SOVIET PHYSICS JETP VOLUME 4, NUMBER 3 APRIL, 1957

Radiative Corrections to the Scattering of Electrons by Electrons and Positrons

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(Submitted to JETP editor June 6, 1955)
J. Exptl. Theoret. Phys. (U.S.S.R.) 31, 449-458 (September, 1956)

The radiative corrections to the scattering of electrons by electrons and positrons are calculated with accuracy up to α^3. Consideration is given to the general case and the limiting case of large energies.

1. ELASTIC SCATTERING OF ELECTRONS BY ELECTRONS

THE scattering of electrons by electrons and positrons, including the radiative corrections of lowest order, can be represented by means of the diagrams shown in Fig. 1, where p_1, p_2, p_1', and p_2' represent the four-dimensional momenta of the first and second electrons before and after scattering, k the four-dimensional momentum of a photon, p the momentum of a virtual electron, μ and ν, the polarizations of virtual photons, and $q = p_1' - p_1$. (We do not consider diagrams containing proper energy parts for a real electron, since on regularization they reduce to zero). The matrix element corresponding to the ith graph will be denoted by U_i. Along with the diagrams shown, one must also consider diagrams obtained from them by means of the interchanges

$$1)\ p_1 \rightleftarrows p_2;\quad p_1' \rightleftarrows p_2';\quad 2)\ p_1' \rightleftarrows p_2';\quad 3)\ p_1 \rightleftarrows p_2.$$

The matrix elements of such diagrams will be denoted by U_i', $\tilde{U}_i$, and $\tilde{U}_i'$, respectively.

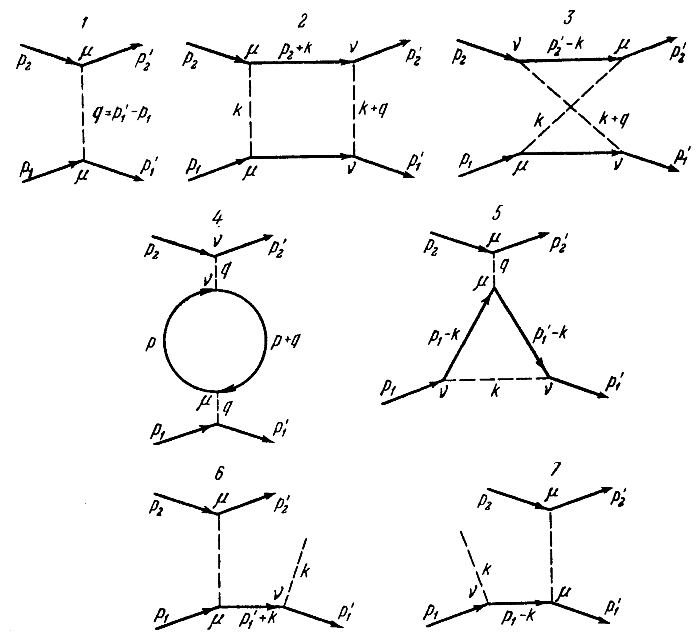

FIG. 1.

On integration the matrix elements corresponding to diagrams 4 and 5 of Fig. 1 diverge in the region of large momenta of the virtual particles. These divergences are regularized in the usual way,[1] and we carry out the integration over a finite relativistically invariant four-dimensional region.[2] The matrix elements corresponding to diagrams 2, 3, and 5 also diverge on integration over the region of small momenta of the virtual particles ("infrared catastrophe"). To avoid this difficulty one proceeds, as is well known, to ascribe to the photon a "mass" λ, which then appears in the cross section for elastic scattering. On combining the cross sections of purely elastic scattering and inelastic scattering (with emission of a long-wavelength photon, with energy not exceeding ΔE), the quantity λ cancels out and does not appear in the final result. The inelastic scattering is represented in diagrams 6 and 7 of Fig. 1.

The matrix elements corresponding to the diagrams of Fig. 1 are formed according to Feynman's rules.[3,2] For example, diagram 3 gives the following matrix element

$$U_3 = -\frac{\alpha^2}{\pi}\delta(p_1 + p_2 - p_1' - p_2') \tag{1}$$
$$\times \int (\bar{u}_1' \gamma_\nu (i\hat{p}_1 - i\hat{k} + 1)^{-1} \gamma_\mu u_1)$$
$$\times (\bar{u}_2' \gamma_\mu (i\hat{p}_2' - i\hat{k} + 1)^{-1} \gamma_\nu u_2)$$
$$\times d^4k/(k^2 + \lambda^2)[(k+q)^2 + \lambda^2],$$

where $\hat{p} = p_\mu \gamma_\mu$, $\bar{u} = u^* \gamma_4$, $\alpha = 1/137$, and the u are the spinor amplitudes of the electrons. (We use a system of units with $\hbar = c = m = 1$.)

The four-vectors p_1, p_2, p_1', and p_2' satisfy the law of conservation of energy and momentum, $p_1 + p_2 = p_1' + p_2'$, from which there result the following equations between the scalar products:

$$p_1 p_1' = p_2 p_2'; \quad p_1 p_2 = p_1' p_2';$$
$$p_1 p_2' = p_1' p_2; \quad p_1 p_2 = p_1 p_1' + p_1 p_2' + 1;$$
$$-p_1 q = p_1' q = -p_2' q = p_2 q = q^2/2.$$

Making use of Dirac's equations

$$(i\hat{p} + 1)u = 0; \quad \bar{u}(i\hat{p} + 1) = 0,$$

we rewrite Eq. (1):

$$U_3 + U_3' = -2\pi i \alpha^2 \delta(p_1 + p_2 - p_1' - p_2') \tag{2}$$
$$\times \{-4(p_1 p_2')(\bar{u}_1' \gamma_\mu u_1)(\bar{u}_2' \gamma_\mu u_2) b$$
$$+ 2[(\bar{u}_1' \gamma_\nu \gamma_\sigma \hat{p}_2' u_1)(\bar{u}_2' \gamma_\nu u_2)$$
$$+ (\bar{u}_1' \gamma_\nu u_1)(\bar{u}_2' \hat{p}_1 \gamma_\sigma \gamma_\nu u_2)] b_\sigma$$
$$- (\bar{u}_1' \gamma_\nu \gamma_\sigma \gamma_\mu u_1)(\bar{u}_2' \gamma_\mu \gamma_\tau \gamma_\nu u_2) b_{\sigma\tau}\},$$

where

$$b_{(\ ,\sigma,\sigma\tau)} = \frac{1}{\pi^2 i} \tag{3}$$
$$\times \int \frac{(1, k_\sigma, k_\sigma k_\tau)\, d^4k}{(k^2 - 2p_1 k)(k^2 - 2p_2' k)(k^2 + \lambda^2)[(k+q)^2 + \lambda^2]}.$$

The symbol $(1, k_\sigma, k_\sigma k_\tau)$ means that in the first case it is to be replaced by 1, in the second by k_σ, and in the third by $k_\sigma k_\tau$.

First we shall show how the scalar integral is calculated.[4] From the formula

$$\frac{1}{ab} \int_0^1 [ax + b(1-x)]^{-2} dx$$

and the formulas obtained from it by differentiating with respect to a and b, together with the four-dimensional integral

$$\int \frac{d^4k}{(k^2 - 2pk + L)^4} = \frac{\pi^2 i}{6(L - p^2)},$$

we obtain

$$b = \int_0^1\int_0^1\int_0^1 \frac{z^2(1-x)\, dy\, dx\, dz}{[xz(1-z)q^2 + \lambda^2 - z(1-x)\lambda^2 - z^2(1-x)^2 p_y^2]^2}, \tag{4}$$

where $p_y = yp_1 + (1-y)p_2'$. The quantity λ^2 appearing in the integral (4) is small in comparison with q^2, p_y^2, and unity. But it cannot be simply neglected, since this would give a divergent integral. Therefore we break the integral (4) up into three integrals, in which the integration with respect to x is carried out over the intervals $(0, \epsilon)$, $(\epsilon, 1-\epsilon_1)$, $(1-\epsilon_1, 1)$, with $\epsilon \ll 1$, $\epsilon_1 \ll 1$. In the second of these integrals we can drop terms containing λ^2, and in the third those in $-z(1-x)\lambda^2$.

The first integral is again broken up into two integrals, in which the integration with respect to z is carried out over the intervals $(0, z_c)$ and $(z_c, 1)$, with $\lambda^2 \ll -z_c^2 p_y^2 \ll z_c q^2 \epsilon$ and $z_c \ll 1$. In the integration over the integral $(0, z_c)$ we drop $z^2 p_y^2$, and in the integral $(z_c, 1)$, the terms containing λ^2. When the expressions obtained are combined, the quantities z_c, ϵ, and ϵ_1 cancel, and we obtain

$$b = \frac{2}{q^2} \ln \frac{q^2}{\lambda^2} M(\Phi_b); \quad M(x) = \frac{x}{\operatorname{sh} 2x}, \tag{5}$$

where Φ_b is defined by the relation

$$(p_2' - p_1)^2 = q_o^2 = 4 \operatorname{sh}^2 \Phi_b.$$

We proceed to the calculation of b_σ. This integral is a four-vector, which can depend only on the vectors p_1, p_2' and q. Since b_σ remains unchanged by the interchange $p_1 \rightleftarrows p_2'$, it has the form

$$b_\sigma = X_b (p_{1\sigma} + p_{2\sigma}') + Y_b q_\sigma, \tag{6}$$

where X_b and Y_b are scalars depending on the scalar products of the vectors p_1, p_2' and q.

Taking the scalar products of Eq. (6) by $p_{1\sigma}$ and q_σ, we obtain two equations for the determination of X_b and Y_b:

$$(-1 + p_1 p_2') X_b - \tfrac{1}{2} q^2 Y_b = \tfrac{1}{2} (F^b - G), \tag{7}$$

$$-q^2 X_b + q^2 Y_b = \tfrac{1}{2} (H^b - F^b - q^2 b).$$

The integrals H^b, F^b, and G appearing in the right members differ from b by the absence from the denominator of the factors $(k+q)^2 + \lambda^2$, $k^2 + \lambda^2$, and $k^2 - 2p_2' k$, respectively. They are calculated in the same way as b:

$$H^b = F^b = -N(\Phi_b) - 2M(\Phi_b) \ln \lambda, \tag{8}$$

$$G = \frac{1}{\operatorname{sh} 2\Phi} \Big[F(e^{-2\Phi} - 1) + \Phi^2 + 2\Phi \ln (1 - e^{-2\Phi}) + \frac{\pi}{2} \Big], \tag{9}$$

where

$$N(x) = \frac{1}{\operatorname{sh} 2x} \Big[x^2 - 2x \ln (2 \operatorname{ch} x) - F(e^{-2x}) + \frac{\pi^2}{12} \Big]; \tag{10}$$

$$F(x) = \int_0^x \ln (1+y) \frac{dy}{y}. \tag{11}$$

The function $F(x)$ satisfies the relations:

$$F(1) = \frac{\pi^2}{12}; \quad F(-1) = -\frac{\pi^2}{6}; \tag{12}$$

$$F(x) + F\left(\frac{1}{x}\right) = \frac{\pi^2}{6} + \frac{1}{2} \ln^2 x;$$

$$F(x-1) + F(1/x - 1) = \tfrac{1}{2} \ln^2 x,$$

and the quantity Φ is defined by the equation

$$(p_1' - p_1)^2 = q^2 = 4 \operatorname{sh}^2 \Phi.$$

We calculate, finally, the tensor integral $b_{\sigma\tau}$. Noting that $b_{\sigma\tau}$ can depend only on the vectors p_1, p_2' and q, and that this quantity is unchanged by the interchange $p_1 \rightleftarrows p_2'$, we find that this integral has the form

$$b_{\sigma\tau} = (p_{1\sigma} + p_{2\sigma}')(p_{1\tau} + p_{2\tau}') K_b + (p_{1\sigma} - p_{2\sigma}')(p_{1\tau} - p_{2\tau}') L_b + [(p_{1\sigma} + p_{2\sigma}') q_\tau + q_\sigma (p_{1\tau} + p_{2\tau}')] W_b + q_\sigma q_\tau Z_b + \delta_{\sigma\tau} T_b, \tag{13}$$

where K_b, L_b, W_b, Z_b, and T_b are scalars depending on the scalar products of the vectors p_1, p_2', and q.

Multiplying Eq. (13) by $p_{1\tau}$, $q_{1\tau}$ and contracting on the indices σ and τ, we obtain a system of three equations, in the right members of which there appear integrals H_σ^b, F_σ^b, G_σ differing from H^b, F^b, G by the presence of the factor k_σ in the numerator. These integrals are calculated in just the same way as b_σ. Then, equating the coefficients of the vectors $p_{1\sigma}$, $p_{2\sigma}'$, q_σ, we obtain a system of six equations for the determination of the five unknowns K_b, L_b, W_b, Z_b, T_b. The extra equation serves as a check on the calculations.

$U_2 + U_2'$ and the other matrix elements are calculated analogously. In the calculation of U_2, additional poles appear in the denominator of the integrand; these are avoided according to the Feynman rule, i.e., an infinitely small negative imaginary part is added to the squares of the four-dimensional momenta: $p^2 \to p^2 - i\epsilon$. Instead of the quantity Φ_b, which occurs in the matrix element U_3, the expression for U_2 contains the quantity $\Phi_a - \pi i/2$, which is defined by the relation $(p_1 - p_2)^2 = q_a^2 = 4\,\mathrm{sh}^2\,\Phi_a$.

The cross section for purely elastic scattering is given by the following formula:

$$d\sigma_{\text{elas.}} = \frac{1}{2v}\frac{1}{4}SS'\Big\{|U_1 + U_1' + \tilde{U}_1 + \tilde{U}_1'|^2 + 2\,\mathrm{Re}\sum_{i=2}^{5}(U_1 + U_1' + \tilde{U}_1 + \tilde{U}_1')^* \times (U_i + U_i + \tilde{U}_i + \tilde{U}_i'\Big\}d\Omega, \tag{14}$$

where v is the velocity of the electron in the center-of-mass system (c. m. s.), S indicates summation over the orientations of the electron spins in the initial state, S' the same for the final state, and $d\,\Omega$ is the element of solid angle into which the electron is scattered.

Since the quantities U_i contain δ-functions, the formula (14) contains a factor $|2\pi\delta(p_1+p_2-p'_1-p'_2)|^2$, which is to be replaced, as shown in Ref. 2, by $2\pi\rho_f$, where ρ_f is the number of final states in unit range of the energy of the system, given by $\rho_f = v/16\pi^3(1-v^2)$.

We shall now show how the summation over the orientations of the electron spins is carried out. Multiplying u by the operator $\beta(-i\hat{p}+1)/2E$ (with $\beta = \gamma_4$ and E the energy of the electron in the c. m. s.) and using the equation $\beta\gamma_\lambda\beta\ldots\beta\gamma_\lambda\beta = \gamma_\lambda\ldots\gamma_\lambda$, we obtain, for example,

$$\begin{aligned} &SS'(\bar{u}_1'\gamma_\lambda u_1)^*(\bar{u}_2'\gamma_\lambda u_2)^*(\bar{u}_1'\gamma_\nu(\hat{p}_1+\hat{p}_2')\gamma_\mu u_1)(\bar{u}_2'\gamma_\mu\hat{q}\gamma_\nu u_2)\\ &= \frac{1}{4E^4}\mathrm{Sp}\{\gamma_\lambda(-i\hat{p}_1'+1)\gamma_\nu(\hat{p}_1+\hat{p}_2')\gamma_\mu(-i\hat{p}_1+1)\}\\ &\quad\times\frac{1}{4}\mathrm{Sp}\{\gamma_\lambda(-i\hat{p}_2'+1)\gamma_\mu\hat{q}\gamma_\nu(-i\hat{p}_2+1)\}, \end{aligned} \tag{15}$$

$$\begin{aligned} &SS'(\bar{u}_2'\gamma_\lambda u_1)^*(\bar{u}_1'\gamma_\lambda u_2)^*(\bar{u}_1\gamma_\nu(\hat{p}_1+\hat{p}_2)\gamma_\mu u_1)(\bar{u}_2\gamma_\mu\hat{q}\gamma_\nu u_2)\\ &= \frac{1}{4E^4}\cdot\frac{1}{4}\mathrm{Sp}\{\gamma_\lambda(-i\hat{p}_2'+1)\gamma_\mu\hat{q}\gamma_\nu(-i\hat{p}_2+1)\gamma_\lambda(-i\hat{p}_1'\\ &\quad+1)\gamma_\nu(\hat{p}_1+\hat{p}_2')\gamma_\mu(-i\hat{p}_1+1)\}. \end{aligned} \tag{16}$$

In expressions of the type of (16) we carry out the summation by the formulas:[5]

$$\gamma_\mu\gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_{2n-1}}\gamma_\mu = -2\gamma_{j_{2n-1}}\cdots\gamma_{j_2}\gamma_{j_1}, \tag{17}$$

$$\gamma_\mu\gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_{2n}}\gamma_\mu = 2\gamma_{j_{2n}}\gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_{2n-1}} + 2\gamma_{j_{2n-1}}\cdots\gamma_{j_2}\gamma_{j_1}\gamma_{j_{2n}}.$$

For the calculation of the traces of the matrices the following rule can be formulated.[2,6] To calculate $1/4\,\mathrm{Sp}(\gamma_{j1}\gamma_{j2}\gamma_{j3}\ldots)$ we draw a circle, and put in correspondence with each matrix γ_i a point i on the circumference. The points are placed on the circumference in the same order as the matrices occur in the product. We join these points in pairs by straight lines. Then each straight line joining points i and k corresponds to a factor δ_{ik}; to each way of joining the points there corresponds in the expansion of the trace a term of the form $(-1)^P\delta_{ik}\delta_{lm}\delta_{np}\ldots$, where P is the number of points of intersection of the straight lines. For example, for the calculation of $1/4\,\mathrm{Sp}(\gamma_i\gamma_k\gamma_l\gamma_m\gamma_n\gamma_p)$ one must draw diagrams, some of which are shown in Fig. 2. To these diagrams there correspond the following terms in the expansion of the trace:

$$\tfrac{1}{4}\mathrm{Sp}(\gamma_i\gamma_k\gamma_l\gamma_m\gamma_n\gamma_p) = +\delta_{im}\delta_{kl}\delta_{np} - \delta_{il}\delta_{km}\delta_{np} + \delta_{in}\delta_{km}\delta_{lp} - \delta_{im}\delta_{kn}\delta_{lp} + \ldots$$

An analogous rule exists for the calculation of the product of two traces.[6] We draw two circles side by side. On each of them we place points $i, k, \ldots$ in the same order as the matrices γ_i, γ_k in one of the products. We join by dotted lines those pairs of points, over which summations are taken [we are considering the case in which these points belong to different traces; otherwise the expression can be simplified by means of the formulas (17)]. Then we join by solid straight lines pairs of points belonging to the same trace (both those with "dummy" and those with "speaking" indices.). To the line joining points with the indices i and k there corresponds a factor δ_{ik} (these points can belong to the same trace or to different traces). To each such diagram there corresponds a term $(-1)^P \delta_{ik}\delta_{lm}\ldots$, where $i-k$, $l-m$, ... are the pairs of points joined by the system of solid and dotted lines and P is the number of intersections of solid lines. If a given assignment of the indices to pairs $i-k$, $l-m$, ... has several diagrams corresponding to it, then the coefficient of the term $\delta_{ik}\delta_{lm}\ldots$ in the expansion of the product of traces is found by combining the coefficients corresponding to the individual diagrams.

By using this method one can, for example, show that

$$\tfrac{1}{4}\mathrm{Sp}(\gamma_\mu\gamma_i\gamma_\nu\gamma_k\gamma_\lambda\gamma_l)\,\tfrac{1}{4}\mathrm{Sp}(\gamma_\mu\gamma_m\gamma_\nu\gamma_n\gamma_\lambda\gamma_p) \tag{18}$$
$$= 6\delta_{im}\delta_{kn}\delta_{lp} + 2(\delta_{im}\delta_{kl}\delta_{np} + \delta_{kn}\delta_{il}\delta_{mp} + \delta_{lp}\delta_{ik}\delta_{mn})$$
$$+ 2(\delta_{im}\delta_{kp}\delta_{ln} + \delta_{kn}\delta_{ip}\delta_{lm} + \delta_{lp}\delta_{in}\delta_{km})$$
$$+ 2(\delta_{in}\delta_{kp}\delta_{lm} + \delta_{ip}\delta_{ln}\delta_{km})$$
$$- 2(\delta_{ik}\delta_{ln}\delta_{mp} + \delta_{ik}\delta_{lm}\delta_{np} + \delta_{kl}\delta_{mn}\delta_{ip}$$
$$+ \delta_{kl}\delta_{mp}\delta_{in} + \delta_{il}\delta_{np}\delta_{km} + \delta_{il}\delta_{mn}\delta_{kp}).$$

Since in using this procedure one has to take into account all ways of joining the given pairs of points, we state the formula[6] determining the number N of these ways:

$$N(n, m, k) = \frac{n!}{(m+k-1)!} \times \sum_{l=0}^{(n-m)/2} \frac{(m+k+l-1)!}{(l!)^2}\,\frac{[(n-m-2l-1)!!]^2}{(n-m-2l)!}, \tag{19}$$

where n is the number of "dummy" indices in each of the traces, m is the number of pairs of "speaking" indices, and k is the number of pairs of "speaking" indices belonging wholly to one trace ($0! = 1$; $(-1)\,!! = 1$).

After calculating the traces and integrals, we obtain the following expression for the cross section for elastic scattering of electrons by electrons:[6]

$$d\sigma^{e-e}_{\text{elas.}} = \frac{1}{8}\alpha^2(1-v^2) \times \mathrm{Re}\sum_{i=1}^{5}(C_i^{(1)} - C_i^{(2)} + \tilde{C}_i^{(1)} - \tilde{C}_i^{(2)})\,d\Omega, \tag{20}$$

where

$$C_1^{(1)} = (q_a^4 + 4q_b^4 + 8q_b^2 + 8)\,q^{-4}; \tag{21}$$
$$C_1^{(2)} = (-q_a^4 + 4)\,q^{-2}q_b^{-2};$$
$$C_2^{(2)} = \frac{4\alpha}{\pi}\frac{\Phi_a - \frac{\pi i}{2}}{\mathrm{th}\,2\Phi_a}\ln\frac{q^2}{\lambda^2}C_1^{(2)}$$
$$- \frac{\alpha}{\pi q_b^2}\Big\{\frac{1}{q^2+4}[2G(-q^2q_a^4 + q_b^4 - 5q_a^4 + 6q^2 - 4q_a^2 + 24) + 8q_b^2\ln q] + 2\Big[N\Big(\Phi_a - \frac{\pi i}{2}\Big)$$
$$+ 2M\Big(\Phi_a - \frac{\pi i}{2}\Big)\ln q\Big](q^2 + q_a^2 + 8) + 4M\Big(\Phi_a - \frac{\pi i}{2}\Big)q_o^2\Big\};$$
$$C_3^{(1)} = -\frac{4\alpha}{\pi}\frac{\Phi_b}{\mathrm{th}\,2\Phi_b}\ln\frac{q^2}{\lambda^2}C_1^{(1)}$$
$$- \frac{\alpha}{\pi q^2}\Big\{\frac{1}{q^2+4}[G(-q^2q_a^4 - 3q^2q_b^4 - 2q^4 - 16\,q_a^2\,q_o^2 - 48\,q_b^2 - 32) + 2\ln q\cdot(q^2q_a^2 + 6q^2 - 4q_a^2 - 8)] + [N(\Phi_b)$$
$$+ 2M(\Phi_b)\ln q](q_b^2 + 2)(q^2 - 2q_a^2) - M(\Phi_b)(q_a^2\,q_b^2 + 6q_b^2 + 8)\Big\};$$
$$C_3^{(2)} = -\frac{4\alpha}{\pi}\frac{\Phi_b}{\mathrm{th}\,2\Phi_b}\ln\frac{q^2}{\lambda^2}C_1^{(2)}$$
$$- \frac{\alpha}{\pi q_o^2}\Big\{\frac{1}{q^2+4}[G(q^2q_b^4 + q^2q_a^4 + 8q_b^4 + 4q^2q_a^2 + 4q^2 - 8q_a^2 - 16) - 2\ln q\,(q^2q_a^2 + 6q^2 + 8)] + [N(\Phi_b) + 2M(\Phi_b)\ln q]$$
$$\times(q_b^4 + q_a^2\,q_b^2 + 2q^2 + 2q_a^2 + 8) + M(\Phi_b)(q_a^2\,q_b^2 + 2q_a^2 - 6q^2 - 8)\Big\};$$
$$C_4^{(1)} = -\frac{2\alpha}{\pi}JC_1^{(1)};\quad C_4^{(2)} = -\frac{2\alpha}{\pi}JC_1^{(2)};$$
$$C_5^{(1)} = -\frac{\alpha}{\pi}\Big[lC_1^{(1)} - \frac{4(q^2-2)}{q^2}M(\Phi)\Big];$$
$$C_5^{(2)} = -\frac{\alpha}{\pi}\Big[lC_1^{(2)} - \frac{2(q^2 - 4q_a^2 - 6)}{q_b^2}M(\Phi)\Big];$$
$$l = 4\Big[(2\Phi\,\mathrm{cth}\,2\Phi - 1)\Big(\ln\frac{1}{\lambda} - 1\Big) + \frac{\Phi}{2}\mathrm{th}\,\Phi\Big] - (4 + 2q^2)M(\Phi);$$
$$J = (1 - \tfrac{1}{3}\mathrm{cth}^2\Phi)(1 - \Phi\,\mathrm{cth}\,\Phi) - \tfrac{1}{9}.$$

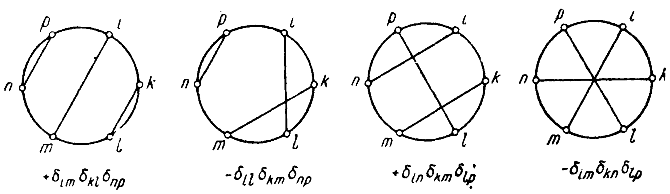

FIG. 2.

$C_2^{(1)}$ is obtained from $C_3^{(1)}$ by the interchange $q_a^2 \rightleftarrows -4-q_b^2$, $\Phi_b \rightleftarrows \Phi_a - \pi i/2$ and change of sign, and $\widetilde{C}_i^{(k)}$ is obtained from $C_i^{(k)}$ by the interchange $q \rightleftarrows q_b$, $\Phi \rightleftarrows \Phi_b$.

If in the expressions (20) we retain only the first terms $C_1^{(k)}$ and $\widetilde{C}_1^{(k)}$, then we obtain the Möller formula.[7]

2. ELASTIC SCATTERING OF ELECTRONS BY POSITRONS

The cross section for scattering of positrons by positrons, including the radiation corrections, is given by Eq. (20).

To find the cross section for scattering of electrons by positrons (or of positrons by electrons) one must put

$$p_1 = p_-; \quad p_1' = p_-'; \tag{22}$$

$$p_2 = -p_+'; \quad p_2' = -p_+;$$

where p_- is the four-vector momentum of the electron and p_+ is the four-vector momentum of the positron. Equation (20) then takes the form:

$$d\sigma_{\mathrm{elas}}^{p-e} = \frac{1}{8}\alpha^2(1-v^2) \tag{23}$$

$$\times \operatorname{Re}\sum_{i=1}^{5}(D_i^{(1)} - D_i^{(2)} + \widetilde{D}_i^{(1)} - \widetilde{D}_i^{(2)})\,d\Omega,$$

where the quantities $D_i^{(k)}$ are obtained from $C_i^{(k)}$ by the interchange $q_b^2 \rightleftarrows -4-q_a^2$, $\Phi_b \rightleftarrows \Phi_a - \pi i/2$, and the quantities $\widetilde{D}_i^{(k)}$ are obtained from $D_i^{(k)}$ by the interchange $q^2 \rightleftarrows -4-q_a^2$, $\Phi \rightleftarrows \Phi_a - \pi i/2$.

If in the expression (23) we retain only the first terms $D_1^{(k)}$ and $\widetilde{D}_1^{(k)}$, then we obtain the Bhabha formula.[8]

3. INELASTIC SCATTERING

In order to eliminate the "mass" λ of the photon, appearing in Eq. (20) in connection with the "infrared catastrophe", it is necessary, as is well known, to consider along with the elastic scattering the inelastic scattering of electrons by electrons with the emission of a photon of energy not exceeding ΔE ($\Delta E \ll 1$; $\Delta E \ll p$). This cross section is given by the following formula:

$$d\sigma_{\mathrm{inelas}}^{e-e} = \frac{\alpha^3(1-v^2)}{32\pi^2}. \tag{24}$$

$$\times (C_1^{(1)} - C_1^{(2)} + \widetilde{C}_1^{(1)} - \widetilde{C}_1^{(2)})$$

$$\times \int_{|\mathbf{k}|\leqslant E}\left(\frac{p_{1\nu}}{p_1\cdot k} + \frac{p_{2\nu}}{p_2\cdot k} - \frac{p_{1\nu}'}{p_1'\cdot k} - \frac{p_{2\nu}'}{p_2'\cdot k}\right)^2$$

$$\times \frac{d^3\mathbf{k}}{(\mathbf{k}^2+\lambda^2)^{1/2}}\,d\Omega.$$

We note that the cross section for inelastic scattering depends in an essential way on the reference system. In the c. m. s. Eq. (24) leads to the following result:[6]

$$d\sigma_{\mathrm{inelas.\,c.\,m.}}^{e-e} \tag{25}$$

$$= \frac{\alpha}{\pi}\,d\sigma_0\Big\{4\ln\frac{2\Delta E}{\lambda}\left(\frac{2\Phi}{\operatorname{th}2\Phi} - \frac{2\Phi_a}{\operatorname{th}2\Phi_a} + \frac{2\Phi_b}{\operatorname{th}2\Phi_b} - 1\right)$$

$$+ \frac{4\Phi_a}{\operatorname{th}\Phi_a} + \frac{4}{\operatorname{sh}2\Phi_a}[H(\pi, v)\operatorname{ch}2\Phi_a$$

$$- H(\vartheta, v)\operatorname{ch}2\Phi - H(\pi-\vartheta, v)\operatorname{ch}2\Phi_b]$$

$$+ 4[N(\Phi_a)\operatorname{ch}2\Phi_a$$

$$- N(\Phi)\operatorname{ch}2\Phi - N(\Phi_b)\operatorname{ch}2\Phi_b]$$

$$+ 4\ln(2\operatorname{ch}\Phi_a)\left(\frac{2\Phi_a}{\operatorname{th}2\Phi_a} - \frac{2\Phi}{\operatorname{th}2\Phi} - \frac{2\Phi_b}{\operatorname{th}2\Phi_b}\right)\Big\},$$

where ϑ is the angle of scattering in the c. m. s., and

$$H(\vartheta, v) = \frac{1}{\sin(\vartheta/2)}\int_{\cos(\vartheta/2)}^{1}\left(\frac{\ln[(1+v\zeta)/2]}{1-v\zeta}\right.$$

$$\left. - \frac{\ln[(1-v\zeta)/2]}{1+v\zeta}\right)\frac{d\zeta}{\sqrt{\zeta^2-\cos^2(\vartheta/2)}}.$$

In the laboratory system (l. s.) the formula is[9]

$$d\sigma^{e-e}_{\text{inelas l.s.}} = \frac{\alpha}{\pi} d\sigma_0 \Big[4\Big(1 - \frac{2\Phi}{\text{th}\,2\Phi} + \frac{2\Phi_a}{\text{th}\,2\Phi_a} - \frac{2\Phi_b}{\text{th}\,2\Phi_b}\Big)\ln\frac{\lambda}{2\Delta E} + 1 + \frac{2\Phi}{\text{th}\,2\Phi} + \frac{2\Phi_a}{\text{th}\,2\Phi_a} + \frac{2\Phi_b}{\text{th}\,2\Phi_b} - \frac{2}{\text{th}\,2\Phi}\int_0^{2\Phi} x\,\text{cth}\,x\,dx + \frac{2}{\text{th}\,2\Phi_a}\int_0^{2\Phi_a} x\,\text{cth}\,x\,dx - \frac{2}{\text{th}\,2\Phi_b}\int_0^{2\Phi_b} x\,\text{cth}\,x\,dx - R(\Phi_a, \Phi_b; \Phi) - R(\Phi_a, \Phi; \Phi_b) + R(\Phi_b, \Phi; \Phi_a)\Big], \quad (26)$$

where

$$R(\Phi_a, \Phi_b; \Phi) = \frac{2\Phi}{\text{th}\,2\Phi}\int_0^1 \mu \ln\frac{\mu+1}{\mu-1}\,dx;$$

$$\mu = A(A^2 - B^2)^{-1/2};$$

$$A = \text{sh}(2\Phi x)\,\text{ch}\,2\Phi_a + \text{sh}\,[2\Phi(1-x)]\,\text{ch}\,2\Phi_b; \quad B = \text{sh}\,2\Phi.$$

The cross section for inelastic scattering of electrons by positrons is obtained from Eq. (25) by the interchange $\Phi_a \rightleftarrows \Phi_b$.

4. SCATTERING IN THE EXTREME RELATIVISTIC CASE

In the limiting case

$$p \gg 1, \quad p\sin(\vartheta/2) \gg 1, \quad p\cos(\vartheta/2) \gg 1,$$

where p is the momentum of the electron in the c. m. s., the scattering cross section (elastic + inelastic) of electrons by electrons in the l. s. is given by the formula

$$d\sigma^{e-e}_{\text{l.s.}} = \frac{\alpha^2(1-v^2)}{4\chi^2(1-\chi)}\Big\{\frac{1}{2}(2 - 3\chi + 3\chi^2 - \chi^3) + \frac{\alpha}{\pi}\Big[2(1 - 2\Phi + 2\Phi_a - 2\Phi_b)\ln\frac{1}{2\Delta E}(2 - 3\chi + 3\chi^2 - \chi^3) + \frac{\Phi}{3}(28 - 48\chi + 48\chi^2 - 17\chi^3) + \Phi_a(2 - 2\chi + \chi^2) + \Phi_b(2 - 2\chi + 3\chi^2 - \chi^3) - \Phi_a^2(2-\chi) - \Phi_b^2(2 - \chi + 2\chi^2 - \chi^3) - \Phi^2(6 - 5\chi + 5\chi^2 - 2\chi^3) + 2\Phi\Phi_a(6 - 7\chi + 6\chi^2 - 2\chi^3) - 2\Phi\Phi_b(10 - 17\chi + 16\chi^2 - 5\chi^3) + 2\Phi_a\Phi_b(2 - 3\chi + 3\chi^2 - \chi^3) - \frac{37}{18}(2 - 3\chi + 3\chi^2 - \chi^3) - \frac{\pi^2}{4}\chi(2-\chi)\Big]\Big\}d\Omega \quad (27)$$

$$+ \text{ terms obtained by the interchange } \chi \to 1-\chi;\ \Phi \rightleftarrows \Phi_b,$$

where $\chi = \sin^2(\vartheta/2)$.

In the same limiting case the cross section for scattering of electrons by positrons is given by the formula:*

$$d\sigma^{e-p}_{\text{l. s.}} = \frac{\alpha^2(1-v^2)}{4\chi^2}\Big\{(1 - \chi + \chi^2)^2 + \frac{\alpha}{\pi}\Big[4(1 - 2\Phi + 2\Phi_b - 2\Phi_a)\ln\frac{1}{2\Delta E}(1 - \chi + \chi^2)^2 - \Phi_a^2(2 - 9\chi + 19\chi^2 - 15\chi^3 + 6\chi^4) - \Phi^2(6 - 15\chi + 19\chi^2 - 9\chi^3 + 2\chi^4) - 2\Phi_b^2(1 - 3\chi + 4\chi^2 - 3\chi^3 + \chi^4) - 2\Phi\Phi_a(10 - 17\chi + 24\chi^2 - 17\chi^3 + 10\chi^4) + 2\Phi\Phi_b(6 - 12\chi + 13\chi^2 - 6\chi^3 + 2\chi^4) + 2\Phi_a\Phi_b(2 - 6\chi + 13\chi^2 - 12\chi^3 + 6\chi^4) + \frac{\Phi}{3}(28 - 42\chi + 51\chi^2 - 23\chi^3 + 6\chi^4) + \frac{\Phi_a}{3}(6 - 23\chi + 51\chi^2 - 42\chi^3 + 28\chi^4) + \Phi_b(2 - 5\chi + 6\chi^2 - 5\chi^3 + 2\chi^4) - \frac{37}{9}(1 - \chi + \chi^2)^2 + \frac{\pi^2}{4}\chi^2(5 - 6\chi + 4\chi^2)\Big]\Big\}d\Omega. \quad (28)$$

In the limiting case of large energies and small

*Equations (20), (21), (27), and (28) have also been obtained in Ref. 9.

scattering angles ($p \gg 1$, $p\sin(\vartheta/2) \gtrsim 1$), the cross section for scattering of electrons in the l. s. has the form

$$d\sigma^{e-e}_{\text{l. s.}} = \frac{\alpha^2(1-v^2)}{4\sin^4(\vartheta/2)}(1-\delta_R)\,d\Omega; \tag{29}$$

$$\delta_R = \frac{\alpha}{\pi}\Big[2\left(1-\frac{1}{3}\operatorname{cth}^2\Phi\right)(1-\Phi\operatorname{cth}\Phi)-\frac{2}{9}$$
$$+4(2\Phi\operatorname{cth}2\Phi-1)\left(\ln\frac{1}{2\Delta E}-1\right)+4\Phi_a(2\Phi\operatorname{cth}2\Phi-1)$$
$$-\left(1+\frac{4\Phi}{\operatorname{sh}4\Phi}\right)+\frac{4}{\operatorname{th}2\Phi}\int_0^{\Phi}x\operatorname{th}x\,dx+\frac{2}{\operatorname{th}2\Phi}\int_0^{2\Phi}x\operatorname{cth}x\,dx\Big].$$

In the c. m. s. a simpler formula is obtained.[10]

Equation (29) also gives the cross section for scattering of electrons by positrons in the limiting case of large energies and small scattering angles.

In the limiting case

$$\ln 2p \gg 1;\quad \ln\left(2p\sin\frac{\vartheta}{2}\right)\gg 1; \tag{30}$$
$$\ln\left(2p\cos\frac{\vartheta}{2}\right)\gg 1$$

the cross section for scattering of electrons by electrons and positrons is of the form

$$d\sigma_{\text{l. s.}} = d\sigma_0(1-\delta_R), \tag{31}$$

where $d\sigma_0$ is the cross section for the main effect, and $\delta_R = \delta_1+\delta_2+\delta_3+\delta_4+\delta_5+\delta_{\text{inelas.}}$ (δ_2 corresponds to diagram 2 in Fig. 1, δ_3 to diagram 3, ... , and $\delta_{\text{inelas.}}$ to diagrams 6 and 7):

$$\delta_2 = -\frac{4\alpha}{\pi}\left(2\Phi_a\ln\frac{1}{\lambda}+\Phi_a^2\right);$$
$$\delta_3 = \frac{4\alpha}{\pi}\left(2\Phi_a\ln\frac{1}{\lambda}+\Phi_a^2\right);$$
$$\delta_4 = -\frac{4\alpha}{\pi}\frac{\Phi_a}{3};\quad \delta_5 = \frac{4\alpha}{\pi}\left(2\Phi_a\ln\frac{1}{\lambda}+\Phi_a^2\right);$$
$$\delta_{\text{inelas}} = \frac{4\alpha}{\pi}\left(2\Phi_a\ln\frac{\lambda}{2\Delta E}+\frac{5}{2}\Phi_a^2\right);\quad \Phi_a = \ln 2p,$$

so that

$$\delta_R = \frac{\alpha}{\pi}\left(8\ln 2p\ln\frac{1}{2\Delta E}+14\ln^2 2p\right). \tag{32}$$

In conclusion, the writer expresses his gratitude to Prof. A. I. Akhiezer for much aid in performing the present research.

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Translated by W. H. Furry
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