$$k\left\{\omega\left[\omega_0^2-\omega^2+i\omega\beta+\frac{5}{3}\frac{\varkappa T_0}{m}k^2+\chi k^2(i\omega+\beta)\right]+k^2\omega_0^2\frac{\varkappa T_0}{m}\frac{3i(\alpha+\beta)-4\omega}{(\alpha+\beta+i\omega)^2}-i\omega_0^2\chi k^2\right\}=0, \quad (4)$$

where $\omega_0=\sqrt{4\pi e^2 n_0/m}$.

This equation has the following solutions:

1. $k=0$,

With

$$n=T=0,\ (i\omega+\beta)u=-e\alpha E/m(\alpha+\beta+i\omega).$$

For $i\omega\neq-\beta$ this gives the current under the influence of the external field

$$j=-en_0u-en_0I_2=n_0e^2E/m(i\omega+\beta).$$

2. $$\omega^2\approx\omega_0^2+i\omega_0\beta+\frac{3\varkappa T_0}{m}k^2$$

(taking into account that $\alpha,\beta\ll\omega_0$); this is the usual expression for the frequency of plasma oscillations with "friction" taken into account.

3. $i\omega\approx-[\chi-\varkappa T_0/m(\alpha+\beta)]k^2,\ u=eE/m(\alpha+\beta)$,

$$T=-T_0n/n_0+i(e/\varkappa k)E,\ n=-(ik/4\pi e)[1+\omega_0^2(\alpha+\beta)^{-2}]E.$$

This solution can be referred to as the electro-entropy wave. For $e\to0$ it reduces to the usual entropy wave.[3].

1 E. P. Gross, Phys. Rev. 82, 232 (1954).
2 A. A. Vlasov, *Theory of many particles*, GITTL, 1950.
3 L. D. Landau and E. M. Lifshitz, *Mechanics of continuous media*, GITTL, Moscow, 1954, p. 410.

Translated by G. M. Volkoff
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The Variational Principle and the Virial Theorem for the Continuous Dirac Spectrum

IU. V. NOVOZHILOV

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THE variational principle for the phase shifts and scattering amplitudes has been investigated for the nonrelativistic case in several works.[1,2] Parzen[3] has formulated the variational principle for the phase shifts in the relativistic case. The present note gives a generalization of the variational principle for all scattering amplitudes to the case of the Dirac equation, and a derivation of the virial theorem for the continuous Dirac spectrum. The results can be applied to the theory of high-energy electron scattering by nuclei.

Let us consider the functional

$$I\{\Psi_1,\Psi_2\}=\int\Psi_2^+(\mathbf{r})[\alpha\mathbf{p}+\beta m+V(\mathbf{r})-E]\Psi_1(\mathbf{r})d\mathbf{r}, \quad (1)$$

where the functions Ψ_i are not in general solutions ψ_i of the Dirac equation

$$[\alpha\mathbf{p}+\beta m+V(\mathbf{r})-E]\psi_i=0.$$

For the exact solutions, $I\{\psi_1,\psi_2\}=0$. Let us restrict ourselves initially to potentials V which decrease faster than $1/r$ as $r\to\infty$. Then, in order that the functional (1) converge at the upper limit, the asymptotic form of the trial functions ψ_1 and ψ_2 should be

$$\Psi_i=u_i\exp[ip\boldsymbol{\nu}_i\mathbf{r}]+G(\boldsymbol{\nu}_i,\mathbf{n})(pr)^{-1}\exp[\pm ipr], \quad (2)$$

where u_i is a unit spinor, $G(\nu,\mathbf{n})$ is a single-valued function of direction, and $\mathbf{n}=\mathbf{r}/r$; the upper sign in the second term of Eq. (2) refers to ψ_1 and the lower one to ψ_2. The function ψ_1 contains a plane wave propagating in the direction ν_1 and an outgoing wave; the function ψ_2, on the other hand, is seen from Eq. (2) to contain, in addition to a plane wave with propagation direction ν_2, an incoming wave. The asymptotic forms of the exact solutions ψ_1 and ψ_2 are similar to Equation (2), but instead of the function $G(\nu,\mathbf{n})$, they contain scattering amplitudes $G^\circ(\nu,\mathbf{n})$. Thus in the asymptotic form, variations $\delta\psi_1$ are due to variations $\delta G(\nu_1,\mathbf{n})$.

The first variation of the functional (1), caused by variations $\delta\psi_1=\psi_i-\psi_i$ of the functions ψ_i about the exact solutions ψ_i, is given by

$$\delta I=\int\psi_2^+\{\alpha\mathbf{p}+\beta m+V-E\}\delta\Psi_1 d\mathbf{r}=-i\int\psi_2^+\alpha\mathbf{n}\delta\Psi_1 dS, \quad (3)$$

where the integration is taken over the surface of an infinite sphere, $r \to \infty$, $\mathbf{n} = \mathbf{r}/r$. Here

$$\delta\Psi_1 = \delta G(\boldsymbol{\nu}_1, \mathbf{n})(pr)^{-1}\exp(ipr).$$

Let us write the spinors u and G in the form

$$u = N\binom{v}{w},\ G = N\binom{F}{F'},\ N^2 = \frac{1}{2}\left[1 + \frac{m}{E}\right] \quad (4)$$

and eliminate the two-component spinors w and F' with the aid of the formulas

$$w_2 = p\boldsymbol{\sigma}\boldsymbol{\nu}_2 v_2/(E+m),\ F' = p\boldsymbol{\sigma}\mathbf{n}F/(E+m).$$

Then, writing $d\omega$ for the element of solid angle, we have from (3)

$$\delta I = -i\frac{r}{p}\int e^{ipr(1-\boldsymbol{\nu}_2\mathbf{n})} u_2^+ \boldsymbol{\alpha}\mathbf{n}\,\delta G(\boldsymbol{\nu}_1, \mathbf{n})\,d\omega \quad (5)$$

$$= \frac{2\pi}{pE}\,\delta\left(v_2^+ F(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2)\right).$$

This Eq. (5) is the variational principle for the scattering amplitude $F^\circ(\nu_1, \nu_2)$. From this it follows that the correct scattering amplitude is the stationary value of the quantity

$$\{(v_2^+ F(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2)) - pEI/2\pi\}.$$

When V is the potential of a central field, the variational principle for the phase shifts is easy to obtain from Eq. (5). In this case we choose trial functions which differ from the exact solutions in their radial parts. Then the functional (1) will converge if the asymptotic forms of the radial parts, which are the same for both trial functions ψ_1 and ψ_2 differ from the asymptotic form of the exact solutions only in the phase $\tilde{\eta}$ and the amplitude $\tilde{A}$. For $r \to \infty$ we thus obtain the radial parts of the trial functions in the form

$$rg_l(r) = \tilde{A}\sqrt{\frac{E}{m}+1}\cos(pr + \tilde{\eta}_{jl});$$

$$rf_l(r) = -\tilde{A}\sqrt{\frac{E}{m}-1}\sin(pr + \tilde{\eta}_{jl}).$$

Then the variational principle for the phase shifts is of the form

$$\delta I = A^2(p/m)\,\delta\eta_{jl}. \quad (6)$$

The virial theorem for the continuous Dirac spectrum can be derived by variation of the length scale, as it has been previously derived for other cases by Fock[4] and later by Demkov.[2] Let us set

$$\Psi_i(\mathbf{r}) = \psi_i(\mathbf{r}+\varepsilon\mathbf{r}),$$

where ψ_i is an exact solution. Since the variation of the length scale changes the asymptotic behavior of ψ_i, we must investigate not Equation (1), to assure the convergence of I, but the functional

$$I'\{\psi(\mathbf{r}+\varepsilon\mathbf{r})\} = \int \psi_2^+(\mathbf{r}+\varepsilon\mathbf{r})\,[\boldsymbol{\alpha}\mathbf{p}$$

$$+ (1+\varepsilon)(\beta m - E) + V]\,\psi_1(\mathbf{r}+\varepsilon\mathbf{r})\,d\mathbf{r}.$$

For $\epsilon \ll 1$ this functional is given up to terms in ϵ^2 by

$$I'\{\psi(\mathbf{r}+\varepsilon\mathbf{r})\} = -\varepsilon\int \psi_2^+(\mathbf{r})\,[V(\mathbf{r}) + \mathbf{r}\nabla V]\,\psi_1(\mathbf{r})\,d\mathbf{r}.$$

On the other hand $I'\{\psi(\mathbf{r}+\epsilon\mathbf{r})\}$ is a functional of the trial function $\psi(\mathbf{r}+\epsilon\mathbf{r}) = \delta\psi + \psi'(\mathbf{r})$ which is hardly different from the function $\psi'(\mathbf{r})$, for which $I\{\psi'(\mathbf{r})\} = 0$. Therefore in order to calculate $I'\{\psi(\mathbf{r}+\epsilon\mathbf{r})\}$ we may also make use of the variational principle (5). The asymptotic form of $\psi'(\mathbf{r})$ differs from that of $\psi(\mathbf{r}+\epsilon\mathbf{r})$ only in the scattering amplitude: the scattering amplitude in $\psi'(\mathbf{r})$ depends on the mass $m(l+\epsilon)$ and the energy $E(l+\epsilon)$, whereas that of $\psi(\mathbf{r}+\epsilon\mathbf{r})$ depends on m and E. Therefore in variation (5) we have the scattering

$$\delta F = -\varepsilon\,[m\partial/\partial m + E\partial/\partial E]\,F.$$

This means that there exists a relation

$$\left[m\frac{\partial}{\partial m} + E\frac{\partial}{\partial E}\right](v_2^+ F(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2)) \quad (7)$$

$$= \frac{pE}{2\pi}\int \psi_2^+\,[V + \mathbf{r}\nabla V]\,\varphi_1\,d\mathbf{r},$$

which is the virial theorem for the continuous Dirac spectrum. The virial theorem is convenient to use at high energies $E \gg m$, when the derivative with respect to mass can be neglected.

For the central field case, the virial theorem is of the form

$$A^2\frac{p}{m}\left[E\frac{\partial\eta}{\partial E} + m\frac{\partial\eta}{\partial m}\right] \quad (8)$$

$$= \int_0^\infty \left[V(r) + r\frac{dV}{dr}\right][g_l^2 + f_l^2]\,r^2 dr.$$

From the variation (5) it is easy to obtain an expression for the change in the scattering amplitude caused by a variation of the potential $V(\mathbf{r}, \lambda) \to V(\mathbf{r}, \lambda + \epsilon\lambda)$, where λ is some parameter. From Equation (5) we have

$$\partial (v_2^+ F(\mathbf{v}_1, \mathbf{v}_2))/\partial\lambda = -\frac{pE}{2\pi}\int \psi_2^+ \frac{\partial V}{\partial\lambda}\psi_1 dr. \qquad (9)$$

Variational principles (5) and (6) and the virial theorem (7), (8) are also valid for a Coulomb field for $r \to \infty$, if we take account of the fact that in this case the asymptotic form of the wave functions should include terms in ln (pr).

1 J. Schwinger, Phys. Rev. 72, 742 (1947). L. Hulthen, Arkiv Mat. Astron. Fys. 35A, 251 (1948). W. Kohn, Phys. Rev. 74, 1763 (1948).

2 Iu. N. Demkov, Dokl. Akad. Nauk SSSR 89, 249 (1953); 97, 1003 (1954).

3 G. Parzen, Phys. Rev. 80, 2 (1950).

4 V. A. Fock, Z. Physik 63, 855 (1940).

Translated by E. J. Saletan
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The Internal Compton Effect

E. G. MELIKIAN
Moscow State University

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SPRUCH and Goertzel[1] have calculated the relative probability of the internal Compton effect for magnetic 22-pole transitions and Iakobson[2] has obtained an approximate relativistic formula for electric 2^j-pole transitions, but only for small gamma-ray energies. In the present work a general formula is derived for the relative probability of this effect for both magnetic and electric transitions in the Born approximation and numerical calculations are carried through for some specific cases.

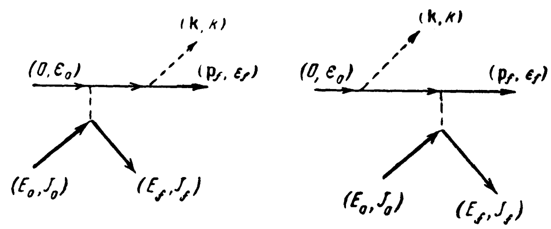

Figure 1.

The process under consideration is characterized by the Feynman diagrams in Fig. 1, where the heavy lines correspond to a nucleus and the thin dashed lines correspond to an electron and photon. In the initial state we have an excited nucleus with charge Z, energy E and angular momentum J_0, and an electron in the K shell with energy ϵ_0. In the final state, the energy and angular momentum of the nucleus are designated by E_f and J_f, and the energy and momentum of the electron and photon by ϵ_f, $\mathbf{p}_f$ and k, $\mathbf{k}$, respectively. Hereinafter, we shall use the system of units in which $h = c = 1$.

Using the general methods of quantum electrodynamics in our calculations (see Ref. 3) we obtain the relative probability of the internal Compton effect (the ratio of the absolute probability of the effect to the probability of a radiation transition of the nucleus) which for the magnetic $2j$-pole transition is expressed by

$$\beta_j^{(0)} = \frac{2\pi\alpha^2 (Zm\alpha)^3 (2j+1)\, p_f}{pk} \,|\, L \,|^2 \Big\{\Big[1 \qquad (1)$$

$$+ \frac{k}{m} + \frac{\Delta E}{mp'}\Big]\frac{p_f k}{p^2}(x^2 - 1)$$

$$- \frac{2\varepsilon_f(\Delta E m + kp')}{mp'} - \Big[\frac{1}{m^2} + \frac{1}{p'^2}\Big][\Delta E m^2$$

$$+ k(m+k)\,p'] \sin\vartheta\, d\vartheta\, dk,$$

$$L = (2/\pi)(p/\Delta E)^{j+1/2}(p^2 - \Delta E^2 - 2iZm\alpha\Delta E)^{-1},$$

$$p' = p_f x - \varepsilon_f, \; x = \cos\vartheta, \; \Delta E = E_0 - E_f, \; \varepsilon_0 \sim m,$$

$$p = |\,\mathbf{p}_f + \mathbf{k}\,|, \; j = |\,I_0 - I_f\,|,$$

where ϑ is the angle between the vectors $\mathbf{p}_f$ and $\mathbf{k}$.

This formula is in agreement with the results of Spruch and Goertzel[1] if we neglect the width of

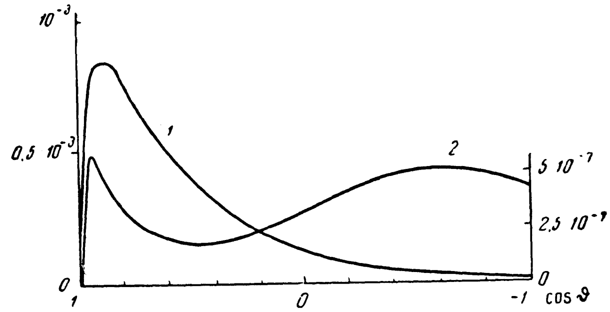

FIG. 2