Possible States of a Two-Graviton System

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(Submitted to JETP editor April 18, 1955)
J. Exptl. Theoret. Phys. (U.S.S.R.) 30, 1098-1103 (June, 1956)

Eigenfunctions are obtained for the angular momentum operator of one and two gravitons. The states of a single graviton are separated, according to their parity, into quasielectric and quasimagnetic states. The possible states of a two-graviton system are investigated. The Clebsch-Gordan coefficients are calculated for $S = 3$ and $S = 4$.

AN investigation by Landau[1] concerning the angular momentum of a system of two photons led to the discovery of certain general selection rules limiting the possible values of angular momentum and parity for such a system. These rules appear as the consequence of specific characteristics of photons, associated with their zero rest mass.

Since gravitons, like photons, have zero rest mass, it would be of interest to look into the analogous problem of determining the limitations on the possible values of angular momentum and parity for a two-graviton system. By using generalized spherical vectors, as in earlier work, it is possible to obtain eigenfunctions of the angular momentum operator for one and two gravitons. The rules prohibiting certain states, characterized by definite values of total angular momentum and parity, follow from the apparent form of these functions.

In the linear approximation of the general theory of relativity, the tensor of the gravitational field $g_{\mu\nu}$, coincident with the metric tensor, departs little from its Galilean values $\delta_{\mu\nu} : g_{\mu\nu} = \delta_{\mu\nu} = h_{\mu\nu}$. Investigating this equation for a particle with zero rest mass and spin 2, Fierz[2] showed that in a gauge transformation of the field

$$h_{\mu\nu} = h'_{\mu\nu} + \partial\Lambda_{\mu}/\partial x_{\nu} + \partial\Lambda_{\nu}/\partial x_{\mu},$$

where Λ_μ is an arbitray vector, only two components of the tensor $h_{\mu\nu}$ can remain different from zero, in accordance with the two possible values of polarization. Thus a weak graviational field is described by a three-dimensional tensor of second rank, satisfying the condition of orthogonality, with a trace equal to zero.

The operator of the total angular momentum $\hat{M}_{ij}$ of a single graviton is defined as a transformation of the tensor function by an infinitesimal three-dimensional rotation

$$(kl \mid \hat{M}_{ij} \mid rs) = \frac{1}{i}\delta_{kr}\delta_{ls}\left(p_i\frac{\partial}{\partial p_j} - p_j\frac{\partial}{\partial p_i}\right) - \frac{1}{i}(\delta_{jk}\delta_{ir}\delta_{ls} - \delta_{ik}\delta_{jr}\delta_{ls} + \delta_{jl}\delta_{is}\delta_{kr} - \delta_{il}\delta_{js}\delta_{kr}) = (kl \mid \hat{L}_{ij} + \hat{S}_{ij} \mid rs), \quad (1)$$

where p_i is the momentum of a graviton, and kl, rs are tensor indices taking the values 1, 2, 3. In configuration space the angular momentum operator (1) operates on the single-graviton wave function $\psi_{rs}(p)$.

The division of the total angular momentum into an orbital part $\hat{L}_{ij}$ and a spin part $\hat{S}_{ij}$ has limited physical meaning, since there is no rest system for a graviton, and therefore the usual definition of spin is not applicable. Furthermore, a state with a definite value of orbital and spin angular momentum does not satisfy the condition of orthogonality, so that only certain superpositions of these states have physical meaning. However, this separation enables us to construct eigenfunctions of the total angular momentum from more simple eigenfunctions of the orbital and spin angular momenta. The spherical harmonics $\Phi_{Lm} = aY_{Lm}(n)$ are eigenfunctions of the operators $\hat{L}^2_{ij}$ and $\hat{L}_z$, where L is the value of the orbital angular momentum, m is its projection on the z axis, and $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$.

$$\int Y^*_{Lm} Y_{L'm'}\, d\omega = \delta_{LL'}\,\delta_{mm'}; \qquad \int_{-\infty}^{\infty} a^2 n^2\, dn = 1.$$

From the equations

$$(kl \mid {}^1/_2\, \hat{S}^2_{ij} \mid rs)\,\chi_{\mu,\,rs} = S(S+1)\,\chi_{\mu,\,kl} = 6\chi_{\mu,\,kl}, \quad (2)$$

$$(kl \mid \hat{S}_z \mid rs)\,\chi_{\mu,\,rs} = \mu\chi_{\mu,\,kl}, \qquad \mu = 0,\ \pm 1,\ \pm 2.$$

(where μ is the projection of the spin $S = 2$ on the z-axis) the following eigenfunctions of the spin operators are obtained:

$$\chi_{0,\,kl}=\frac{1}{\sqrt{6}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix};\tag{3}$$

$$\chi_{\pm1,\,kl}=\mp\frac{1}{2}\begin{pmatrix}0&0&1\\0&0&\pm i\\1&\pm i&0\end{pmatrix};$$

$$\chi_{\pm2,\,kl}=-\frac{1}{2}\begin{pmatrix}1&\pm i&0\\\pm i&-1&0\\0&0&0\end{pmatrix}$$

normalized so that $\chi^*_{\mu,kl}\,\chi_{\mu',\,kl}=\delta_{\mu\mu'}$. We write the wave function of a single graviton in compliance with the Clebsch-Gordan formula for the separation of the direct product of the representations $D_L\times D_S$ into irreducible parts:

$$\psi_{kl}^{JLM}=a\sum_{\mu}C_{JM}^{L,\,M-\mu;\,2,\,\mu}\,Y_{L,\,M-\mu}\,\chi_{\mu,\,kl},\tag{4}$$

where J is the value of the total angular momentum, M is its projection on the z-axis, L is the value of the orbital angular momentum and the $C^{L,M-\mu;\,2,\mu}_{J,M}$ are the Clebsch-Gordan coefficients, given, for example, in Ref. 3.

Taking advantage of the definition of the generalized spherical S-vectors of type λ[4]

$$(Y^S_{J,\,J+\lambda,\,M})_\mu\tag{5}$$
$$=(-1)^\lambda\left[\frac{2J+1}{2J+2\lambda+1}\right]^{1/2}C^{J,\,M;\,S,\,\mu}_{J+\lambda,\,M+\mu}\,Y_{J+\lambda,\,M+\mu},$$

we rewrite the eigenfunction of a single graviton in the form

$$\psi_{kl}^{JLM}=a\sum_{\mu}(Y^2_{JLM})^{\mu}\,\chi_{\mu,\,kl},\tag{6}$$
$$(Y^2_{JLM})^{\mu}=(-1)^{\mu}\,(Y^2_{JLM})_{-\mu}.$$

From the functions ψ^{JLM}_{kl} it is possible to construct two linearly independent combinations, satisfying the orthogonality condition

$$n_k\,\psi^{JM}_{kl}=0;\quad \psi^{JM}_{kl}=a\sum_{\mu}(Y^{JM})^{\mu}\,\chi_{\mu,\,kl};\tag{7}$$
$$Y^{JM}=\sum_{L}\rho_L\,Y_{JLM};$$
$$Y^{(1)\,JM}=\left[\frac{J+2}{2J+1}\right]^{1/2}Y_{J,\,J-1,\,M}+\left[\frac{J-1}{2J+1}\right]^{1/2}Y_{J,\,J+1,\,M};$$
$$Y^{(2)\,JM}=-\left[\frac{(J+1)(J+2)}{(2J+1)(2J-1)}\right]^{1/2}Y_{J,\,J-2,\,M}+\left[\frac{6(J-1)(J+2)}{(2J-1)(2J+3)}\right]^{1/2}Y_{J,\,J,\,M}+\left[\frac{J(J-1)}{(2J+1)(2J+3)}\right]^{1/2}Y_{J,\,J+2M}.$$

The two solutions (7) differ in parity. For tensor fields the inversion operator I is defined in the following way:

$$I\,\psi_{kl}(\mathbf{n})=\psi_{kl}(-\mathbf{n}).$$

Noting the relationship $Y_{Lm}(-\mathbf{n})=(-1)^L Y_{Lm}(\mathbf{n})$, we obtain

$$IY^{(1)\,JM}=(-1)^{J+1}\,Y^{(1)\,JM},$$
$$IY^{(2)\,JM}=(-1)^{J}\,Y^{(2)\,JM}.$$

By analogy with the solutions for a single photon[5] we call $Y^{(1)\,JM}$ a state of quasimagnetic type, $Y^{(2)JM}$ a state of quasielectric type.

As was to be expected, both solutions (7) are meaningful only for values of $J\geq 2$, that is, the total angular momentum carried away by a graviton cannot be less than two. This follows from the quadrupole character of graviton emission.

Two free gravitons are desribed by a tensor of fourth rank

$$\psi_{klmn}(\mathbf{p}_1\mathbf{p}_2)=\psi_{mnkl}(\mathbf{p}_2\mathbf{p}_1).$$

In the center-of-mass system $\mathbf{p}_1+\mathbf{p}_2=0$ the wave function will depend only on the relative momentum $\mathbf{p}_1-\mathbf{p}_2=2\mathbf{p}$, and the orbital angular momentum can be represented in the form

$$\hat{L}_{ij}=-i\left(p_i\frac{\partial}{\partial p_j}-p_j\frac{\partial}{\partial p_i}\right).\tag{8}$$

We obtain the following expression for the spin operator of two gravitons:

$$(klmn\,|\,\hat{S}_{ij}\,|\,pq\,rs)\tag{9}$$
$$=(kl\,|\,\hat{S}_{ij}\,|\,pq)\,\delta_{mr}\,\delta_{ns}+(mn\,|\,\hat{S}_{ij}\,|\,rs)\,\delta_{kp}\,\delta_{lq}$$
$$=-i\,(\delta_{jp}\,\delta_{ik}\,\delta_{ql}-\delta_{ip}\,\delta_{jk}\,\delta_{ql}-\delta_{iq}\,\delta_{pk}\,\delta_{jl}+\delta_{jq}\,\delta_{pk}\,\delta_{il})\,\delta_{rm}\,\delta_{sn}$$
$$-i\,(\delta_{jr}\,\delta_{im}\,\delta_{sn}-\delta_{ir}\,\delta_{jm}\,\delta_{sn}-\delta_{is}\,\delta_{rm}\,\delta_{jn}+\delta_{js}\,\delta_{rm}\,\delta_{in})\,\delta_{pk}\,\delta_{ql}.$$

Taking advantage of the fact that the eigenfunctions of the orbital angular momentum are the spherical harmonics Y_{Lm}, and that those of the spin angular momentum are the quadratic combinations $\chi_{\mu_1,kl}\,\chi_{\mu_2,\,mn}$, we construct solutions for two gravitons from the Clebsch-Gordan formula:

$$\psi^{JLMS}_{klmn}=a\sum_{\mu}C^{L,\,M-\mu;\,S,\,\mu}_{JM}\,Y_{L,\,M-\mu}\,\chi^S_{\mu,\,klmn},\tag{10}$$

$$\chi^S_{\mu,\,klmn}=\sum_{\mu_2}C^{2,\,\mu-\mu_2;\,2\mu_2}_{S,\,\mu}\,\chi_{\mu-\mu_2,\,kl}\,\chi_{\mu_2,\,mn}.$$

Here J is the total angular momentum of the two gravitons, L is the orbital angular momentum, M is the projection of J on the z-axis, S is the spin of the two-graviton system and μ is the projection of S on the z-axis. In view of the normalization properties

$$\chi^{*S}_{\mu,\,klmn}\,\chi^{S'}_{\mu',\,klmn} = \delta_{SS'}\,\delta_{\mu\mu'},$$
$$\int (Y^{*S}_{JLM})^{\mu}\,(Y^{S}_{J'L'M'})^{\mu}\,d\omega = \delta_{JJ'}\,\delta_{LL'}\,\delta_{MM'},$$

it can be shown that the functions (10) are normalized in the following way:

$$\int \psi^{*JLMS}_{klmn}\,\psi^{J'L'M'S'}_{klmn}\,dp = \delta_{JJ'}\,\delta_{LL'}\,\delta_{MM'}\,\delta_{SS'}.$$

If we take into account the orthogonality requirement

$$n_k\psi^{JM}_{klmn} = 0;\quad \psi^{JM}_{klmn} = \sum_{L,\,S}\rho_{L,\,S}\,\psi^{JLMS}_{klmn} \tag{11}$$
$$= a\sum_{\mu,S}(Y^{JMS})^{\mu}\,\chi^{S}_{\mu,\,klmn},\quad Y^{JMS} = \sum_{L}\rho_{L,\,S}\,(Y^{S}_{JLM})$$

we can determine the apparent form of all 25 coefficients $\rho_{L,S}$ for the five possible values of L : J, $J \pm 1$, $J \pm 2$, and the five values of S: 0, 1, 2, 3, 4. It should be pointed out that for two identical particles, the parity of the orbital angular momentum conforms with that of the total spin.

For determining the coefficients $\rho_{L,S}$ we require the Clebsch-Gordan coefficients for $S = 3$, calculated in Ref. 6 for $S = 4$, which can be calculated by the usual methods from the formulas (3.110) and (3.111) in Ref. 3. By comparing the coefficients of identical spherical harmonics with zero, we obtain from the requirement (11) the result that, for odd states with $L = 2k + 1$, and for odd total angular momenta J,

$$\rho_{J,\,1} = \rho_{J,\,3} = \rho_{J\pm 2,3} = 0.$$

Thus the state $L = 2k + 1$, $J = 2n + 1$ is forbidden for any J.

In an anlogous manner we obtain the coefficients $\rho_{L,S}$ for odd states with even total angular momentum, that is, we determine the following functions, normalized to unity:

$$\begin{aligned} Y^{IJ,\,M,\,3} &= \left[\frac{3J(J-1)(J+1)}{4(2J-3)(2J+1)(2J+3)}\right]^{1/2} Y^3_{J,\,J-1,\,M} \\ &- \left[\frac{3J(J+1)(J+2)}{4(2J-1)(2J+1)(2J+5)}\right]^{1/2} Y^3_{J,\,J+1,\,M} \\ &- \left[\frac{5J(J-1)(J-2)}{4(2J-3)(2J-1)(2J+1)}\right]^{1/2} \times Y^3_{J,\,J-3,\,M} \\ &+ \left[\frac{5(J+1)(J+2)(J+3)}{4(2J+1)(2J+3)(2J+5)}\right]^{1/2} Y^3_{J,\,J+3,\,M}. \end{aligned} \tag{12}$$

This solution admits odd states with even angular momentum for any J.

Even states with odd total angular momentum can be described by the following functions, normalized to unity:

$$\begin{aligned} Y^{\mathrm{II}\,J,\,M,\,4} &= \left[\frac{7(J-3)(J+3)(J+4)}{2(2J+3)(2J-3)(2J+1)}\right]^{1/2} Y^4_{J,\,J-1,\,M} \\ &+ \left[\frac{7(J-3)(J-2)(J+4)}{2(2J+5)(2J+1)(2J-1)}\right]^{1/2} Y^4_{J,\,J+1,\,M} \\ &+ \left[\frac{(J+4)(J+3)(J+2)}{2(2J-1)(2J-3)(2J+1)}\right]^{1/2} \times Y^4_{J,\,J-3,\,M} \\ &+ \left[\frac{(J-3)(J-2)(J-1)}{2(2J+5)(2J+3)(2J+1)}\right]^{1/2} Y^4_{J,\,J+3,\,M}. \end{aligned} \tag{13}$$

There will be no state here with $S = 2$, since $\rho_{J\pm 1,2} = 0$. For $J = 1$ the rules for the addition of angular momentum admit $L = 4$, but for $J = 1$, $\rho_{J\pm 3\,,4} = 0$; Similarly, for $J = 3$

$$\rho_{J-1,4} = \rho_{J+1,4} = \rho_{J+3,4} = 0,$$

that is, even states of two gravitons with $J = 1$ and $J = 3$ are forbidden.

For even states with even total angular momenta the solution $\rho_{J,2} = \rho_{J\pm 2,2} = 0$ is obtained:

$$\begin{aligned} Y^{III\,J,M,4} &= \left[\frac{(J+1)(J+2)(J+3)(J+4)}{4(2J+1)(2J-1)(2J-3)(2J-5)}\right]^{1/2} Y^4_{J,J-4,M} \\ &+ \left[\frac{7(J-3)(J+2)(J+3)(J+4)}{(2J+3)(2J+1)(2J-1)(2J-5)}\right]^{1/2} Y^4_{J,J-2,M} \\ &+ \left[\frac{35(J-3)(J-2)(J+3)(J+4)}{2(2J+5)(2J+3)(2J-1)(2J-3)}\right]^{1/2} Y^4_{J,J,M} \\ &+ \left[\frac{7(J-3)(J-2)(J-1)(J+4)}{(2J+7)(2J+3)(2J+1)(2J-1)}\right]^{1/2} Y^4_{J,J+2,M} \\ &+ \left[\frac{J(J-1)(J-2)(J-3)}{4(2J+7)(2J+5)(2J+3)(2J+1)}\right]^{1/2} Y^4_{J,J+4,M}. \end{aligned} \tag{14}$$

For the case under discussion $S = 4$: for $J = 0$, L can be equal only to 4, and the corresponding $\rho_{J+4,4}$ is found to be equal to zero; for $J = 2$, $\rho_{J,4} = \rho_{J+2,4} = \rho_{J+4,4} = 0$.

It can be easily verified that even states with even total angular momentum can be described by the following equation, which satisfies the orthogonality condition (11):

$$\psi_{klmn}^{JVJM} = \frac{1}{2\sqrt{6}} Y_{JM} (\eta_{knlm}^0 + \eta_{kmln}^0 - \eta_{klmn}^0) = \psi_0 \chi_{klmn}^0 = \psi_{klmn}^0, \qquad (15)$$

where

$$\eta_{klmn}^0 = (\delta_{kl} - p_k p_l / p^2)(\delta_{mn} - p_m p_n / p^2).$$

Substituting $\varphi_0 \chi_{klmn}^0$ into the equation for the eigenfunction of the square of the angular momentum operator:

$$\begin{aligned} M^2 \psi_{klmn}^0 = {} & (1/2\, \hat{\mathscr{M}}_{ij}^2 + 8)\, \psi_{klmn}^0 \\ & + 2i\, (\hat{\mathscr{M}}_{kp} \psi_{plmn} + \hat{\mathscr{M}}_{lq} \psi_{kqmn}^0 \\ & + \hat{\mathscr{M}}_{mr}\, \psi_{klrn}^0 + \hat{\mathscr{M}}_{ns}\, \psi_{klms}^0) \\ & + 2\, (2\psi_{mlkn}^0 + 2\psi_{nlmk}^0 + 2\psi_{klmn}^0 \\ & - \delta_{km} \psi_{plpn}^0 - \delta_{kn} \psi_{plmp}^0 - \delta_{lm} \psi_{kppn}^0 \\ & - \delta_{ln} \psi_{kpmp}^0 - \delta_{mn} \psi_{klpp}^0 - \delta_{kl} \psi_{ppmn}^0), \end{aligned} \qquad (16)$$

we obtain an equation for ψ_0,

$$M^2 \psi^0 = 1/2\, \mathscr{M}_{ij}^2 \psi^0, \qquad (17)$$

that is, $M^2 = J(J+1)$; $\psi_0 = p_{JM} e^{iM\varphi} = Y_{JM}$, a spherical harmonic. From the symmetry condition for the two-graviton solution

$$\psi_{klmn}^0(p) = \psi_{mnkl}^0(-p),$$

noting that $\chi_{klmn}^0 = \chi_{mnkl}^0$, we obtain

$$Y_{JM}(\mathbf{p}) = Y_{JM}(-\mathbf{p}) = (-1)^J Y_{JM}(\mathbf{p}),$$

that is, $J = 2n$.

All results concerning the number of states are brought together in the following table:

J	Even States	Odd States
0	1	1
1	0	0
2	1	1
3	0	0
$2n$	2	1
$2n+1$	1	0

Here it is obvious that for two gravitons, the appearance of states with total angular momentum $J = 1$ and $J = 3$, as well as the appearance of odd states with odd total angular momentum, is strictly forbidden.

These results are in complete agreement with the rules ascertained by Shapiro[7], by which it is forbidden for a particle to disintegrate into two identical bosons of zero rest mass and spin S if 1) the spin of the disintegrating particle is odd and less than $2S$, or if 2) the spin of the disintegrating particle is odd and its wave function is odd.

The method used here makes it possible to determine the number of states with predetermined angular momentum and parity, and also to obtain eigenfunctions for one and two gravitons.

[1] L. D. Landau, Dokl. Akad. Nauk SSSR **60**, 207 (1948).

[2] M. Fierz, Helv. Phys. Acta **12**, 3 (1939).

[3] E. Condon and G. Shortley, *Theory of Atomic Spectra*, IIL, 1949.

[4] Berestetskii, Dolginov and Ter-Martirosian, J. Exptl. Theoret. Phys. (U.S.S.R.) **20**, 527 (1950).

[5] A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics*, GTTI, 1953.

[6] M. Yamada and M. Morita, Progr. Theor. Phys. **8**, 431 (1952).

[7] I. S. Shapiro, J. Exptl. Theoret. Phys. (U.S.S.R.) **27**, 393 (1954).

Translated by W. M. Whitney
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