The Theory of Nonlinear Effects in the Ionosphere

S. A. ZHEVAKIN AND V. M. FAIN

(Submitted to JETP editor July 8, 1954)
J. Exptl. Theoret. Phys. (U.S.S.R.) 30, 518-527 (March, 1956)

Nonlinear effects which arise in the propagation in the ionosphere of modulated and unmodulated electromagnetic waves of arbitrary amplitude have been computed with consideration of the earth's magnetic field.

1 The elementary theory of the Luxembourg-Gorkov effect has been given by Bailey and Martin.[1] Later, the theory was improved by Ginzburg, who gave a rigorous analysis of nonlinear effects in the ionosphere by means of the kinetic equation.[2]

The kinetic equation was solved by Ginzburg by means of successive approximations, in which analysis he, for well understood reasons, restricted himself to a consideration of the first approximation only. In the same paper the results were applied only for the case of sufficiently weak electromagnetic fields.* Meanwhile, the use of the velocity distribution function of the electrons, obtained in closed form in Ref. 3 for the case of propagation in a plasma of an amplitude modulated high-frequency electromagnetic wave of arbitrary intensity (in the presence of a constant magnetic field) permit us to carry out the calculation of the magnitude of cross modulation and other ionospheric nonlinear effects without any assumption as to the smallness of the field intensity. As a result of such a calculation, an essential simplification of the theory of nonlinear effects in the ionosphere can be achieved, inasmuch as we can avoid the direct application of the unwieldy method of the kinetic equation.

Suppose that a high power transmitter produces in the ionosphere the field

$$E_1 = E_0(1 + \mu\cos\Omega t)\cos\omega_1 t, \qquad (1)$$

where μ is the depth of the modulation, Ω the modulation frequency and ω_1 the carrier frequency. Further, let us suppose the existence of an additional disturbing electromagnetic wave with frequency ω_2. In the quasi-stationary approximation, which exists for $\Omega << \omega_2$, and upon fulfillment of the conditions

$$\omega_2^2 \gg \nu_{ef}^2; \ |\omega_2 - \omega_H| \gg \nu_{ef}, \qquad (2)$$

where $\omega_H = |(e/mc)H|$, the absorption coefficient k is proportional to ω_2. Therefore, the field of the wave with frequency ω_2 at the receiver is proportional to the expression[4]

$$E \sim \exp\{-(\omega_2/c)\int k_0 ds - (\omega_2/c)\bar{\eta}\int k_0 ds\}. \quad (3)$$

Here $(\omega_2/c)\int k_0\, ds$ is the total absorption in a wavelength for the weak field. For brevity, the notation $\eta = \Delta\nu_{ef}/\nu_{ef,0}$ is introduced, where $\nu_{ef,0}$ is the effective number of collisions in the weak field; $\Delta\nu_{ef} = \nu_{ef} - \nu_{ef,0}$ is the increment in the effective number of collisions under the action of an electric field of frequency ω_1, radiated by the transmitter; $\bar{\eta}$ is some mean value of the quantity η along the direction of propagation. We emphasize that, because of the conditions (2), Eq. (3) is equally applicable for both small and large values of $\Delta\nu_{ef}$.

We seek the quantity η, which characterizes the cross modulation, in two limiting cases. In the first case,

$$\Omega \ll \delta\nu_{ef}, \qquad (4)$$

while in the second,

$$\Omega \gg \delta\nu_{ef}, \qquad (5)$$

where δ characterizes the mean energy lost by the electron in a collision with a molecule or an ion. Below we shall consider only elastic collisions. Then

$$\delta = 2m/M, \qquad (6)$$

where m is the mass of the electron and M is the mass of the molecule or ion.

2. We consider the case (4). If the condition

*We note that, as a consequence of the omission by Ginzburg of a solution for arbitrary magnitude of the field intensity, the very meaning of a "weak field" therein remains insufficiently defined.

[1] V. A. Bailey and D. F. Martin, Phil. Mag. 18, 369 (1934).

[2] V. L. Ginzburg, Izv. Akad. Nauk SSSR, Ser. Fiz. 12, 293 (1948).

[3] V. M. Fain, J. Exptl. Theoret. Phys. (U.S.S.R.) 28, 422 (1955). Soviet Phys. JETP 1, 205 (1955).

[4] Ia. L. Al'pert, V. L. Ginzburg and E. L. Feinberg, *The propagation of radiowaves*, Moscow, 1953.

$$\frac{(1-z)^2}{(1+z)^2}\frac{\omega_1^2}{\nu_{ef}^2} \gg 1, \qquad (7)$$

is satisfied, where $z=\omega_H^2/(\omega_1^2+\nu^2)$, the isotropic part of the velocity distribution function of the electron is a Maxwell function with effective temperature[3]

$$T_{ef} = T\left[1+\frac{Me^2}{6kT\omega_1^2m^2}\left(E_\perp^2\frac{1+z}{(1-z)^2}+E_\parallel^2\right)(1+\mu\cos\Omega t)^2\right], \qquad (8)$$

where $E_\perp$ and $E_\parallel$ are the components of the electric field respectively perpendicular and parallel to the earth's magnetic field H, T is the temperature of the ionosphere (more precisely, the temperature of the molecules and ions).

In a weak field, in the case of collisions with molecules,[4]

$$\nu_{ef,m,0} = \frac{4\pi a^2}{3}N_m\bar{v}, \qquad (9)$$

where $\bar{v}=(8kT/\pi m)^{1/2}$ a is the molecular radius, N_m the molecular concentration. For collisions with ions,[4]

$$\nu_{ef,i,0} = \frac{2\pi e^4 N_i\bar{v}\ln\alpha}{3(kT)^2}, \qquad (10)$$

where N_i is the ionic concentration, $\alpha=2kT\rho_m/e^2$ (ρ_m = the maximal impact parameter[4]).

In the strong field of the transmitter with frequency ω_1, the quantity ν_{ef} is found from Eqs. (9), (10) by simple substitution of T_{ef} for T:

$$\nu_{ef,m} = \frac{4\pi a^2}{3}N_m\sqrt{\frac{8kT_{ef}}{\pi m}}; \qquad (9')$$

$$\nu_{ef,i} = \frac{2}{3}\pi\frac{e^4N_i\ln\alpha_{ef}}{(kT_{ef})^2}\sqrt{\frac{8kT_{ef}}{\pi m}}, \qquad (10')$$

where $\alpha_{ef}=2kT_{ef}\rho_m/e^2$. Hence, for the quantities η_m and η_i, which characterize the cross modulation, in the case of collisions with molecules and in collisions with ions, we get, respectively,

$$\eta_m = \frac{\Delta\nu_{ef,m}}{\nu_{ef,m,0}} = \frac{\nu_{ef,m}-\nu_{ef,m,0}}{\nu_{ef,m,0}} = \sqrt{\frac{T_{ef}}{T}}-1; \qquad (11)$$

$$\eta_i = \frac{\Delta\nu_{ef,i}}{\nu_{ef,i,0}} = \frac{T^{3/2}}{T_{ef}^{3/2}}\frac{\ln\alpha_{ef}}{\ln\alpha}-1. \qquad (12)$$

In the case of propagation of the transmitter wave (frequency ω_1) close to the gyromagnetic frequency ω_H ($\omega_1\approx\omega_H$), we must consider that this wave, generally speaking, is elliptically polarized. The electric field at an arbitrary point of space is then written as

$$E_x = E_{0x}\cos\omega t\,(1+\mu\cos\Omega t),$$
$$E_y = E_{0y}\cos(\omega t-\beta)(1+\mu\cos\Omega t),$$
$$E_z = E_{0z}\cos(\omega t-\gamma)(1+\mu\cos\Omega t).$$

The velocity distribution function of electrons in the presence of such a field and a constant magnetic field H. directed along the x axis, is obtained by the method set forth in Ref. 3. It has the form (in this calculation, D. Ia. Kaushanskii took part) $f(\mathbf{v})=f_0(v)+\mathbf{v}\cdot\mathbf{f}_1(v)$, where the isotropic part of the distribution function is equal (we recall that we now assume the quasi-stationary condition (4) to be satisfied) to

$$f_0(v) = C\exp\left[-\int_0^v dv\,mv\left\{kT+(1+\mu\cos\Omega t)^2\frac{e^2}{m^2}\left(\frac{(E_{0y}^2+E_{0z}^2)Ml}{6A}+\frac{E_{0x}^2Ml^2}{6\omega^2l^2+v^2}+\frac{2E_{0y}E_{0z}\omega eH\,Ml\sin(\beta-\gamma)}{3Amc(\omega^2+\nu^2)(1+z')}\right)\right\}^{-1}\right],$$

where

$$A=\frac{l^2\omega^2(1-z')^2+v^2(1+z')^2}{l(1+z')},$$

$$z'=\frac{\omega_H^2}{\omega^2+\nu^2}, \quad \nu=\frac{v}{l}$$

(l is the mean free pathlength of the electron, $e=-|e|$ is the electronic charge) and

$$\mathbf{f}_1 = \frac{e}{m}(1+\mu\cos\Omega t)\{[d_1(E_{0y}\cos\beta\mathbf{j}+E_{0z}\cos\gamma\mathbf{k})+q_1(-HE_{0z}\cos\gamma\mathbf{j}+HE_{0y}\cos\beta\mathbf{k}) + r_1(HE_{0z}\sin\gamma\mathbf{j}-HE_{0y}\sin\beta\mathbf{k})+q_1(-E_{0y}\sin\beta\mathbf{j}-E_{0z}\sin\gamma\mathbf{k})+h_1E_{0x}\mathbf{i}]\cos\omega t + [g_1(E_{0y}\cos\beta\mathbf{j}+E_{0z}\cos\gamma\mathbf{k})+r_1(-HE_{0z}\cos\gamma\mathbf{j}+HE_{0y}\cos\beta\mathbf{k}) + q_1(-HE_{0z}\sin\gamma\mathbf{j}+HE_{0y}\sin\beta\mathbf{k})+d_1(E_{0y}\sin\beta\mathbf{j}+E_{0z}\sin\gamma\mathbf{k})+p_1E_{0x}\mathbf{i}]\sin\omega t\},$$

where

$$d_1 = -\frac{1}{A}\frac{\partial f_0}{\partial v};\quad g_1 = \frac{\omega l^2(z'-1)}{v[\omega^2 l^2(1-z')^2+v^2(1+z')^2]}\frac{\partial f_0}{\partial v};$$

$$h_1 = -\frac{l}{\omega^2 l^2+v^2}\frac{\partial f_0}{\partial v};\quad q_1 = \frac{e(\omega g_1-\nu d_1)}{mc(\omega^2+\nu^2)};$$

$$p_1 = -\frac{\omega}{(\omega^2+\nu^2)v}\frac{\partial f_0}{\partial v};\quad r_1 = -\frac{e(\omega d_1+\nu g_1)}{mc(\omega^2+\nu^2)}.$$

We note that in the isotropic part of the distribution function, the term $\frac{2MlE_{0y}E_{0z}\omega eH\sin(\beta-\gamma)}{3Amc(\omega^2+\nu^2)(1+z')}$ is positive if the electric field has the direction of rotation corresponding to the so-called extraordinary wave, and negative for the direction of rotation corresponding to the ordinary wave. If $\beta=\gamma=0$, i.e., if the electric field is linearly polarized, then the velocity distribution function of the electrons coincides with the function obtained in Ref. 3.

Let the condition

$$\omega_H = \omega_1;\quad \omega_1^2 \gg \nu_{ef}^2. \tag{13}$$

be satisfied. Then the isotropic part of the distribution function takes the form

$$f_0 = C\exp\left[-\int_0^v dv\, mv\left\{kT+(1+\mu\cos\Omega t)^2\frac{e^2}{m^2}\left(\frac{(E_{0y}^2+E_{0z}^2)M}{12\nu^2} + \frac{E_{0x}^2 M}{6\omega_1^2} - \frac{E_{0y}E_{0z}MH\sin(\beta-\gamma)}{6\nu^2[H]}\right)\right\}^{-1}\right]. \tag{14}$$

In this case, $e=-|e|$.

The effective collision number is found from the formula[4]

$$\nu_{ef} = -\frac{4\pi}{3}\int_0^\infty \frac{v^4}{l}\frac{\partial f_0}{\partial v}dv. \tag{15}$$

Normalizing f_0 to unity and substituting in Eq. (15), we find, for $l=$ const (as will be the case in the collision of electrons with molecules), that

$$\nu_{ef,m} = \frac{4\pi a^2}{3}N_m\sqrt{\frac{8kT_1}{\pi m}}\,g(u), \tag{16}$$

where

$$T_1 = T\left[1+\frac{Me^2E_{0x}^2}{6m^2kT\omega_1^2}(1+\mu\cos\Omega t)^2\right],$$

$$u = \frac{Me^2l^2}{24m^2(kT_1)^2}\Big(E_{0y}^2+E_{0z}^2 - 2E_{0y}E_{0z}\frac{H}{|H|}\sin(\beta-\gamma)\Big)(1+\mu\cos\Omega t)^2,$$

and the function $g(u)$ is determined by the expression

$$g(u) = \frac{\sqrt{\pi}}{2}\frac{\int_0^\infty xe^{-x}(1+x/u)^u dx}{\int_0^\infty \sqrt{x}e^{-x}(1+x/u)^u dx}. \tag{17}$$

For integral values of u, the integrals which appear in the function $g(u)$ are expressed in elementary functions (see Ref. 5). The values of $g(u)$ for integral u are the following:

[5] I. M. Ryzhik and I. S. Fradshtein, *Table of integrals, sums, series and derivatives*, Moscow, 1951.

$u=0$	1	2	3	4	5	7	10
$g(u)=1{,}000$	1,200	1,314	1,379	1,431	1,475	1,571	1,629

From Eqs. (9) and (16) for the case of collision of electrons with molecules, we find

$$\eta_m = \sqrt{\frac{T_1}{T}}\, g(u) - 1; \quad (\omega_1 \approx \omega_H). \tag{18}$$

3. We proceed to the calculation of the magnitude of the cross modulation for the case $\Omega >> \delta\nu_{ef}$. In view of the fact that in this case the quasi-stationary condition (in relation to the frequency of the low power transmitter ω_2) is essentially made use of by us, the modulation frequency Ω must be limited also: $\delta\nu_{ef} << \Omega << \omega_2$.

By the method developed in Ref. 3, it is easy to find the isotropic part of the distribution function in this case. It has the form

$$f_0(v) = \Phi_{0,0} + F_{0,1}\cos\Omega t + G_{0,1}\sin\Omega t + F_{0,2}\cos 2\Omega t + G_{0,2}\sin 2\Omega t + F_{0,3}\cos 3\Omega t + G_{0,3}\sin 3\Omega t + F_{0,4}\cos 4\Omega t + G_{0,4}\sin 4\Omega t + \cdots \tag{19}$$

Expressions for $\Phi_{0,0}$, $G_{0,1}$, $F_{0,1}G_{0,2}$, $F_{0,2}$, etc. have been developed in Ref. 3. When conditions (5) and (7) are satisfied, these expressions take the form

$$\Phi_{0,0} = \left(\frac{m}{2\pi k T'_{ef}}\right)^{3/2} \exp\left\{-\frac{mv^2}{2kT'_{ef}}\right\},$$

where

$$T'_{ef} = T\left[1 + \frac{Me^2\left(1+\frac{1}{2}\mu^2\right)}{6kT\omega_1^2 m^2}\left(E_{0\perp}^2\frac{(1+z)}{(1-z)^2} + E_{0\parallel}^2\right)\right];$$

$$G_{0,1} = \frac{1}{3\Omega v^2}\frac{\partial}{\partial v}\left[\frac{\mu e^2 v^3}{m^2\omega_1^2 l}\left(E_{0\perp}^2\frac{(1+z)}{(1-z)^2} + E_{0\parallel}^2\right)\frac{\partial\Phi_{0,0}}{\partial v}\right];$$

$$G_{0,2} = \frac{\mu}{8}G_{0,1};$$

$$F_{0,1} = -\frac{1}{6v^2\Omega}\frac{\partial}{\partial v}\left[v^3\frac{e^2}{m^2\omega_1^2 l}\left(E_{0\perp}^2\frac{(1+z)}{(1-z)^2} + E_{0\parallel}^2\right)\left(1+\frac{3}{8}\mu^2\right)\frac{\partial G_{0,1}}{\partial v}\right] - \frac{1}{\Omega v^2}\frac{\partial}{\partial v}\left[\frac{m}{M}\frac{v^4 G_{0,1}}{l} + \frac{kTv^3}{Ml}\frac{\partial G_{0,1}}{\partial v}\right];$$

$$F_{0,2} = -\frac{1}{12v^2\Omega}\frac{\partial}{\partial v}\left[v^3\frac{e^2}{m^2\omega_1^2 l}\left(E_{0\perp}^2\frac{(1+z)}{(1-z)^2} + E_{0\parallel}^2\right)\mu\left(\frac{9}{8}+\frac{\mu^2}{16}\right)\frac{\partial G_{0,1}}{\partial v}\right] - \frac{\mu}{16\Omega v^2}\frac{\partial}{\partial v}\left[\frac{m}{M}\frac{v^4 G_{0,1}}{l} + \frac{kT}{M}\frac{v^3}{l}\frac{\partial G_{0,1}}{\partial v}\right];$$

$$F_{0,3} = -\frac{1}{18\Omega v^2}\frac{\partial}{\partial v}\left[v^3\frac{e^2}{m^2\omega_1^2 l}\left(E_{0\perp}^2\frac{(1+z)}{(1-z)^2} + E_{0\parallel}^2\right)\frac{3\mu^2}{8}\frac{\partial G_{0,1}}{\partial v}\right];$$

$$F_{0,4} = \frac{\mu}{16}F_{0,3};$$

$$G_{0,3} = \frac{1}{18\Omega v^2}\frac{\partial}{\partial v}\left[v^3\frac{e^2}{m^2\omega_1^2 l}\left(E_{0\perp}^2\frac{(1+z)}{(1-z)^2} + E_{0\parallel}^2\right)\left(\frac{\mu^2}{4}\frac{\partial F_{0,1}}{\partial v} + \mu\frac{\partial F_{0,2}}{\partial v} + \left(1+\frac{\mu^2}{2}\right)\frac{\partial F_{0,3}}{\partial v} + \frac{\mu^2}{4}\frac{\partial F_{0,4}}{\partial v}\right)\right] + \frac{1}{3\Omega v^2}\frac{\partial}{\partial v}\left[\frac{m}{M}\frac{v^4 F_{0,3}}{l} + \frac{kTv^3}{Ml}\frac{\partial F_{0,3}}{\partial v}\right];$$

$$G_{0,4} = \frac{1}{24\Omega v^2}\frac{\partial}{\partial v}\left[v^3\frac{e^2}{m^2\omega_1^2 l}\left(E_{0\perp}^2\frac{(1+z)}{(1-z)^2} + E_{0\parallel}^2\right)\left(\frac{\mu^2}{4}\frac{\partial F_{0,2}}{\partial v} + \mu\frac{\partial F_{0,3}}{\partial v} + \left(1+\frac{\mu^2}{2}\right)\frac{\partial F_{0,4}}{\partial v}\right)\right] + \frac{1}{4\Omega v^2}\frac{\partial}{\partial v}\left[\frac{m}{M}\frac{v^4 F_{0,4}}{l} + \frac{kTv^3}{Ml}\frac{\partial F_{0,4}}{\partial v}\right].$$

Substituting all these expressions in Eq. (15), and setting l=const there, we find the effective number of electronic collisions with molecules:

$$\nu_{ef,m} = \nu_0 + \nu'_\Omega \cos \Omega t + \nu''_\Omega \sin \Omega t + \nu'_{2\Omega} \cos 2\Omega t + \nu''_{2\Omega} \sin 2\Omega t + \nu'_{3\Omega} \cos 3\Omega t + \nu''_{3\Omega} \sin 3\Omega t + \nu'_{4\Omega} \cos 4\Omega t + \nu''_{4\Omega} \sin 4\Omega t, \quad (20)$$

where

$$\nu_0 = \frac{4}{3l}\sqrt{\frac{8kT'_{ef}}{\pi m}};\quad \nu'_\Omega = \frac{8e^2\mu}{3\Omega^2 l^2 m^2\omega_1^2}\frac{m}{M}\left(E^2_{0\perp}\frac{(1+z)}{(1-z)^2} + E^2_{0\parallel}\right)\nu_0;$$

$$\nu''_\Omega = \frac{4\mu e^2}{3\Omega l^2 m^2\omega_1^2}\left(E^2_{0\perp}\frac{(1+z)}{(1-z)^2} + E^2_{0\parallel}\right);\quad \nu'_{2\Omega} = \frac{\mu}{16}\nu'_\Omega;\quad \nu''_{2\Omega} = \frac{\mu}{8}\nu''_\Omega;$$

$$\nu'_{3\Omega} = 0;\quad \nu''_{3\Omega} = \frac{m}{M}\frac{e^2\mu^2}{18\Omega^2 m^2\omega_1^2 l^2}\left(E^2_{0\perp}\frac{(1+z)}{(1-z)^2} + E^2_{0\parallel}\right)\nu''_\Omega;\quad \nu'_{4\Omega} = 0;\quad \nu''_{4\Omega} = \frac{3\mu}{64}\nu''_{3\Omega}.$$

Hence

$$\eta_m = \eta_0 + \eta'_\Omega \cos \Omega t + \eta''_\Omega \sin \Omega t + \eta'_{2\Omega} \cos 2\Omega t + \eta''_{2\Omega} \sin 2\Omega t + \eta'_{3\Omega} \cos 3\Omega t + \eta''_{3\Omega} \sin 3\Omega t + \eta'_{4\Omega} \cos 4\Omega t + \eta''_{4\Omega} \sin 4\Omega t, \quad (21)$$

where

$$\eta_0 = \sqrt{T'_{ef}/T} - 1;$$

$$\eta'_\Omega = \nu'_\Omega/\nu_{ef,m,0};\quad \eta''_\Omega = \nu''_\Omega/\nu_{ef,m,0};$$

$$\eta'_{2\Omega} = \nu'_{2\Omega}/\nu_{ef,m,0};\quad \eta''_{2\Omega} = \nu''_{2\Omega}/\nu_{ef,m,0} \text{ и т. д.}$$

We shall not consider the collisions of electrons with ions, because of the small practical interest of these collisions when condition (5) is satisfied.[4]

4. All the formulas we have obtained agree in the linear approximation relative to E^2 and, in the absence of a magnetic field, with the formulas obtained in Ref. 4.* The difference between our formulas and the corresponding formulas of Ref. 4 are illustrated in Figs. 1 and 2.

In Figs. 1 and 2 the quantity $x = \frac{e^2}{m^2}\frac{E_0^2}{\omega_1^2}\frac{M}{6kT} \times (1+\mu\cos\Omega t)^2$ is plotted along the abscissa and the quantity $\eta = \Delta\nu_{ef}/\nu_{ef,0}$ along the ordinate; the earth's magnetic field was not considered.

We now estimate magnitudes. In this case, we consider that we need to replace the quantity $2m/M = \delta_{el}$ which characterizes the energy lost by the electron in an elastic collision, for the calculation of inelastic collisions by a certain mean quantity δ_0. We assume, in accordance with Ref. 6, that $\delta_0 = 2\times 10^{-3}$. Assuming that the system of the transmitter radiates upwards with a gain of $g=1.5$ relative to radiation isotropically radiated in the upper halfspace, we shall compute the field intensity E_0 in the ionosphere by means of the formula,

$$E_0 = \left(10^{-5}\sqrt{\frac{W_{\text{W}}}{r_{\text{KM}}}}\right) \exp\left\{-\frac{\omega_1}{c}\int k(\omega_1)\, ds\right\} \text{ CGSE},$$

*In Ref. 2 the isotropic part of the distribution function $f_{00}+f_{01}$ for the case $\Omega \ll \delta\nu_{ef,0}$ was incorrectly not normalized to unity. Therefore, in this case our formulas do not agree in the linear approximation with the formulas of Ref. 2. In our work, $\Delta\nu_{ef}$ for the collision of electrons with ions has the same sign as in the elementary theory, while in Ref. 2, $\Delta\nu_{ef}$ has a different sign, because of the incorrect normalization.

[6] L. G. Huxley, Nuovo Cim. Suppl. 9, 59 (1952).

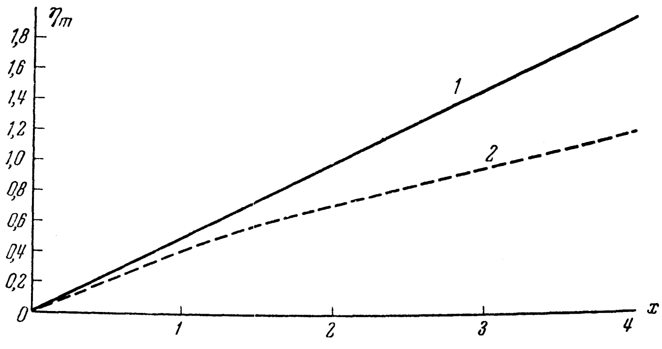

FIG. 1. Relative increase of the effective number of electronic collisions with molecules $\eta_m = \Delta\nu_{ef,m}/\nu_{ef,m0}$ as a function of the parameter x, which is proportional to the square of the amplitude of the high frequency electromagnetic wave. *1*-according to the approximate theory with consideration of terms only of order E^2; *2*-according to the exact theory.

where W_{kW} is the transmitter power in kilowatts, r_{km} is the distance in kilometers. Assuming the distance from the power transmitter to the region of the ionosphere, where the cross modulation takes place, to be equal to 85 km, the modulation factor of the field of the power station to $\mu=0.5$, the temperature of the ionosphere $T=300$ °K, the circular frequency of the power station $\omega_1 = 10^6$ rad/sec, and considering that

$$\frac{\omega_1}{c}\int k(\omega_1)\,ds \approx \frac{\omega_2}{c}\int k_0(\omega_2)\,ds \approx 1,$$

we obtain a table for the comparison of the magnitudes of the cross modulation according to the exact theory and in the linear approximation relative to E_0^2.

Power W in kW	x	η				$\exp\left\{-\frac{\omega}{c}\,\bar{\eta}\int k_0 ds\right\}$			
		linear approximation		exact theory		linear approximation		exact theory	
		Collisions							
		with molecules	with ions	with molecules	with ions	with molecules	with ions	with molecules	with ions
125	0.53	0.26	0.72	0.24	0.44	0.77	0.49	0.79	0.645
250	1.06	0.53	1.44	0.44	0.62	0.59	0.237	0.645	0.54
500	2.12	1.06	2.88	0.77	0.79	0.35	0.056	0.465	0.455
1000	4.24	2.12	5.75	1.29	0.90	0.12	0.003	0.276	0.41

Thus, even for transmitter powers of ~250 kW, noticeable deviations from the results of the linear theory are obtained. The above applies particularly to the case of collisions of electrons with ions. As a result, the linear approximation relative to E_0^2 is no longer satisfactory for transmitter powers ≥250 kW. We note that the amount of cross modulation in Eq. (3) is characterized by the factor $\exp\{-\underline{w}^2\,\eta - \int ko(w_2)do\}$, and that it is impossible in the case of strong fields to limit oneself to the first two terms of the expansion of this exponent in a power series, as was done in Ref. 4.

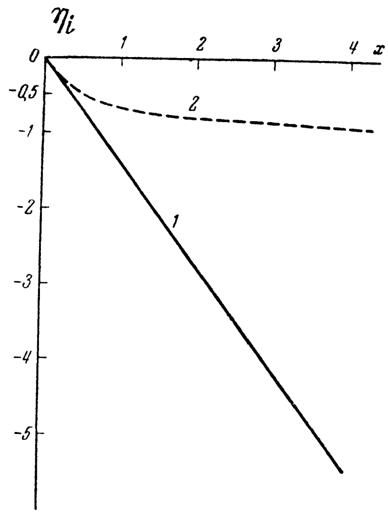

FIG. 2. Relative increase of the effective number of electronic collisions with ions $\eta_i = \Delta\nu_{ef,i}/\nu_{ef,i0}$ as a function of the parameter x, which is proportional to the square of the amplitude of the high frequency electromagnetic wave. *1*-according to the approximate theory with consideration of terms only of order E^2; *2*-according to the exact theory.

It is natural that in the two limiting cases (4) and (5), our formulas, in contrast to the formulas of Ref. 4, contain the frequencies 3Ω, 4Ω, etc., in addition to the frequencies Ω and 2Ω. It is curious that the coefficients for $\sin\Omega t$ and $\sin 2\Omega t$ in Eq. (21) (in the case l=const) agree in accuracy with the corresponding coefficients* obtained in Ref. 4. We also note that the coefficients η'_Ω, $\eta'_{2\Omega}$ which already appear in the linear approximation relative to E_0^2, but which are $\sim \delta\nu_{ef}/\Omega$ times smaller than the coefficients η''_Ω and $\eta''_{2\Omega}$, are not considered in Ref. 4.

5. Vilenskii[7] considered the phenomenon of phase and amplitude self-modulation of an electromagnetic wave which arises in its propagation through the ionosphere. This phenomenon was studied experimentally in Refs. 8,9 at a frequency close to the gyromagnetic. In these works, the phenomenon was given the name "automodulation" (see also Ref. 10). However, in Ref. 7, the normalization of the distribution function of Ref. 2 was applied incorrectly, as a result of which the author of Ref. 7 obtained wrong results for the case $\Omega \ll \delta\nu_{ef,0}$. For the correct calculation, the effect of phase self-modulation is some 10^3 times smaller than that found in Ref. 7.

Actually, let the transmitter radiate a modulated wave which, upon its entry into the ionosphere, corresponds to the field $E = E_0(1+\mu\cos\Omega t)\cos\omega t$. We set up the condition that the coordinate s runs along the path of the wave so that on the lower boundary of the ionosphere (for entry into it of the electromagnetic wave) $s=0$. The field of the wave at a specific point s in the ionosphere can then be represented as

$$\mathbf{E} = \mathbf{E}_1(s,t)(1+\mu\cos\Omega t)\cos(\omega t - \varphi(s,t)). \quad (22)$$

We now calculate the current arising under the action of this field. In such a case, in contrast to Ref. 7, we shall neglect not the quantity $\nu^2/\omega^2 \ll 1$, but the ratio ν^4/ω^4. The distribution function in the absence of a magnetic field and upon fulfillment of the condition $\Omega \ll \delta\nu_{ef}$ will be:[3]

$$f(\mathbf{v},t) = f_0(v,t) + \mathbf{v}\mathbf{f}_1, \quad (23)$$

where

$$f_0 = \left(\frac{m}{2\pi k T_{ef}}\right)^{3/2} e^{-mv^2/2kT};$$

$$T_{ef} = T\left[1 + \frac{M}{6kT}\frac{e^2E_1^2}{m^2\omega^2}(1+\mu\cos\Omega t)^2\right];$$

$$\mathbf{f}_1 = -\mathbf{E}_1\frac{e}{m}\left[\frac{1}{v}\frac{\nu}{\omega^2+\nu^2}\cos(\omega t - \varphi(s)) + \frac{1}{v}\frac{\omega}{\omega^2+\nu^2}\sin(\omega t-\varphi(s))\right] \times \frac{\partial f_0}{\partial v}(1+\mu\cos\Omega t)$$

$$\approx -\mathbf{E}_1\frac{e}{m}\left[\frac{\cos(\omega t-\varphi(s))}{\omega^2 l} + \frac{1}{v\omega}\left(1-\frac{\nu^2}{\omega^2}\right)\sin(\omega t - \varphi(s)\right]\frac{\partial f_0}{\partial v}(1+\mu\cos\Omega t).*$$

*Provided that the presence of a magnetic field is not taken into account.

*In the isotropic part of the distribution function, we have neglected the quantity $(\nu/\omega)^2$, because in the calculation of the current density this is equivalent to throwing away terms of order ν^4/ω^4. Just for simplicity, we have neglected the ratio $(\nu/\omega)^2$ in the coefficient for $\cos(\omega t - \varphi(s))$ also, since this leads only to an insignificant change in the calculation of amplitude modulation.

[7] I. M. Vilenskii, Dokl. Akad. Nauk SSSR 92, 525 (1953).

[8] M. Cutolo, Nuovo Cim. 9, 687 (1952).

[9] M. Cutolo, Nature 167, 315 (1951).

[10] *A new nonlinear effect in the ionosphere*, Usp. Fiz. Nauk 49, 484 (1953).

We find the total current from the formula

$$\mathbf{j}_t = Ne \int_{-\infty}^{+\infty} \mathbf{v} f(\mathbf{v},t)\, d\mathbf{v} = \frac{4\pi Ne}{3} \int_0^{\infty} \mathrm{f}_1 v^4 dv, \tag{24}$$

which yields

$$\mathbf{j}_t = \boldsymbol{E}_1 \Big\{\sigma \cos(\omega t - \varphi(s)) - \omega \frac{\varepsilon - 1}{4\pi} \sin(\omega t - \varphi(s))\Big\} (1 + \mu \cos \Omega t) + \boldsymbol{E}_1^3 \sigma_1 \{\cos(\omega t - \varphi(s)) + A \sin(\omega t - \varphi(s))\} (1 + \mu \cos \Omega t)^3, \tag{25}$$

where N_m is the molecular concentration,

$$\sigma = \frac{Ne^2 \nu_{ef,0}}{m\omega^2}; \qquad \nu_{ef,0} = \frac{4}{3l} N_m \bar{v};$$

$$\sigma_1 = \frac{M}{m} \frac{e^2 \sigma}{12kT\omega^2 m}; \varepsilon = 1 - \frac{4\pi e^2}{m\omega^2}\left(1 - 1{,}10 \frac{\nu_{ef,0}^2}{\omega^2}\right);$$

$$\bar{v} = \left(\frac{8kT}{\pi m}\right)^{1/2}; \quad A = -2{,}21 \frac{\nu_{ef,0}}{\omega}.$$

In the following we shall for simplicity consider the region of the ionosphere in which self-modulation arises which is homogeneous in an isotropic medium. Then the quantities $k(\omega)$, n, σ etc., which enter into the theory are certain averages corresponding to the quantities of the real ionosphere.

For a homogeneous and isotropic medium, we get from Maxwell's equations

$$\frac{\partial^2 E}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \frac{4\pi}{c^2}\frac{\partial j_t}{\partial t} = 0. \tag{26}$$

Substituting the current (25) and the field (22) in Eq. (26) and taking into account that $\Omega << \omega$, we can equate separately the terms in $\sin(\omega t - \varphi(s))$ and $\cos(\omega t - \varphi(s))$. As a result we get two equations for $E'_1 = E_1(s,t)(1 + \mu \cos \Omega t)$ and $\varphi(s,t)$:

$$\frac{\partial E_1'}{\partial s^2} - E_1'\left(\frac{\partial \varphi}{\partial s}\right)^2 + \frac{\omega^2 \varepsilon}{c^2} E_1' - \frac{4\pi\sigma_1 A\omega}{c^2} E_1'^3 = 0; \tag{27}$$

$$2\frac{\partial E_1'}{\partial s}\frac{\partial \varphi}{\partial s} + E_1' \frac{\partial^2 \varphi}{\partial s^2} + \frac{4\pi\sigma\omega}{c^2} E_1' + \frac{4\pi\sigma_1\omega}{c^2} E_1'^3 = 0.$$

It is easy to convince oneself by the proper substitution that the solution, with accuracy to E_0^3, which satisfies the boundary conditions $(\varphi)_{s=+0} = 0$ and $(E'_1)_{s=+0} = E_0(1 + \mu \cos \Omega t)$ has the form:

$$E_1' = \Big\{E_0(1 + \mu \cos \Omega t) - \frac{\sigma_1}{2\sigma}\frac{1 + 2Ak/n}{1 + 4(k/n)^2} E_0^3 (1 + \mu \cos \Omega t)^3\Big\} e^{-\omega ks/c} + \frac{\sigma_1}{2\sigma}\frac{1 + 2Ak/n}{1 + 4(k/n)^2} E_0^3 (1 + \mu \cos \Omega t)^3 e^{-3\omega ks/c};$$

$$\varphi(s) = \frac{\omega}{c} ns + \frac{\sigma_1}{2\sigma}\Big\{\frac{k}{n}\frac{1 + 2Ak/n}{1 + 4(k/n)^2} - \frac{A}{2}\Big\} E_0^2 (1 + \mu \cos \Omega t)(1 - e^{-2\omega ks/c}),$$

or

$$E = E_0 e^{-\omega ks/c}(1 - \gamma - {}^3/_2\gamma\mu^2) \times \{1 + \mu_\Omega \cos \Omega t - \mu_{2\Omega} \cos 2\Omega t - \mu_{3\Omega} \cos 3\Omega t\} \times \cos\{\omega t - (\omega ns/c) - \alpha(1 + \mu^2/2) - \beta_\Omega \cos \Omega t - \beta_{2\Omega} \cos 2\Omega t\},$$

where

$$\alpha = \frac{M}{2m}\frac{e^2}{6kT\omega^2 m}\Big\{\frac{k}{n}\frac{1 - 4{,}42\nu_{ef,0}k/\omega n}{1 + 4(k/n)^2} + 1{,}10\frac{\nu_{ef,0}}{\omega}\Big\} E_0^2 (1 - e^{-2\omega ks/c});$$

$$\gamma = \frac{M}{2m}\frac{e^2}{12kT\omega^2 m} \times \frac{1 - 4{,}42\nu_{ef,0}k/\omega n}{1 + 4(k/n)^2} E_0^2 (1 - e^{-2\omega ks/c});$$

$$\mu_\Omega = \frac{\mu(1 - 3\gamma - {}^3/_4\gamma\mu^2)}{1 - \gamma - {}^3/_2\gamma\mu^2};$$

$$\mu_{2\Omega} = \frac{{}^3/_2\gamma\mu^2}{1 - \gamma - {}^3/_2\gamma\mu^2};$$

$$\mu_{3\Omega} = \frac{{}^1/_4\gamma\mu^3}{1 - \gamma - {}^3/_2\gamma\mu^2}; \quad \beta_\Omega = 2\alpha\mu;$$

$$\beta_{2\Omega} = \alpha\mu^2/2.$$

If we assume that $k << n$, as is done in Ref. 7, and use the final results of this work, then the quantity α which determines the phase modulation is 1.2×10^3 times smaller than in Ref. 7, and the quantity which determines the amplitude modulation is about 4 times smaller.

Assuming the distance to the lower bound of the ionosphere to be $r \approx 100$ km, the temperature of the ionosphere to be $T = 300°$ K, the power of the transmitter $W = 200$ kw, the angular frequency of the transmitter $W = 3 \times 10^6$ rad/sec, the effective number of collisions $\nu_{ef,0} = 10^6$,, the angular frequency of modulation $\Omega = 300$ rad/sec, and replacing the value $2m/M = \delta_{el}$ (which characterizes the energy transfer in an elastic collision with a molecule) by 2×10^{-3} (Ref. 6), we get for the frequency shift $\Delta\omega$ (upon satisfying the condition $2\omega ks/c >> 1$): $d\varphi/dt = \omega + \Delta\omega = \omega - \beta_{\Omega}\Omega \sin \Omega t$, a quantity of the order of 2 rad/sec.

We note in conclusion that the experimental investigation of the phase self-modulation, arising in the passage through the ionosphere of an amplitude modulated electromagnetic wave would be useful for the determination of the effective number of collisions in this region of the ionosphere in which the phase self-modulation arises.

Translated by R. T. Beyer
96