The Role of Spin in the Radiation from a "Radiating" Electron

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It is shown that in the case of energies which are not too great the magnitudes of the spin corrections are of the second order of smallness with respect to the magnitudes of the quantum corrections, while in the case of extreme relativistic energies the spin essentially changes the character of the differential spectrum and the magnitude of the total energy of radiation.

1. INTRODUCTION

THE opinion exists that the spin does not play an essential role in the radiation of a radiating electron. This opinion is based on the consideration that the spin contribution to the wave function of an electron in a magnetic field decreases with increase in the energy of the electron. However, one cannot conclude from this that the spin is nonessential, since in the calculation of the matrix elements of radiative transitions not only the wave functions, which are distinguished from each other only by small spin contributions , are different, but the various operators whose matrix elements are being calculated are also different. Hence the question of the role of spin cannot be decided on the basis of general considerations[1-3]. An actual calculation is necessary.

A remark is necessary concerning the statement of the problem. Simply raising the question of isolating the "role of spin" within the framework of relativistic theory appears artificial in itself. A statement of the question which has an exact sense is the following: to compare under the same external conditions the radiation from an electron and from a boson with zero spin and with mass and charge equal to the mass and charge of an electron.

The present work is devoted to the clarification of the question of the role of spin in the radiation of an electron moving in a constant magnetic field. The calculation is carried out without taking account of damping.

2. QUANTUM MECHANICAL FORMULAS FOR THE INTENSITY OF RADIATION, WITH AND WITHOUT TAKING ACCOUNT OF SPIN

In order to obtain by a single method formulas characterizing the radiation with and without taking account of spin, it is convenient to carry out the calculation in the scheme of Lorentz forces rather than in the Hamiltonian scheme.

From the covariant definition of the four-dimensional energy-momentum vector $p_\mu(\mathbf{p}, iW/c)$

$$p_\mu[\sigma] = \frac{1}{c}\int_\sigma T_{\mu\nu}\, d\sigma_\nu \tag{2.1}$$

and the relationship

$$\delta p_\mu[\sigma] / \delta\sigma(x) = c^{-1}\partial T_{\mu\nu}/\partial x_\nu, \tag{2.2}$$

which follows from it, one obtains for Δp_μ, the change in the energy-momentum during a transition from the hypersurface σ_1 to the hypersurface σ_2, the following expression:

$$\Delta p_\mu = c^{-2}\int_{\sigma_1}^{\sigma_2} dx\, F_{\mu\nu} j_\nu, \tag{2.3}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor, j_ν is the four-dimensional current, and the fact that

$$\partial T_{\mu\nu}/\partial x_\nu = c^{-1} F_{\mu\nu} j_\nu \tag{2.3a}$$

has been taken into account.

The operators in (2.3) are taken in the Heisenberg representation. Removing the hypersurfaces $\sigma_1(t_1)$ and $\sigma_2(t_2)$ in (2.3) to infinity ($t_2 \to \infty$, $t_1 \to -\infty$, $t_2 - t_1 = T \to \infty$), averaging the change in momentum over the time interval, and changing from bound operators to free field operators, we obtain from (2.3)

$$\overline{\Delta p_\mu} = \frac{1}{c^3 T}\int_{-\infty}^{\infty} dx \sum_{n=0}^{\infty}\left(\frac{-i}{\hbar c}\right)^n \int_{-\infty}^{t} dx_1 \int_{-\infty}^{t_1} dx_2 \ldots \int_{-\infty}^{t_{n-1}} dx_n, \tag{2.4}$$
$$[j_{\lambda_n}^{(0)}(x_n) A_{\lambda_n}^{(0)}(x_n), \ldots [j_{\lambda_2}^{(0)}(x_2) A_{\lambda_2}^{(0)}(x_2),$$
$$[j_{\lambda_1}^{(0)}(x_1) A_{\lambda_1}^{(0)}(x_1), \quad F_{\mu\nu}^{(0)}(x)\, j_\nu^{(0)}(x)],]\ldots].$$

In this formula it is to be kept in mind that

$$\frac{1}{T}\int_{-\infty}^{\infty} dt \ldots = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} dt \ldots, \tag{2.4a}$$

and that the square brackets denote commutators.

Averaging (2.4) over the state of a single-electron in the absence of photons, and limiting ourselves to the first nondisappearing term, we obtain

$$\langle \overline{\Delta p_\mu}\rangle_{1,0} = -\frac{i}{hc^4 T}\int_{-\infty}^{\infty} dx \tag{2.5}$$

$$\times \int_{-\infty}^{t} dx' \langle 1,0 \mid [j_\lambda(x') A_\lambda(x'),\ F_{\mu\nu}(x) j_\nu(x)] \mid 1,0\rangle.$$

For simplicity the indices (0) used on the free operators have been omitted.

An average over the photonic vacuum gives (in Gaussian units)

$$\langle [j_\lambda(x') A_\lambda(x'),\ F_{\mu\nu}(x) j_\nu(x)]\rangle_0 \tag{2.6}$$

$$= 2\pi ch\left\{j_\lambda(x') j_\lambda(x)\frac{\partial}{\partial x_\mu} D^{(+)}(x'-x)\right.$$

$$\left. - j_\lambda(x) j_\lambda(x')\frac{\partial}{\partial x_\mu} D^{(+)}(x-x')\right\},$$

where $D^{(+)}$ signifies the positive-frequency D function.

The integral in (2.5) in the surface of changing (t, t') is taken over the half-surface with boundary $t = t'$. For definiteness let us take the first component of (2.6). If we make the exchange $x \leftrightarrow x'$ in the integral, then the region of integration with respect to (t, t') will be the other half-surface with the same boundary $t = t'$. This exchange also makes the first expression under the integral sign coincide with the second expression, with the exception that a derivative with respect to x'_μ occurs instead of a derivative with respect to x_μ. However,

$$(\partial/\partial x_\mu) D^{(+)}(x-x') \tag{2.7}$$

$$= -(\partial/\partial x'_\mu) D^{(+)}(x-x').$$

Hence, the integrands coincide completely and a change in sign occurs. Consequently,

$$\langle \overline{\Delta p_\mu}\rangle_{1,0} = \frac{2\pi i}{c^3 T}\int_{-\infty}^{\infty} dx \tag{2.8}$$

$$\times \int_{-\infty}^{\infty} dx' \langle 1 \mid j_\lambda(x) j_\lambda(x') \mid 1\rangle \frac{\partial}{\partial x_\mu} D^{(+)}(x-x').$$

The radiation W is equal (with reversed sign) to the magnitude of the fourth component of the change in momentum $\overline{\Delta p_\mu}$ multiplied by c/i. Hence,

$$W = \frac{1}{c^2 T}\frac{1}{4\pi^2}\int d^3\varkappa \langle 1 \mid ([\varkappa_0, \mathbf{j}(-\varkappa)], \tag{2.9}$$

$$[\varkappa_0, \mathbf{j}(\varkappa)]) \mid 1\rangle,$$

where the brackets [ ] indicate vector derivation and $\varkappa_0 = \varkappa/|\varkappa|$. The changing of the four-dimensional current

$$j(\varkappa) = \int_{-\infty}^{\infty} dx e^{-i\varkappa x} j(x) \tag{2.10}$$

into its Fourier components is carried out in Eq. (2.9), and the fourth component of the current is excluded with the aid of the law of conservation of charge

$$\partial_\nu j_\nu(x) = 0. \tag{2.11}$$

The current operators have the following form:

a) For the Dirac equation

$$\mathbf{j}(x) = ec\Psi^+\alpha\Psi = iec\overline{\Psi}\gamma\Psi, \tag{2.12a}$$

b) For the scalar equation

$$\mathbf{j}(x) = \frac{e}{2m}\left\{\Phi^*(\mathbf{p}\Phi)\right. \tag{2.12b}$$

$$\left. -(\mathbf{p}\Phi^*)\Phi - 2\frac{e}{c}\Phi^*\mathbf{A}\Phi\right\},$$

where $\mathbf{A}$ is the vector potential of the external field (c is a number).

The wave functions of an electron in a constant magnetic field may be found by iteration. They may be given in the following form (the components of the vector potential being $A_x = -\frac{1}{2}Hy$, $A_y = \frac{1}{2}Hx$, $A_z = 0$):

a) For the Dirac equation

$$\Psi = \Psi_{n,l,k_z,\varepsilon,s} \tag{2.13}$$

$$= L_z^{-1/2} e^{-ic\varepsilon K_n t + k_z z}\,\Psi_{n,l,k_z,\varepsilon,s}(x,y);$$

$$\Psi_{n,l,k_z,\varepsilon,s}(x,y) \tag{2.13a}$$

$$= \sqrt{\frac{\gamma}{\pi}}\left|\begin{array}{l} sf(\varepsilon,s)\, e^{i(n-1-l)\varphi} I_{n-1,l}(\gamma\rho^2), \\ if(\varepsilon,-s)\, e^{i(n-l)\varphi} I_{n,l}(\gamma\rho^2), \\ \varepsilon f(-\varepsilon,s)\, e^{i(n-1-l)\varphi} I_{n-1,l}(\gamma\rho^2), \\ is\varepsilon f(-\varepsilon,-s)\, e^{i(n-l)\varphi} I_{n,l}(\gamma\rho^2), \end{array}\right.$$

where (ρ, φ, z) are the cylindrical coordinates of the point (x, y, z);

$$I_{n, n'}(\xi) = \frac{1}{\sqrt{n!n'!}} e^{-\xi/2}\xi^{(n-n')/2} Q_{n'}^{(n-n')}(\xi),$$

$$Q_n^{(m)}(\xi) = \frac{\Gamma(n+m+1)}{\Gamma(m+1)} {}_1F_1(-n, m+1, \xi),$$

$$f(\varepsilon, s) = \frac{1}{2}\left(1+\varepsilon\frac{k_0}{K_n}\right)^{1/2}\left(1+s\frac{k_z}{\sqrt{K_n^2-k_0^2}}\right)^{1/2},$$

$$\gamma = \frac{eH}{2ch}, \quad K_n = \sqrt{k_0^2+4\gamma n+k_z^2}, \quad k_0 = \frac{mc}{h},$$

n is the principal quantum number, l is the radial quantum number, $\epsilon = 1$ denotes the electronic state, $\epsilon = -1$, the positron state, s is the spin variable, and L_z is the normalized length in the direction of the z axis;

b) for the scalar equation

$$\Phi_{n, l, k_z} = (k_0/K_n^{(1)}L_z)^{1/2} \times \exp\{-icK_n^{(1)}t + ik_z z\}\varphi_{n, l}(x, y); \quad (2.14)$$

$$\varphi_{n, l}(x, y) = \sqrt{\gamma/\pi}\, e^{i(n-l)\varphi} I_{n, l}(\gamma\rho^2), \quad (2.14a)$$

where $K_n^{(1)} = \sqrt{k_0^2 + 4\gamma(n+1/2) + k_z^2}$, and the significance of the other quantities is the same as in case a). The factor $(k_0/K_n^{(1)})^{1/2}$ occurs because of normalization to the current of a single charge. Introducing the notation

$$\tilde{\alpha} = \int \Psi^{+}_{n', l', k_z', 1, s'} \times \alpha e^{-i\varkappa_x x - i\varkappa_y y}\Psi_{n, l, 0, 1, s}\, dx\, dy, \quad (2.15a)$$

$$\tilde{\mathbf{P}} = \int \varphi^{*}_{n', l'} e^{-i\varkappa_x x - i\varkappa_y y}\mathbf{P}\varphi_{n, l}\, dx\, dy, \quad (2.15b)$$

where the α are the Dirac matrices and $\mathbf{P} = \mathbf{p} - (e/c)\mathbf{A}$, we obtain from Eq. (2.9) [on taking account of Eqs. (2.12a) and (2.12b)], the following formulas characterizing the radiation with and without taking account of spin

$$W = \frac{ce^2}{2\pi}\frac{1}{2} \times \sum_{n', l', s, s'} \sum \int d^3\varkappa\, ([\varkappa_0, \tilde{\alpha}^{+}][\varkappa_0, \tilde{\alpha}])\,\delta(K - K' - \varkappa), \quad (2.16a)$$

$$K = K_n, \quad K' = K_{n'},$$

$$W^{(1)} = \frac{ce^2}{2\pi}\sum_{n', l'}\int d^3\varkappa \frac{1}{h^2 K^{(1)'}K^{(1)}} \times ([\varkappa_0, \tilde{\mathbf{P}}^{*}][\varkappa_0, \tilde{\mathbf{P}}])\,\delta(K^{(1)} - K^{(1)'} - \varkappa), \quad (2.16b)$$

$$K^{(1)} = K_n^{(1)}, \quad K^{(1)'} = K_{n'}^{(1)},$$

where in formula (2.16a) the initial state has been averaged over the spin states and in formula (2.16b) the fact that $[\varkappa_0, \varkappa] = 0$ has been used.

On carrying out the calculation intended in formula (2.16) and using the well-known recurrence relations between the Laguerre polynomials, we can, after a series of transformations, obtain expressions for the intensity of radiation in the following form:

a) $W = \sum_{m=0}^{n} W_m;$

$$W_m = \frac{\gamma}{\pi}ce^2 A\int\frac{\Phi_m(\xi)}{A-\xi}\delta[\varphi_m(\xi)]\,d^3\xi, \quad (2.17a)$$

$$\xi = \varkappa/2\sqrt{\gamma}, \quad A = K/2\sqrt{\gamma},$$

$$\varphi_m(\xi) = \xi - A + \sqrt{A^2 - m + \xi^2\cos^2\vartheta};$$

$$\Phi_m(\xi) = \mathrm{ctg}^2\vartheta(I^2_{n, n-m} + I^2_{n-1, n-1-m}) + \frac{\xi^2\sin^2\vartheta}{A^2}(I'^2_{n, n-m} + I'^2_{n-1, n-1-m}) + \frac{\xi^2\sin^2\vartheta}{A^2}(I_{n, n-m}I'_{n, n-m} - I_{n-1, n-1-m}I'_{n-1, n-1-m}),$$

where the argument of the function I is equal to $\xi^2\sin^2\vartheta$, and the primes indicate derivatives with respect to that argument;

b) $W^{(1)} = \sum_{m=0}^{n} W_m^{(1)};$

$$W_m^{(1)} = \frac{\gamma}{\pi}ce^2 A\int\frac{\Phi_m^{(1)}(\xi)}{A^{(1)}-\xi}\delta[\varphi_m^{(1)}(\xi)]\,d^3\xi; \quad (2.17b)$$

$$\frac{1}{2}\Phi_m^{(1)}(\xi) = \mathrm{ctg}^2\vartheta I^2_{n, n-m} + \frac{\xi^2\sin^2\vartheta}{A^{(1)2}}I'^2_{n, n-m},$$

where

$$A^{(1)} = K^{(1)}/2\sqrt{\gamma}, \quad \varphi_m^{(1)}(\xi) = \xi - A^{(1)} + \sqrt{A^{(1)2} - m + \xi^2\cos^2\vartheta}.$$

A comparison of formulas (2.17a) and (2.17b) allows us to draw conclusions concerning the "role of spin" without any approximation whatsoever. The difference in the magnitudes of $A^{(1)}$ and A leads to a difference in magnitudes of order $1/n$ in these formulas and is completely non-essential, since to our specified accuracy $A^{(1)}$ can be replaced by A and $\varphi^{(1)}$ by φ in formula (2.17b). Moreover, on taking into account the equalities

$$I_{n-1,n'-1}(x) = \frac{n+n'-x}{2\sqrt{nn'}} I_{n,n'}(x) - \frac{x}{\sqrt{nn'}} I'_{n,n'}(x), \tag{2.18}$$

$$I'_{n-1,n'-1}(x) = \frac{n+n'-x}{2\sqrt{nn'}} I'_{n,n'}(x) + \frac{1}{2x}\left[2\sqrt{nn'} - \frac{(n+n'-x)^2}{2\sqrt{nn'}}\right] I_{n,n'}(x),$$

we may show that in the first and second parentheses in the expression for Φ_m in Eq. (2.17a) $I_{n,n'}$ and $I_{n-1,n'-1}$ can be equated with an accuracy of magnitude $\sim\sqrt{1-\beta^2}$, so that these terms coincide with the corresponding terms for $\Phi_m^{(1)}$ in Eq. (2.17b). Thus the "role of spin" is determined by the term in the third parenthesis in the expression for $\Phi_m(\xi)$ in Eq. (2.17a). As will be shown, this role is by no means small in the extreme relativistic case.

In the entire following exposition, in order to shorten the notation in the formulas, magnitudes characterizing the radiation from a "radiating" electron will be represented as the sum of two components. The first component, designated by the index (1), gives the "spinless" part of the radiation". More exactly, this first part characterizes the corresponding physical magnitude of the radiation from a "radiating" boson (with spin 0).

In order to simplify the formulas, it is convenient to go over to a continuous spectrum. Corresponding to what was said above, further calculations can be carried out with formula (2.17a), in which the "spinless part " of the radiation is contained.

Let us transform the integral for W_m in (2.17a) into a surface integral by means of the well-known formula

$$\int \ldots \delta(\varphi)\, d\tau = \int\limits_{\varphi=0} \ldots \frac{d\sigma}{|\nabla\varphi|}, \tag{2.19}$$

where $d\sigma$ is an element of the surface $\varphi = 0$. Then, instead of (2.17a), we obtain

$$W_m = \frac{\gamma}{\pi} ce^2 A \int\limits_{S_m} \frac{\Phi_m(\xi)}{(\partial\varphi_m/\partial \mathbf{n})(A-\xi)}\, d\sigma, \tag{2.20}$$

where S_m is a surface in (ξ) space determined by the equation $\varphi_m(\xi) = 0$, $d\sigma$ is an element of the surface, and $\mathbf{n}$ is the outward normal.

In the entire frequency region in which we are interested we can direct our considerations to the continuous sepctrum. Hence, for the production of the total energy of radiation we change from a sum over m to the integral

$$W = \int\limits_0^n W_m\, dm. \tag{2.21}$$

In this transition the family of surfaces S_m fills the entire region, the boundary of which is determined by the equation $\varphi_n = 0$. Let us take into account the fact that during the transition from one surface to another the identity

$$\varphi_m \equiv 0, \tag{2.22}$$

holds, and this leads to the equations

$$dm = 2(A-\xi)(\partial\varphi_m/\partial\mathbf{n})\, dn, \tag{2.23}$$

$$m = 2A\xi - \xi^2 \sin^2\vartheta. \tag{2.24}$$

On putting Eq. (2.23) into Eq. (2.21), taking account of the value of W_m as given by Eq. (2.20), and noting that $d\sigma\, dn = d^3\xi$ is an element of volume in the infinite space in which we are operating, we finally obtain

$$W = \frac{2\gamma}{\pi} ce^2 A \int \Phi_{\nu'}(\xi)\, d^3\xi, \tag{2.25}$$

where

$$\nu' = 2A\xi - \xi^2\sin^2\vartheta.$$

For what is to follow, it is convenient to go over to different units. In the units used here the wave number of the basic classical oscillation ("first harmonic" in the sense of the classical theory of the "radiating" electron) is equal to

$$\xi_0 = 1/2A. \tag{2.26}$$

As a new independent variable we take the ratio of ξ to ξ_0, that is, that which in the classical theory of the radiating electron is called the number of the harmonic. Setting

$$\nu = 2A\,\xi, \tag{2.27}$$

we finally obtain

$$W = \frac{ce^2\beta^2}{4\pi R^2}\int\limits_{Vn}\Phi_{\nu'}\left(\frac{\nu}{2A}\right)d^3\nu, \tag{2.28}$$

where $R = \sqrt{n/\gamma}$ is the classical radius of the trajectory of the electron,

$$\nu' = \nu\,[1 - (\nu/4n)\,\beta^2\sin^2\vartheta],$$

$$\beta = \sqrt{n}\,/\,A,\quad \sqrt{1-\beta^2} = mc^2\,/\,E.$$

The region V_n is the former region, but $\nu = 2A\,\xi$ in the new units.

3. THE DIFFERENTIAL SPECTRUM

In what follows we shall use approximations of the functions $I_{n,n'}$ which will be correct for the entire region of variation of the arguments and indices of these functions. The approximations were obtained by Klepnikov[4] and have the following form:

$$I_{n,n'}(x) = \frac{1}{\pi\sqrt{3}}\sqrt{1-\frac{x}{(\sqrt{n}-\sqrt{n'})^2}}\;K_{1/3}\left\{\frac{2}{3}(nn')^{1/4}(\sqrt{n}-\sqrt{n'})\times\left[1-\frac{x}{(\sqrt{n}-\sqrt{n'})^2}\right]^{3/2}\right\}, \tag{3.1}$$

$$I'_{n,n'} = \frac{1}{\pi\sqrt{3}}\frac{(nn')^{1/4}}{\sqrt{n}-\sqrt{n'}}\left[1-\frac{x}{(\sqrt{n}-\sqrt{n'})^2}\right]\times K_{2/3}\left\{\frac{2}{3}(nn')^{1/4}(\sqrt{n}-\sqrt{n'})\left[1-\frac{x}{(\sqrt{n}-\sqrt{n'})^2}\right]^{3/2}\right\}. \tag{3.2}$$

On putting Eqs. (3.1) and (3.2) into formula (2.28), we obtain

$$W = W^{(1)} + W^{(2)}; \tag{3.3}$$

$$W^{(1)} = \frac{ce^2}{R^2}\frac{1}{3\pi^2}\int\frac{\nu^2\,d\nu}{1-\nu/2n}\sin\vartheta\,d\vartheta\left\{\varepsilon_\vartheta\cos^2\vartheta\,K^2_{1/3}\left(\frac{1}{3}\frac{\nu}{1-\nu/2n}\varepsilon_\vartheta^{3/2}\right) + \varepsilon_\vartheta^2 K_{2/3}\left(\frac{1}{3}\frac{\nu}{1-\nu/2n}\varepsilon_\vartheta^{3/2}\right)\right\}; \tag{3.3a}$$

$$W^{(2)} = \frac{ce^2}{R^2}\frac{1}{3\pi^2}\int\frac{\nu^2\,d\nu}{1-\nu/2n}\sin\vartheta\,d\vartheta\,\frac{1}{2}\left(\frac{\nu}{2n}\right)^2\left(1-\frac{\nu}{2n}\right)^{-1}\varepsilon_\vartheta^2\times\left[K^2_{1/3}\left(\frac{1}{3}\frac{\nu}{1-\nu/2n}\varepsilon_\vartheta^{3/2}\right) + K^2_{2/3}\left(\frac{1}{3}\frac{\nu}{1-\nu/2n}\varepsilon_\vartheta^{3/2}\right)\right], \tag{3.3b}$$

where

$$\varepsilon_\vartheta = 1 - \beta^2\sin^2\vartheta.$$

The fact that in spherical coordinates $d^3\nu = \nu^2 d\nu\,\sin\theta\,d\theta\,d\varphi$ has been taken account of in formula (3.3), and the integration has been carried out with respect to the angle φ, on which, as a consequence of axial symmetry, the expression under the integral sign does not depend. The region of integration is set by the law of conservation of energy-momentum. As stated above, the component $W^{(1)}$ corresponds to the radiation of the spinless particle.

In order to obtain the differential spectrum it is necessary to carry out the integration in expressions (3.3) with resepct to the angle θ. Because of the exponential fall-off in the expression under the integral sign as θ departs from $\pi/2$, we may carry out an exchange of the variables $\cos\theta = x$ and extend the limits of integration to infinity. We encounter here a number of integrals which may be evaluated with the aid of Mellin transformation theory. In order for this to be done, a substitution must be made for the squares of the functions K with the aid of Nicholson's integral.

$$K_\mu(z)\,K_\nu(z) = 2\int\limits_0^\infty K_{\mu-\nu}(2z\,\mathrm{ch}\,t)\,\mathrm{ch}\,(\mu+\nu)\,t\,dt, \tag{3.4}$$

and then the function K_0 must be expressed with the aid of a relation obtained by means of Mellin's theorem from the well-known equation

$$\int_0^\infty K_\nu(x)\, x^{\mu-1}\, dx = 2^{\mu-2}\,\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\mu+\nu}{2}\right). \quad (3.5)$$

The result obtained is an absolutely convergent integral in which we may interchange the order of the integrations. Then with the aid of the multiplication formula for Γ-functions

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{n}\right)\ldots\Gamma\left(z+\frac{n-1}{n}\right) = (2\pi)^{(n-1)/2}\, n^{1/2-nz}\,\Gamma(nz) \quad (3.6)$$

the expressions under the integral signs may be considerably simplified and the integrals evaluated. For example, for the integral

$$I = \int_0^\infty \varepsilon_x^2\,[K_{1/3}^2(p\varepsilon_x^{3/2}) + K_{2/3}^2(p\varepsilon_x^{3/2})]\, dx, \quad (3.7)$$

where $\epsilon_x = 1-\beta^2+x^2$, $p>0$, there results, after the evaluations just mentioned, the expression

$$I = \frac{\varepsilon_0^{5/2}}{8\pi i}\int_{k-i\infty}^{k+i\infty} p_0^{-\mu}\, 2^{\mu-2}\,\varphi(\mu)\, d\mu; \quad (3.8)$$

$$\varphi(\mu) = \frac{4\pi}{\sqrt{3}}\, 2^{-\mu}\,\Gamma\left(\frac{\mu}{2}-\frac{1}{6}\right)\Gamma\left(\frac{\mu}{2}-\frac{5}{6}\right),$$

$$p_0 = p\varepsilon_0^{3/2}.$$

The path of integration goes parallel to the imaginary axis, to the right of the poles of the integrand. From Eq. (3.8) it follows immediately that

$$\int_0^\infty \varepsilon_x^2\,[K_{1/3}^2(p\varepsilon^{3/2}) + K_{2/3}^2(p\varepsilon_x^{3/2})]\, dx = \frac{\pi}{\sqrt{3}}\frac{\varepsilon_0}{p}\, K_{2/3}(2p\varepsilon_0^{3/2}). \quad (3.9)$$

The other integrals are evaluated analogously.

On carrying out the evaluations just mentioned, we obtain the following formulas for the differential spectrum:

$$W = \int_{\nu=0}^{2n} dW_\nu, \quad dW_\nu = dW_\nu^{(1)} + dW_\nu^{(2)}, \quad (3.10)$$

$$dW_\nu^{(1)} = \frac{1}{\pi\sqrt{3}}\frac{ce^2}{R^2}\,\varepsilon_0\nu\, d\nu \int_{\frac{2}{3}\frac{\nu}{1-\nu/2n}\varepsilon_0^{3/2}}^{\infty} K_{5/3}(x)\, dx, \quad (3.10a)$$

$$dW_\nu^{(2)} = \frac{1}{\pi\sqrt{3}}\frac{ce^2}{R^2}\,\varepsilon_0\nu\, d\nu\left(\frac{\nu}{2n}\right)^2\left(1-\frac{\nu}{2n}\right)^{-1} \times K_{2/3}\left(\frac{2}{3}\frac{\nu}{1-\nu/2n}\varepsilon_0^{3/2}\right). \quad (3.10b)$$

These formulas are correct for energies $E \ll E_{1/2} = mc^2(2Rmc/3h)^{1/2}$, as well as for energies $E \sim E_{1/2}$ and $E \gg E_{1/2}$, for the radiation of the entire spectrum. The formulas for the differential spectrum which were obtained earlier by Sokolov and Ternov[5] and Schwinger[6], and which are exact to within magnitudes of the first order in h, are special cases of formula (3.10) and may easily be obtained from it. For example, taking account of the fact that

$$\nu = \omega/\omega_0, \quad 1/2n = (h/Rmc)(mc^2/E),$$

$$\varepsilon_0 = 1-\beta^2 = (mc^2/E)^2,$$

$$\omega_c = {}^3/_2\,\omega_0\,(E/mc^2)^3,$$

we immediately obtain from Eq. (3.10a)

$$dW^{(1)} \approx \frac{3\sqrt{3}}{4\pi}\frac{e^2}{R}\left(\frac{E}{mc^2}\right)^4\frac{\omega_0\omega}{\omega_c^2}\, d\omega \times \int_{(\omega/\omega_c)(1+h\omega/E)}^{\infty} K_{5/3}(x)\, dx, \quad (3.11)$$

for $(h\omega/E) \ll 1$, and this coincides with Schwinger's formula. Similarly, we may obtain Sokolov and Ternov's formula, for which the same limiting condition holds.

In order to analyze the spectra it is convenient to transform to another variable $\xi = h\omega/E$. Then the formula for the differential spectrum of the "radiating" electron takes the following form:

$$dW = \frac{ce^2}{\pi\sqrt{3}}\left(\frac{mc}{h}\right)^2\xi\, d\xi\left[\int_{\xi/(1-\xi)\zeta}^{\infty} K_{5/3}(x)\, dx + \frac{\xi^2}{1-\xi}K_{2/3}\left(\frac{\xi}{1-\xi}\frac{1}{\zeta}\right)\right], \quad (3.12a)$$

while the formula characterizing the radiation of the spinless particle is given by the equation

$$dW^{(1)} = \frac{ce^2}{\pi\sqrt{3}}\left(\frac{mc}{h}\right)^2\xi\, d\xi \times \int_{\xi/(1-\xi)\zeta}^{\infty} K_{5/3}(x)\, dx, \quad \zeta = \frac{3}{2}\frac{h}{Rmc}\left(\frac{E}{mc^2}\right)^2. \quad (3.12b)$$

In the cases which are most interesting from the point of view of the influence of spin, namely, those of extremely relativistic energies ($\zeta \gg 1$), the following formulas, applicable to almost the entire spectrum, excluding the part in immediate proximity to the high frequency end, are obtained from Eq. (3.12):

a) for the electron

$$dW \approx \frac{ce^2}{R^2}\frac{3^{1/6}\,\Gamma(2/3)}{\pi}\left(\frac{Rmc}{h}\right)^{4/3}\left(\frac{E}{mc^2}\right)^{4/3} \times d\xi\,\xi^{1/3}(1-\xi)^{2/3}\left(1+\xi^2/2(1-\xi)\right), \tag{3.13a}$$

b) for the spinless particle

$$dW^{(1)} \approx \frac{ce^2}{R^2}\frac{3^{1/6}\,\Gamma(2/3)}{\pi}\left(\frac{Rmc}{h}\right)^{4/3} \times \left(\frac{E}{mc^2}\right)^{4/3} d\xi\,\xi^{1/3}(1-\xi)^{2/3}. \tag{3.13b}$$

The formulas obtained allow us to investigate the differential spectrum of the electron and the spinless particle for any energies. From the formulas for the differential spectrum it is immediately clear that for arbitrary energies and for the entire spectral region the spectrum of the radiating boson lies below the spectrum of the radiating electron. For increasing energies ($\zeta \to \infty$) the maximum density of the radiation of the electron comes together at the far end of the spectrum. The maximum density of the boson's radiation does not coalesce at the end of the spectrum in this case, but comes together at a point of the spectrum lying at approximately one third the distance from the beginning. A very typical case, showing the difference between the spectra of the radiating electron and boson, is shown in Fig. 1. This

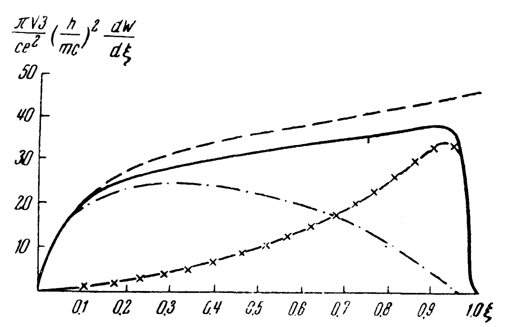

FIG. 1.

Figure shows the spectra for $\zeta = 100$. The spectrum of the electron is shown by the continuous line, while the "dash-dot" line shows the spectrum of the boson, and the "dash-cross" line shows the "difference" in density of the radiation of the electron and boson (the "role of spin"). The dashed line shows the spectrum given by classical theory. The remaining part of the spectrum given by classical theory, that on the high-frequency side, is not shown in the Figure, since, in virtue of the law of conservation of energy, these higher frequencies cannot actually be radiated.

It is clear from Fig. 1 that the difference in density of the radiation of the electron and boson is quite significant in this extreme relativistic case, and that the main contribution to the density of radiation is produced in the second half of the spectrum "at the expense of spin".

4. TOTAL ENERGY OF RADIATION

In order to obtain the total energy of radiation we must carry out an integration over ν in Eq. (3.10). Changing the variable of integration to

$$x = \frac{2}{3}\frac{\nu}{1-\nu/2n}\varepsilon_0^{3/2}, \tag{4.1}$$

we can represent the total energy of radiation in the form

$$W = W_{\mathrm{kl}}\,\varphi(\zeta), \quad \varphi(\zeta) = \varphi^{(1)}(\zeta) + \zeta^2\varphi^{(2)}(\zeta); \tag{4.2}$$

$$\varphi^{(1)}(\zeta) = \frac{9\sqrt{3}}{16\pi}\int_0^\infty \frac{x^2K_{5/3}(x)\,dx}{(1+\zeta x)^2}; \tag{4.3}$$

$$\varphi^{(2)}(\zeta) = \frac{9\sqrt{3}}{8\pi}\int_0^\infty \frac{x^3K_{2/3}(x)\,dx}{(1+\zeta x)^4}. \tag{4.4}$$

These integrals may be evaluated with the help of Mellin transformation theory. The integral for $\varphi^{(1)}(\zeta)$ with respect to ζ is absolutely convergent ($\zeta > 0$). The integral

$$Q(\zeta) = \int_0^\infty \frac{xK_{5/3}(x)\,dx}{1+\zeta x} \tag{4.5}$$

is also absolutely convergent. Hence, Eq. (4.3) can be written in the form

$$\varphi^{(1)}(\zeta) = -(9\sqrt{3}/16\pi)\,\partial Q/\partial\zeta. \tag{4.6}$$

Using the same method used in obtaining the formulas for the differential spectrum, we can show that

$$Q = \frac{\pi}{4}\alpha^2\frac{1}{2\pi i} \times \int_{k-i\infty}^{k+i\infty} \frac{\Gamma(s/2-5/6)\,\Gamma(s/2+5/6)}{\sin\pi s}\left(\frac{\alpha}{2}\right)^{-s} ds, \tag{4.7}$$

$$\alpha = 1/\zeta, \quad 5/3 < k < 2.$$

On evaluating this integral we obtain

$$Q(\zeta) = -\frac{2}{3}\pi^2 \frac{1}{\zeta^2}\left[\Phi_{5/3}\left(\frac{i}{\zeta}\right) + \frac{\sqrt{3}}{2\pi}\zeta\right], \tag{4.8}$$

where we have introduced the notation

$$\Phi_\nu(z) = i^{-\nu}[J_\nu(z) - \mathbf{J}_\nu(z)] + i^\nu[J_{-\nu}(z) - \mathbf{J}_{-\nu}(z)], \tag{4.8a}$$

and J_ν and $\mathbf{J}_\nu$ are the well-studied functions of Bessel and Anger.

An evaluation of (4.4) by an analogous method leads to the integral

$$\int_0^\infty \frac{K_{2/3}(x)\,dx}{1+\zeta x} = \frac{\pi^2}{2\sin^2(2\pi/3)}\frac{1}{\zeta}\Phi_{2/3}\left(\frac{i}{\zeta}\right). \tag{4.9}$$

Hence, we finally obtain the following expressions for the total energy of radiation:

a) for the electron

$$W = W_{kl}\varphi(\zeta), \quad \varphi(\zeta) = \frac{3\sqrt{3}}{8}\pi\left\{\frac{\partial}{\partial\zeta}\frac{\Phi_{5/3}(i/\zeta)}{\zeta^2} - \frac{\zeta^2}{3}\frac{\partial^3}{\partial\zeta^3}\frac{\Phi_{2/3}(i/\zeta)}{\zeta} - \frac{\sqrt{3}}{2\pi}\frac{1}{\zeta^2}\right\}, \tag{4.10a}$$

b) for the spinless particle

$$W^{(1)} = W_{kl}\varphi^{(1)}(\zeta), \quad \varphi^{(1)}(\zeta) = \frac{3\sqrt{3}}{8}\pi\left\{\frac{\partial}{\partial\zeta}\frac{\Phi_{5/3}(i/\zeta)}{\zeta^2} - \frac{\sqrt{3}}{2\pi}\frac{1}{\zeta^2}\right\}. \tag{4.10b}$$

In order to find the asymptotic expressions for the total energy of radiation for $\zeta \ll 1$, it is necessary to use the well-known asymptotic expansion of the Anger functions[8], with the aid of which we obtain

$$\varphi(\zeta) \approx 1 - \frac{55\sqrt{3}}{24}\zeta + \frac{64}{3}\zeta^2 - \frac{8855\sqrt{3}}{108}\zeta^3 + \frac{89600}{81}\zeta^4 - \ldots \tag{4.11a}$$

$$\varphi^{(1)}(\zeta) \approx 1 - \frac{55\sqrt{3}}{24}\zeta + \frac{56}{3}\zeta^2 - \frac{6545\sqrt{3}}{108}\zeta^3 + \frac{56000}{81}\zeta^4 - \ldots \tag{4.11b}$$

The asymptotic expansion (4.11a) of the exact formula (4.10a) coincides with the asymptotic expansion found in Ref. 4 by direct evaluation of the total energy of radiation in the asymptotic sense.

The first correction term was evaluated by Sokolov, Klepikov and Ternov[7]. This term has also recently been obtained by Schwinger[6].

For $\zeta \gg 1$ we may use series for the Bessel and Anger functions and keep the desired number of terms. As follows directly from Eq. (4.10), the main terms in the total energy of radiation have the following form:

a) for the electron

$$W^{(\infty)} \approx (32\,\Gamma(2/3)/(27\cdot 3^{4/3})) \times (ce^2/R^2)(Rmc/h)^{4/3}(E/mc^2)^{4/3}; \tag{4.12a}$$

b) for the spinless particle

$$W^{(1)(\infty)} \approx (2\Gamma(2/3)/(3\cdot 3^{4/3})) \times (ce^2/R^2)(Rmc/h)^{4/3}(E/mc^2)^{4/3}. \tag{4.12b}$$

It is clear from this that in the extreme relativistic case ($\zeta \gg 1$) the spinless particle radiates only approximately 9/16th as much as the electron. Thus the "role of spin" is very significant in the radiation in the extreme relativistic case.

In the case of $\zeta \ll 1$ the "role of spin" is determined by the ratio obtained from Eqs. (4.11a) and (4.11b).

$$\frac{W}{W^{(1)}} \approx 1 + \frac{8}{3}\zeta^2 - \frac{275\sqrt{3}}{18}\zeta^3 + \ldots \tag{4.13}$$

Thus it is clear that in this case the spin corrections are of second order of smallness as compared to the quantum corrections (second order in h). This is the reason that Schwinger[6], in calculating the first quantum correction for the radiation from the spinless particle, obtained the same result which Sokolov, Klepikov and Ternov[7] obtained earlier from the Dirac electron.

It should be remarked that for $h \to 0$ the exact formulas (2.17a) and (2.17b) go over into the exact formulas of the classical theory of the radiation from a "radiating" electron. This may be shown with the help of the relation

$$\lim_{n\to\infty;\,\nu,z\neq 0} I_{n,n-\nu}\left(\frac{z^2}{4n}\right) = J_\nu(z). \tag{4.14}$$

It follows from this that there must be no factor of type $1 + (mc^2/E)^2$ in the formulas, since this factor does not depend on h and does not disappear for a transition to the classical limit $h \to 0$.

With respect to the physical reason for the difference in the radiation of the electron and the spinless particle, Sokolov (see Ref. 3) has shown that in the case of extreme relativistic energies this difference evidently depends on the behavior characteristics of the spin magnetic moment of the electron. It should be noted, first of all, that in the relativistic case the electron, in a manner of speaking, "loses" its magnetic moment in accordance with the formula

$$\mu \approx \mu_0 mc^2 / E, \quad \mu_0 = eh/2mc. \tag{4.15}$$

However, on the other hand, with increasing energy the interaction with the high-frequency parts of the virtual field of the photons plays an increasingly significant role in the radiation.

The matrix elements characterizing the radiation at the expense of the magnetic moment and at the expense of the charge interaction are proportional to the magnitudes $\sim \mu H \sim \mu \varkappa A$ and $\sim eA$, respectively. Consequently, the ratio of the energy of radiation W_μ at the expense of the magnetic moment to the energy of radiation W_e at the expense of the charge interaction is equal to $(W_\mu/W_e) \sim (\mu\omega_{max}/ec)^2$ in order of magnitude. In the case of $\zeta \ll 1$, the maximum frequency is given by $\omega_{max} \approx \omega_0 (E/mc^2)^3$, while for $\zeta \gg 1$ it is given by $\omega_{max} \approx E/h$. Hence, we obtain at once

$$W_\mu / W_e \sim \begin{cases} \zeta^2 & \text{for } \zeta \ll 1, \\ 1 & \text{for } \zeta \gg 1. \end{cases}$$

Thus the statement of Sokolov corresponds completely with the results of the present work.

I wish to express my gratitude to Professors A. A. Sokolov, N. P. Klepikov and I. M. Ternov for many discussions of the questions considered in the present work.

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Two-Nucleon Potential of Intermolecular Type and Nuclear Saturation

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A study is made of the statistical model of the nucleus with uniform density distribution of the nucleons, on the basis of a two-nucleon interaction potential of the type of the Lennard-Jones intermolecular potential with a hard barrier. It is shown that saturation can be obtained with a certain choice of the parameters in the potential.

1. THE explanation of the stability of atomic nuclei is one of the main problems of the theory of nuclear structure, directly related to the explanation of saturation, which consists of the fact that in medium-weight and heavy nuclei the density of nucleons and the binding energy per nucleon are roughly constant. The existence of saturation has always placed restrictions on the choice of one or another kind of theory of the nuclear forces, which it is still impossible to determine uniquely. At first it seemed possible to achieve saturation by means of exchange forces of various kinds.[1] But the data on the scattering of nucleons ($n-p$ and $p-p$) at moderately high energies ($\gtrsim$ 100 mev)