 H/\partial r^3$ and $(\Delta r)_{\max}$

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1 Hammer, Pidd and Terwilliger, Rev. Sci. Instr. 26, 555 (1955).

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On the Construction of the Scattering Matrix. II. The Theory with Non–Local Interaction

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N. N. Bogoliubov's method for constructing the scattering matrix is generalized to the case of a theory with non–local interaction. For such a theory, a scattering matrix is constructed which satisfies the physically necessary requirements.

1. INTRODUCTION

ATTEMPTS, having their origin in the "struggle with divergences", to avoid the use of point interactions in the quantum theory of fields and to replace it by an extended interaction, are as old as quantum electrodynamics itself.[1–3] However, elaborate investigations of such theories,[4–5] undertaken within the framework of the description of a many–electron system by means of the many–time formalism or the Tomonoga–Schwinger equation in the interaction representation, have shown that the introduction of a form factor violates the conditions for solvability of the equations of motion, since the Hamiltonians at points with space–like separation no longer commute. Consequently, the non–local theory is incompatible with the Hamiltonian method. The physical reason for this is that the introduction of a form factor actually results in propagation of signals (at least, in the small) with super–light velocity. Thus the requirement that there exist a wave function describing the state of the system at a definite time loses its meaning.

In the hope of avoiding the difficulties of the Hamiltonian method, attempts have been made to go directly to the Euler–Lagrange integro–differential equations which follow from the variational principle with non–local interaction.[6] In quantum theory this procedure leads to the considera-