It is evident from (6) and (7) that, thanks to a consideration of the field of virtual photons, the classical quantities x and p_x are operators which do not commute with each other:

$$p_x x - x p_x = \frac{\hbar}{i}\frac{2}{\pi}\gamma\omega_0 J, \qquad (8)$$

$$J = \int_0^\infty \frac{y^2 dy}{(y^2-1)^2 + \gamma^2\omega_0^2 y^6}.$$

In deriving the last formula, we took into consideration the permutation relations

$$[a_s a_{s'}^+]_- = \delta_{ss'} - \varkappa_s^0 \varkappa_{s'}^0.$$

The integral J does not diverge and has the value

$$J = \pi/2\gamma\omega_0 + O(\gamma\omega_0). \qquad (9)$$

Therefore, in first approximation, we obtain permutation relations which coincide with the permutation relations (1) of wave theory.

Making use of the operator expressions (6) and (7) for x and p_x, we can obtain the energy levels for the harmonic oscillator. If the momentum is given, not in the form (8), but in the form

$$p_x = m\dot{x}, \qquad (10)$$

as was done, for example, in Ref. 8, then an additional factor of 2 appears on the right side of Eq. 1. It is of interest to note that when photons are absent ($a_s^+ a_{s'} = 0$) it is better to use Eq. (7) for the momentum in the expression of zero point energy

$$E_0 = (p_x^2/2m) + \tfrac{1}{2}m\omega_0^2 x^2, \qquad (11)$$

rather than Eq. (10) as would have been more natural in the given case.

Giving the momentum by Eq. (7), we find that the zero-point energy will automatically contain the necessary subtraction terms, leaving the finite quantity

$$E_0 = \frac{\hbar\omega_0}{2} + \frac{\hbar\omega_0^2 e^2}{3\pi c^3 m}\left(\ln\frac{3c^3 m}{2e^2\omega_0} - 1\right). \qquad (12)$$

The first term is the well-known expression for the zero-point energy without vacuum terms, and the second is an additional energy caused by the vacuum action.

A strict quantum electrodynamical derivation of the corresponding formula gives an expression very close to Eq. (12):

$$E_0^{qm} = \frac{\hbar\omega_0}{2} + \frac{\hbar\omega_0^2 e^2}{3\pi c^3 m}\left(\ln\frac{mc^2}{\hbar\omega_0} - 0{,}2\right). \qquad (13)$$

It is of interest to observe that if we limit the integral of type (9) (which is also encountered in the calculation of the zero-point energy), to relativistic frequencies of vibration $\omega_{max} = 2mc^2/\hbar$, as was done by Weisskopf[9], then we obtain Eq. (13) for the additional energy, with the help of the fluctuation method.

Thus a classical system which describes the motion of an electron in its interaction with the second quantized field of photons (actually radiated, or only virtual) is the same as in quantum mechanics.

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Radiation Resonance in Synchrotrons

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THE present research deals with a resonance which can take place under certain conditions in synchrotrons, due to the presence of radiation, and which can bring about amplification of the amplitude of betatron oscillations. The main points are the following:

1. In the absence of radiation, the betatron oscillations in synchrotrons are described by an equation of the form

$$(d^2x / d\zeta^2) + F(\zeta)x = 0, \tag{1}$$

where x is the departure from the equilibrium orbit, $F(\zeta)$ is some periodic function which is determined by the nature of the change in the magnetic field, ζ is a parameter which characterizes the position of a particle in the synchrotron. We shall consider that the synchrotron consists of N identical sectors; in each of the sectors, the parameters ζ changes by 2π. Then one revolution of the particle in the synchrotron corresponds to a change of ζ by $2N\pi$, $F(\zeta + 2\pi) = F(\zeta)$.

As is known from the theory of Flock, solutions of Eq. (1) take the form

$$x(\zeta) = e^{i\mu\zeta} g(\zeta), \quad g(\zeta + 2\pi) = g(\zeta), \tag{2}$$

where μ is a parameter which is determined by the behavior of $F(\zeta)$. The problem of the investigation of the stability of the motion described by Eq. (1) consists of the calculation of the value of μ. Choosing μ in the form of a function of the parameters which characterize the synchrotron, we obtain the conditions for which the motion is stable, i.e,. is suitable for practical use.

Equation (1) does not hold in the presence of radiation. In such a case we must consider the effect on the system of the radiation and of the accelerating interval and subsequently, investigate the motion of the system towards stability.

2. We assume that we can neglect the quantum character of the radiation and consider the radiation as a continuous process of energy loss. Then Eq. (1) is somewhat modified by the replacement of $F(\zeta)$ by another function $F_1(\zeta)$ with the same period, and a periodic function of ζ with period $2\pi N$ (for definiteness, we consider the case of a single accelerating interval) appears on the right-hand side. If, for the modified homogeneous equation, the parameter of stability is denoted by μ_1 (μ_1 is always close to μ), then for

$$\mu_1 N = p, \quad p = 0, \pm 1, \pm 2, \ldots \tag{3}$$

in the solution of the equation with the right-hand side, there appear long-lasting terms, i.e., resonance takes place. The amplitude of the betatron oscillations begins to increase rapidly.

By way of an example, we consider a synchrotron with four sections with soft focussing. We denote by l the lengths of the rectalinear sections, by R the radius of curvature of the curved sections and by $n = -(r/H)(dH/dr)$ the power of the decay of the magnetic field. Then the condition for stability of motion for the calculation without consideration of radiation is given by the following inequality:

$$\frac{l}{R} < 2(1-n)^{-1/2} \times \left[1 + \cos\frac{\pi}{2}\sqrt{1-n}\right] \Big/ \sin\frac{\pi}{2}\sqrt{1-n}. \tag{4}$$

Radiation resonance occurs when

$$\frac{l}{R} = 2(1-n+\varepsilon)^{-1/2} \times \cos\left(\frac{\pi}{2}\sqrt{1-n+\varepsilon}\right) \Big/ \sin\left(\frac{\pi}{2}\sqrt{1-n+\varepsilon}\right). \tag{5}$$

In this equality, ϵ is a very small quantity which changes slowly in the process of the acceleration cycle. Thus at different stages of the acceleration, different values of l/R are dangerous, i.e., there is a region of resonance values. This region of resonance values lies within the region of stability.

3. What has been demonstrated above is valid when the quantum character of the radiation is neglected. It is now clear what changes in this treatment are brought about by consideration of the quantum character of the radiation, because it is known[1] that the latter influences the motion of the electron.

The physical picture of the resonance described above is that the particle passes successively through accelerating regions in one and the same phase of its betatron oscillation. It can be calculated that in one revolution the mean phase change $\langle(\Delta\varphi^2)\rangle_{av}^{1/2}$ is in the thousandths of a radian. Thus for $E = 1$ bev, $A = 1$ cm and $n = 0.6$, $\langle(\Delta\varphi)^2\rangle_{av}^{1/2} \approx 1.2 \times 10^{-3}$ radian. This means that during one thousand revolutions, the quantum character of the radiation does not lead to the departure of the particle from resonance.

On the basis of all that has been said, it can be concluded that the resonance described above is dangerous for the operation of the synchrotron. In practice, one should stay as far away from this resonance as possible. We have called it radiative resonance because it is determined by radiation and is more intense the more intense the radiation.

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