only for $j = 0$ is the commonly used condition[4] necessary that $\int \lambda^{-2}\rho(\lambda^2)\,d\lambda^2$ should not diverge at the upper limit. For a collision with an angular momentum $j > 0$ the requirements imposed on the function $\rho(\lambda^2)$ may be relaxed.

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On the Reduction of Wave Equations for Spin 0 and 1 to the Hamiltonian Form

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RECENTLY Schrödinger[1] and Case[2] have shown that equations for particles of spin 0 and 1

$$(\beta_k \nabla_k + \varkappa)\psi = 0, \tag{1}$$

the matrices β_k in which satisfy the well-known conditions of Duffin

$$\beta_i\beta_k\beta_l + \beta_l\beta_k\beta_i = \delta_{ik}\beta_l + \delta_{kl}\beta_i \tag{2}$$

may be reduced to the Hamiltonian form

$$H\psi = i\dot{\psi}, \tag{3}$$

where H is the Hamiltonian operator which has the form

$$H = \gamma_\alpha p_\alpha + \varkappa\gamma_4 \quad (\alpha = 1, 2, 3), \tag{4}$$

while the matrices γ_k ($k = 1, 2, 3, 4$) also satisfy conditions (2). To Eq. (3) we must also add the initial condition of the form

$$(H\gamma_4 - \varkappa)\psi = 0, \tag{5}$$

which, in consequence of (3), holds at any arbitrary instant of time. These results are evidently of interest, in particular because they allow one to formulate a theory of particles with spin 0 and 1 (to a large extent by analogy with the well-developed theory of Dirac). In the work by Schrödinger[1] and Case[2] the Hamiltonian form (3) is obtained by resolving Eqs. (1) into their components for a certain specific choice of the matrices β_k. Such a noninvariant method of derivation is unnecessarily awkward and requires separate calculations for spin 0 and spin 1. Moreover, the connection between the matrices γ_k and the initial matrices β_k remains unclear in this method of procedure. The object of the present note consists of showing the method by means of which the reduction to the form (3)-(5) may be carried out simultaneously for spin 0 and 1 without introducing components, and solely on the basis of Eq. (1) and the algebra (2). In the course of the derivation the relation between the matrices γ_k and β_k is also established. The results are generalized to the case of zero rest mass.

Let us write (1) in the form $(\beta_\alpha\nabla_\alpha + \beta_4\nabla_4 + \varkappa)\psi = 0$ and multiply it by $(1 - \beta_4^2)$ and by β_4^2. Taking into account the fact that $\beta_4\beta_\alpha\beta_4 = 0$ we shall obtain

$$[((\beta_4\beta_\alpha - \beta_\alpha\beta_4)\nabla_\alpha + \varkappa\beta_4)\beta_4 - \varkappa]\psi = 0, \tag{6}$$

$$\beta_4[((\beta_4\beta_\alpha - \beta_\alpha\beta_4)\nabla_\alpha + \varkappa\beta_4) + \nabla_4]\psi = 0. \tag{7}$$

On the basis of (2) it is easily seen that the matrices

$$\gamma_\alpha = i(\beta_4\beta_\alpha - \beta_\alpha\beta_4), \quad \gamma_4 = \beta_4 \tag{8}$$

satisfy the same conditions (2) as do β_k and are likewise Hermitian. We note that (8) is easily solved with respect to β_α and gives $\beta_\alpha = -i(\gamma_4\gamma_\alpha - \gamma_\alpha\gamma_4)$. Evidently (6) can be rewritten with the aid of (8) in the form (5) where H is defined by (4) and $p_\alpha = -i\nabla_\alpha$. Thus the initial condition (5) is the result of multiplying (1) by $(1 - \beta_4^2)$. The relation (7) takes on the form

$$\gamma_4(H + \nabla_4)\psi = 0. \tag{9}$$

We operate on (5) by the operator $\nabla_4 = -i\,\partial/\partial t$, on (9) by the operator H and subtract the results. Taking into account that

$$H\gamma_4 H = \varkappa H, \tag{10}$$

we shall obtain the fundamental Eq. (3).

The method presented above may also be extended to the generalized wave equations of Lehmann[3]

$$(\beta_k \nabla_k + \beta_0)\psi = 0, \quad |\beta_0| = 0 \tag{11}$$

for particles with zero rest mass and with spin 0 and 1. The matrices β_k of these equations satisfy conditions (2), while the matrix β_0 may always be chosen in a projective form: $\beta_0^2 = \beta_0$ (see Ref. 3). For spin 0, Eqs. (11) take on the form

$$\nabla_i \psi_i = 0, \quad \nabla_i \psi_0 + \psi_i = 0 \tag{12}$$

and for spin 1 (Maxwell's equations)

$$\nabla_k \psi_{ik} = 0, \quad \nabla_k \psi_i - \nabla_i \psi_k + \psi_{ik} = 0 \tag{13}$$

[see Ref. 3, Eqs. (8) and (10)]. It is easy to show that in these cases β_0 has the following properties:

$$\beta_0\beta_k + \beta_k\beta_0 = \beta_k; \quad \beta_0\beta_k = \beta_k(1-\beta_0); \tag{14}$$

$$(1-\beta_0)\beta_k = \beta_k\beta_0;$$

from this and from $\beta_0^2 = \beta$ the following relations are also obtained

$$\beta_0\beta_k\beta_0 = 0, \quad \beta_0\beta_i\beta_k = \beta_i\beta_k\beta_0. \tag{15}$$

Multiplying, as before, (11) by $(1-\beta_4^2)$ and by β_4^2, we will correspondingly obtain [taking into account (14) and (15)],

$$[(\gamma_\alpha p_\alpha + \beta_0\gamma_4)\gamma_4 - \beta_0]\psi = 0, \tag{16}$$

$$\gamma_4[(\gamma_\alpha p_\alpha + \gamma_4\beta_0) + \nabla_4]\psi = 0. \tag{17}$$

Here γ_k are defined by (7), and therefore $\beta_0\gamma_\alpha = \gamma_\alpha\beta_0$, $\beta_0\gamma_4 = \gamma_4(1-\beta_0)$. Further,

$$(\gamma_\alpha p_\alpha + \beta_0\gamma_4)(17) - \nabla_4(16) \rightarrow (\gamma_\alpha p_\alpha + \nabla_4)\beta_0\psi = 0. \tag{18}$$

On the other hand,

$$(1-\beta_0)(16) \rightarrow \gamma_\alpha p_\alpha \gamma \beta_0 \psi = 0. \tag{19}$$

We introduce the notation

$$\beta_0\psi = \psi_0, \tag{20}$$

$$\gamma_\alpha p_\alpha = H. \tag{21}$$

As a result of this, Eqs. (18) and (19) take on the form

$$H\psi_0 = i\dot{\psi}_0, \tag{22}$$

$$H\gamma_4\psi_0 = 0. \tag{23}$$

Since in this case

$$H\gamma_4 H = 0, \tag{24}$$

the initial condition (23) is maintained for all time in consequence of (22).

Thus also in the case of particles of zero rest mass one may obtain from Eqs. (11) the Hamiltonian form (22) with the initial condition (23), not for the complete wave function ψ but only for its part $\psi_0 = \beta_0\psi$.

The following circumstance is of interest. Evidently (11) cannot be obtained from (1) by equating to zero the constant $\varkappa = mc/h$ which is proportional to the mass of the particle, even though (11) [in contrast to (1)] describes particles of zero rest mass (see Ref. 4). The situation is different if we adopt a description by means of equations in the Hamiltonian form (3), (5) and correspondingly (21), (23). Indeed, (22) and (23) are directly obtained from (3) and (5) by setting $\varkappa = 0$ (and replacing ψ by ψ_0). The same also holds for relations (22) and (24),[0] which are special cases of the corresponding relations (4) and (10) with $\varkappa = 0$.

By decomposing Eq. (22) for the case of spin 0 [see (12)], we shall obtain

$$-p_\alpha\psi_4 = \dot{\psi}_\alpha, \quad p_\alpha\psi_\alpha = \dot{\psi}_4, \tag{25}$$

while (23) gives identically zero, i.e., the initial conditions in this case disappear. It may be easily seen that the four equations (25) are equivalent to the five equations (12). Similarly, relation (22) for the case of Maxwell's equations (13) takes on the form

$$p_\alpha\psi_{\beta_4} - p_\beta\psi_{\alpha 4} = \dot{\psi}_{\alpha\beta}, \quad p_\beta\psi_{\alpha\beta} = \dot{\psi}_{\alpha 4}, \tag{26}$$

while (23) gives $p_\alpha\psi_{\alpha 4} = 0$. Introducing the three-dimensional notation $\mathrm{E} = (i\psi_{\alpha 4})$, $\mathrm{H} = (\psi_{\alpha\beta})$, we obtain for the Hamiltonian equations for the electromagnetic field curl $\mathrm{H} = \dot{\mathrm{E}}$, $-$ curl $\mathrm{E} = \dot{\mathrm{H}}$ with the "initial condition" div E = 0.

Note added in proof: After this letter had been submitted for publication the author learned of an analogous article by Case (see Ref. 5) in which, however, the case of zero rest mass is not considered.

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On the Theories of Multiple Meson Production

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IN the calculations of effective cross sections for multiple production of mesons, integrations are carried out over the momenta (or over the energies) and over the angles of emergence of the product particles. Since exact calculations, taking into account all the conservation laws, are accompanied by great difficulties, various approximate methods of calculation are used. In such calculations the maximum limit for the energy of one of the particles is taken to be equal to the value $\epsilon_n = \mathcal{E} - \sum_{i=1}^{n-1} m_i$ ($\mathcal{E}$ is the total energy of colliding particles, Σm_i is the sum of the rest masses of all the product particles except the given one). The object of this article is to show that these values should be diminished as required by the laws of conservation of energy and momentum. The calculation of the maximum energy and momentum for each product particle is reduced to the problem of finding a conditional extremum.

We shall carry out the calculation in the system of the center of mass. We assume that after the collision n particles in all are formed. We denote the total energy, the momentum and the velocity of the ith product particle by ϵ_i, $\mathbf{k}_i$, v_i, and the corresponding quantities for the product particle being investigated we shall denote by the index n. Then the laws of the conservation of energy and momentum will take on the form:

$$\varepsilon_n + \sum_1^{n-1} \varepsilon_i = \mathcal{E}, \quad \mathbf{k}_n + \sum_1^{n-1} \mathbf{k}_i = 0. \tag{1}$$

It is evident that in order to make k_n a maximum it is necessary that the momenta of all the $(n-1)$ particles should be directed opposite to the momentum $\mathbf{k}_n$; then

$$k_n = -\sum_1^{n-1} k_i. \tag{2}$$

Let us find v_n:

$$v_n = k_n / \varepsilon_n = \sum_1^{n-1} k_i / (\mathcal{E} - \sum_1^{n-1} \varepsilon_i). \tag{3}$$

It is necessary to find the maximum of the function $v_n(k_1, \ldots, k_{n-1})$ which is defined by Eq. (3) under the condition

$$\Big[m_n^2 + \Big(\sum_1^{n-1} k_i\Big)^2\Big]^{1/2} + \sum_1^{n-1} (m_i^2 + k_i^2)^{1/2} = \mathcal{E}, \tag{4}$$

which follows from (1) and from the relations $\epsilon_i = m_i^2 + k_i^2$. The problem is reduced to the solution of the system of equations

$$\partial F(k_1, \ldots, k_{n-1}) / \partial k_j = 0, \quad (j = 1, \ldots, n-1), \tag{5}$$

where $F \equiv v_n + \lambda \mathcal{E}$.

In its expanded form Eq. (5) takes on the form:

$$\frac{1}{\mathcal{E} - \Sigma\varepsilon_i} + \frac{v_j \Sigma k_i}{(\mathcal{E} - \Sigma\varepsilon_i)^2} + \lambda \left[\frac{\Sigma k_i}{\mathcal{E} - \Sigma\varepsilon_i} + v_j\right] = 0. \tag{6}$$

Since λ must satisfy all the equations of the system the equations must be satisfied identically; from this it follows that $v_1 = v_2 = \ldots = v_{n-1} = v$. Then the $(n-1)$ particles may be considered as a single particle with the mass $M = \sum_{i=1}^{n-1} m_i$ and the velocity v.

From (4) we find

$$k_{n\,\max} = [(\mathcal{E}^2 - M^2 + m_n^2)^2 - 4m_n^2 \mathcal{E}^2]^{1/2} / 2\mathcal{E}, \tag{7}$$

$$\varepsilon_{n\,\max} = {}^1/_2 [\mathcal{E} - (M^2 - m_n^2) / \mathcal{E}],$$

$$v_{n\,\max} = [(\mathcal{E}^2 - M^2 + m_n^2)^2 - 4m_n^2\mathcal{E}^2]^{1/2} / (\mathcal{E}^2 - M^2 + m_n^2).$$

Sternheimer[1] has calculated the maximum angle of recoil of the nucleon after a collision.

$$\operatorname{tg} \theta_{\max} = (1 - v_c^2)^{1/2} \, (v_c^2 / v_{\max}^2 - 1)^{-1/2} \tag{8}$$