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Application of Singular Integral Equation Theory to Problems of the Scattering of Particles in an External Field*

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The possible application of singular integral equation theory to the calculation of the scattering phases of particles in an external field is given. The case of scattering by a δ-potential is considered as an example.

IT has been shown in a series of cases relating the scattering of particles in an external field, for example, the scattering of electrons by heavy nuclei, that, on the one hand, the Born approximation for the external field is quite inadequate, and on the other hand, radiation corrections do not yet exist. For phase calculations in this case, one must use numerical integration of the differential equation for the wave function of the scattered particles. However, determination of the phases by this method requires a great deal of work on very accurate calculation of the wave functions for a considerable distance from the scatterer, and the computation must be repeated for each value of the momentum and energy of the particles. Much simpler calculations are obtained by application of the method of Drukarev[1], where an auxiliary function approaches the phase monotonically. However, even in this case, the calculation must be repeated for each value of the energy and momentum. It is therefore desirable to point out a method in which the amplitude (or phase) of the scattering would be found without calculation of the auxiliary function, and would be found for a considerable energy interval at one stroke.

Such a possibility is presented in a series of cases in the use of integral equations of scattering investigated by several authors[2-4]. For direct derivation of the integral equation of scattering, we consider the equation for the S matrix under the condition that the interaction is initiated at $t = 0$:

$$i\partial S(t)/\partial t = H(t)S(t) + i\delta(t), \qquad (1)$$

where $H(t) = \exp(iH_0t)H\exp(-iH_0t)$ is the interaction operator in the interaction representation (if the energy spectrum of all states is continuous, then we can take the initial time at $t=-\infty$ and omit $\delta(t)$ in the equation). Having in view those problems in which there are only diverging waves in the final state, we shall look for the solution in the form

$$S(t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} e^{-i(E-H_0)t} \times [1+\xi(E-H_0)U(E)]\frac{dE}{E-H_0+(i/2)\Gamma(E)}, \qquad (2)$$

where $\Gamma(E)$ is a diagonal operator and $U(E)$ is nondiagonal, and

$$\xi(x) = \frac{1}{i}\int_0^{\infty} e^{itx}\,dt = \frac{P}{x} - \pi i\delta(x) = -2\pi i\delta_+(x) \qquad (3)$$

(P = principal value). Here $x\xi(x) = 1$ and

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{itx}\xi(x)\,dx = \eta(t) = \begin{cases}1, & t>0,\\ 0, & t<0.\end{cases} \qquad (4)$$

With the aid of Eq. (4), we can put Eq. (2) in the form

$$S(t)=\eta(t)-\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-i(E-H_0)t}\xi(E-H_0) \times\left[U(E)-\frac{i}{2}\Gamma(E)\right]\frac{de}{E-H_0+(i/2)\Gamma(E)}. \qquad (5)$$

We substitute Eq. (5) on the left hand side of Eq. (1) and Eq. (2) on the right. Then

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-i(E-H_0)t} \times\left[U(E)-\frac{i}{2}\Gamma(E)-H-H\xi(E-H_0)U(E)\right] \times\frac{dE}{E-H_0+(i/2)\Gamma(E)} = 0 \qquad (6)$$

for arbitrary $t > 0$, whence

$$U-(i/2)\Gamma-H-H\xi U = 0. \qquad (7)$$

The diagonal terms give

$$\Gamma = 2i(H+H\xi U)_d, \qquad (8)$$

and the nondiagonal terms,

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$$U = (H + H\xi U)_{n,d}. \quad (9)$$

The probability of transition from state i to state f in this case is

$$W_{fi} = |S_{fi}(\infty)|^2 \quad (10)$$

$$= \frac{|\dot{U}_{fi}(E_f)|^2}{\{E_f - E_i - {}^1/_2 \operatorname{Im}\Gamma_{ii}(E_f)\}^2 + \{{}^1/_2 \operatorname{Re}\Gamma_{ii}(E_f)\}^2}.$$

Equation (9) serves as a fundamental equation for finding the operator U. As is shown in Ref. 3, it has the operator solution

$$U = \frac{1}{1 - H\xi}\left\{H - \left[\left(\frac{1}{1 - H\xi}\right)_d\right]^{-1}\left(\frac{1}{1 - H\xi}H\right)_d\right\}, \quad (11)$$

wherein $Re\Gamma(E) = 2\pi[U^+(E)\delta(E - H_0)U(E)]_d$. In the transition to a continuous spectrum $\Gamma \to 0$, we get the simplification:

$$U = H + H\xi U, \quad (12)$$

$$U = (1 - H\xi)^{-1}H. \quad (13)$$

In this case, the transition probability per unit time is

$$W = \sum_f \lim_{t \to \infty} \frac{\partial}{\partial t}|S_{fi}(t)|^2 \quad (14)$$

$$= 2\pi \sum_f \delta(E_f - E_i)|U_{fi}(E_f)|^2.$$

Carrying out a formal expansion of Eqs. (11) and (13) in powers of H, we can show the complete identity of the solution (2) of Eq. (1) with the usual solutions in the form of series. It was shown in Ref. 3 that $S(t) = 0$ for $t < 0$ and $\lim_{t \to 0+0} S(t) = 1$, whence the unitary character of $S(t)$ follows for arbitrary $t > 0$.

Computing the matrix elements of the operators on the right and left sides of Eq. (12), we find

$$U_{fi} = H_{fi} + H_{ff'}\xi(E - E_{f'})U_{f'i}, \quad (15)$$

where the summation over f' is carried out over all states whose wave functions form a complete set of functions of the operator H_0 and of operators that associated with it in obtaining a complete set of operators.

Since, according to Eq. (3), the integral in Eq. (15) enters in the sense of a principal value, this equation is a linear singular integral equation. The theory of such equations was worked out by Muskhelishvili and his coworkers[5,6].

We consider the scattering of the particle without spin on a central field, where $H(\mathbf{r}) = (g^2/r_0)\Phi(r/r_0)$. We represent the solution of Eq. (15) in the form

$$U(E,\theta) \quad (16)$$

$$= \frac{4\pi g^2 r_0^2}{L^3}\frac{1}{x\sqrt{x^2 - 1}}\sum_{l=0}^{\infty}\frac{2l+1}{2}P_l(\cos\theta)\psi_l(x),$$

where $x = E/mc^2$, and decompose the matrix element H_{if} in Legendre polynomials. Then we get the following equations:

$$\psi_l(x)\frac{1}{x\sqrt{x^2 - 1}} = f_l(x,x) + \lambda\int_1^{\infty}\psi_l(y) \quad (17)$$

$$\times\left\{\frac{1}{x - y} - \pi i\delta(x - y)\right\}f_l(x,y)\,dy,$$

$$f_l(x,y) = \frac{1}{r_0 k_0}[(x^2 - 1)(y^2 - 1)]^{-1/4} \quad (18)$$

$$\times\int_0^{\infty}J_{l+1/2}(k_0 r_0\rho\sqrt{x^2 - 1})J_{l+1/2}$$

$$\times(k_0 r_0\rho\sqrt{y^2 - 1})\Phi(\rho)\rho\,d\rho,$$

where $\lambda = (g^2/\hbar c)(r_0 k_0)^2$ $(k_0 = mc/\hbar)$, and $J_{l+1/2}$ is a Bessel function.

If the solution of this equation is known, then, from a comparison of Eqs. (16) and (14) with the formulas of the general theory of scattering, we obtain

$$f(\theta) = \frac{\lambda}{k_0\sqrt{x^2 - 1}}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\psi_l(x), \quad (19)$$

$$\sigma = \frac{4\pi}{k_0^2}\frac{\lambda^2}{x^2 - 1}\sum_{l=0}^{\infty}(2l+1)|\psi_l(x)|^2. \quad (20)$$

Comparison of Eq. (19) with the expression $f(\theta)$ through the phase of the scattering gives

$$\delta_l = \frac{1}{2}\operatorname{arctg}\frac{2\lambda\operatorname{Re}\psi_l}{\sqrt{1 - (2\lambda\operatorname{Re}\psi_l)^2}}, \quad (21)$$

where use is made of the relation

$$\operatorname{Im}\psi_l(x) = \lambda|\psi_l(x)|^2, \quad (22)$$

which follows from the unitary nature of the S matrix, and which guarantees the real nature of the phases and satisfaction of the equation

$$\sigma = (4\pi/k_0)(x^2 - 1)^{-1/2}\operatorname{Im}f(0). \quad (23)$$

In the more general case, Eq. (15) separates into individual equations which correspond to definite values of the total moment and total isotopic spin, or also to their projections (depending on the type of interaction).

As an example of the application of singular integral equation theory, let us consider the scattering of a particle without spin on a point potential. Let $H(\mathbf{r}) = g^2 r_0{}^2 \delta(\mathbf{r})$. Then $f_l(x,y) = g^2 r_0{}^2 L^{-3} \delta_{l0}$ and, in terms of the variables

$$x = \frac{E}{mc^2}, \quad U(x) = \frac{\varphi(x)}{x\sqrt{x^2-1}} \frac{g^2 r_0^2}{L^3}$$

Eq. (17) takes the form ($l = 0$)

$$\frac{\varphi(x)}{x\sqrt{x^2-1}} - \lambda \int_1^\infty \left(\frac{1}{x-y} - \pi i \delta(x-y)\right)\varphi(y)\,dy = 1, \tag{24}$$

where $\lambda = (1/2\pi^2)(g^2/\hbar c)(k_0 r_0)^2$. For the limiting case $m = 0$, we get (for $x = (E/g^2) r_0$)

$$\varphi(x)\frac{1}{x^2} - \lambda \int_0^\infty \left(\frac{1}{x-y} - \pi i \delta(x-y)\right)\varphi(y)\,dy = 1, \tag{25}$$

where $\lambda = (1/2\pi^2)(g^2/\hbar c)^3$.

Equations (24) and (25) belong to a type of equation considered in Ref. 5:

$$A(z)\varphi(z) + \frac{1}{\pi i}\int_L \frac{K(z,\zeta)\varphi(\zeta)\,d\zeta}{\zeta - z} = f(z), \tag{26}$$

where $K(z, e)$ does not become zero or infinite for $z = e$, L is a certain line in the complex plane e and z lies on this line. It was shown in Ref. 5 that the solution of the equation

$$A(x)\varphi(x) + \frac{B(x)}{\pi i}\int_L \frac{\varphi(y)}{y-x}\,dy = f(x), \tag{27}$$

if $A(x) \pm B(x) \neq 0$ on the line L, has the form

$$\varphi(x) = \frac{A(x)}{A^2(x) - B^2(x)} - \frac{1}{\pi i}\frac{B(x)}{A^2(x)-B^2(x)}\int_L \frac{f(t)\,dt}{t-x}\frac{Z(x)}{Z(t)}, \tag{28}$$

$$Z(x) = \sqrt{A^2(x) - B^2(x)} \times \exp\left\{\frac{1}{2\pi i}\int_L \frac{d\tau}{\tau - x}\ln\frac{A(\tau)-B(\tau)}{A(\tau)+B(\tau)}\right\}. \tag{29}$$

In accordance with this theory, the solution of Eq. (24) has the form

$$\frac{\varphi(x)}{x\sqrt{x^2-1}} = \frac{1+\pi i\lambda x\sqrt{x^2-1}}{1+2\pi i\lambda x\sqrt{x^2-1}} - \frac{\lambda}{(1+2\pi i\lambda x\sqrt{x^2-1})^{1/2}} \times \int_1^\infty \frac{t\sqrt{t^2-1}\,dt}{(t-x)(1+2\pi i\lambda t\sqrt{t^2-1})^{1/2}} \times \exp\left\{\frac{t-x}{2\pi i}\int_1^\infty \frac{\ln(1+2\pi i\lambda\tau\sqrt{\tau^2-1})}{(\tau-x)(\tau-t)}\,d\tau\right\}, \tag{30}$$

and the solution of Eq. (25) is given in the form $\varphi(x) = x^2 f(\pi\lambda x^2)$, where

$$f(z) = \frac{1+iz}{1+2iz} - \frac{iz}{\sqrt{1+2iz}} \times \exp\left\{-\frac{1}{2\pi i}\int_0^\infty \frac{\ln(1+2iz\tau^2)}{\tau(\tau-1)}\,d\tau\right\} \times \frac{1}{\pi i}\int_0^\infty \frac{t^2\,dt}{(t-1)\sqrt{1+2izt^2}} \times \exp\left\{\frac{1}{2\pi i}\int_0^\infty \frac{\ln(1+2izt^2\tau^2)}{\tau(\tau-1)}\,d\tau\right\}. \tag{31}$$

For our case, when the interval of integration is infinite, the solutions of the two possible classes (with indices 0 and –1) coincide, and all the solutions of the homogeneous equation are equal to zero.

It is easy to show that the convergence in Eqs. (30) and (31) is uniform, and is determined only by the presence of the integral in the exponent. Therefore it is appropriate to consider the integrals in Eq. (31) and show that they converge. It is easy to show that

$$F(2zt^2) \equiv \frac{1}{2\pi i}\int_0^\infty \frac{\ln(1+2izt^2\tau^2)}{\tau(\tau-1)}\,d\tau = -\frac{\pi}{4}\frac{1}{2\pi i} \times \int_{-1/2-i\infty}^{-1/2+i\infty} \frac{ds}{s}\frac{\operatorname{ctg}\pi s}{(2zt^2)^{s/2}}\left(\sec\frac{\pi}{4}s + i\operatorname{cosec}\frac{\pi}{4}s\right). \tag{32}$$

Completing, for $z > 0$, $2zt^2 > 1$, the contour on the right, and carrying out the calculation only for

$s = 0$, we obtain the asymptotic expression for $F(2zt^2)$ as $t \to \infty$:

$$F(2zt^2) \sim -\frac{1}{4} \ln t \sqrt{2z} + i\left(\frac{1}{2\pi} \ln^2 t \sqrt{2z} - \frac{31}{96}\pi\right). \tag{33}$$

Then the outer integral in Eq. (31) converges for $z > 0$, since the integral

$$\int_1^\infty \exp\left\{-\frac{1}{4}\ln x + \frac{i}{2\pi}\ln^2 x\right\} dx = \pi\sqrt{\frac{i}{2}}\, e^{9\pi i/32}\left(1 - \Phi\left(\frac{3}{8}\sqrt{2\pi i}\right)\right), \tag{34}$$

converges. Here $\Phi(x) = 2\pi^{-1/2}\int_0^x e^{-t^2}dt$. For this reason, the solutions (30) and (31) of Eqs. (24) and (25) exist, though they cannot be expanded in powers of $g^2/\hbar c$. This is particularly clear in the example (31), since Eq. (31) depends only on the product $\pi\lambda x^2$, and cannot be decomposed into powers of this quantity without disrupting the convergence of the integral. This coincides with the fact that the integral in (31), in correspondence with (33), is proportional to $z^{-1/2} \sim (g^2/\hbar c)^{-3/2}$ and is not analytic in g^2/hc as $g^2/hc \to 0$. Furthermore, this also agrees with the result of the application of perturbation theory where, for the case $m \neq 0$,

$$U(x) = \frac{g^2 r_0^2}{L^3}\left\{1 + \lambda\int_1^\infty \left[\frac{1}{x-y} - \pi i\delta(x-y)\right] \times y\sqrt{y^2-1}dy + \lambda^2\int_1^\infty \left[\frac{1}{x-y} - \pi i\delta(x-y)\right] \times y\sqrt{y^2-1}dy\int_1^\infty\left[\frac{1}{y-t} - \pi i\delta(y-t)\right] \times t\sqrt{t^2-1}dt + \ldots\right\} \tag{35}$$

and all the integrals diverge.

It is easy to see that the Fredholm theory of scattering gives the same result as perturbation theory. Hence, it is also inapplicable to such a singular potential as the δ-potential. Thus the theory of the solution of singular integral equations according to Muskhelishvili makes it possible to obtain solutions without decomposition in the coupling parameter, which is especially important when this solution is not expanded in such a series (but this, according to the Fredholm theory, always takes place when the potential is sufficiently singular at zero, and consequently its Fourier components diminish too slowly with an increase in energy).

In the case of a more complicated potential, the singular equations will not achieve as clear a description of the solution, as in the case considered, but, with the help of the operation of regularization reported in Ref. 5, these equations can be converted into Fredholm equations with regular kernels which can, in turn, be solved numerically.

For quantized fields there arise infinite sets of equations of the type (15). In this case, on the one hand, it is fundamental to assume as fruitful the application of the theory of systems of singular integral equations to obtain solutions without decomposition in powers of the coupling parameter but, on the other hand, the mathematics for infinite systems is still insufficiently worked out to draw more fundamental conclusions.

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