Matrix Aspects of Boson Theory

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Certain relations are presented which make it possible to change at any stage of a calculation from anticommuting matrices to Kemmer matrices. The paper shows the connection between the Γ_μ, the matrices of the irreducible representation of the Kemmer and Dirac algebra, and also the Tamm matrices.

1. TRANSITION FROM MATRIX TO FIELD DESCRIPTION

IN a previous paper[1] we have introduced the basic operators which convert from undor matrix notation to the usual tensor notation. The scalar projection operator R, which has the property

$$k_0^{-2} R\Psi = \psi, \tag{1}$$

can obviously be expressed in terms of the reflection matrices $R_\mu = 2\beta_\mu^2 - I$,

$$R = (1/2)^4 \prod_{\mu=1}^{4} (I - R_\mu) \tag{2}$$

(Reference 4 contains an evident error in the coefficient). By expanding the product, one can also obtain for R the formula

$$R = (1/2)^4 ((I + R_5)(I + M_1) + M_2), \tag{3}$$

where

$$M_1 = \sum_{\mu=1}^{4} R_\mu, \quad M_2 = \sum_{\mu\neq}^{4} \sum_{\nu=1}^{4} R_\mu R_\nu, \tag{4}$$

$$M_3 = R_5 M_1 = \sum_{\mu\neq}^{4} \sum_{\nu\neq}^{4} \sum_{\rho=1}^{4} R_\mu R_\nu R_\rho. \tag{5}$$

The matrices I, M_1, M_2, $R_5 M_1$ form the center of the Kemmer algebra, the quantities which commute with all elements of G_{16} being[2]

$$I,\ M = M_1 - M_2, \quad N = R_5 (I - M_1).$$

The traces of the matrices M_1, M_2, M_3 vanish.

The pseudoscalar operator $\widetilde{R}$, with the property

$$k_0^{-2} \widetilde{R}\Psi = \widetilde{\psi}, \tag{6}$$

can be obtained from R by means of the generalized Larmor transformation[1] $\widetilde{R} = \Gamma_5 R \Gamma_5$.

Both these operators, by their definition, satisfy the obvious relations

$$R^n = R, \quad \widetilde{R}^n = \widetilde{R}, \quad \mathrm{Sp}\, R = \mathrm{Sp}\, \widetilde{R} = 1. \tag{7}$$

In working out the commutation relations between the Γ_μ and the operators M_1, M_2, one is forced to introduce the group of matrices $\overline{\Gamma}_\mu$, $\overline{G}_{16}$; this is reasonable since the commutative group R_μ connects Γ_μ and $\overline{\Gamma}_\mu$($\overline{\Gamma}_\mu = R_\mu \Gamma_\mu$, n.s.*):

$$\begin{gathered} \{\Gamma_\mu M_1\} = 2\overline{\Gamma}_\mu, \quad \{\overline{\Gamma}_\mu M_1\} = 2\Gamma_\mu, \\ [\Gamma_\mu M_2] = -2(M_1 \overline{\Gamma}_\mu - \Gamma_\mu), \\ [\overline{\Gamma}_\mu M_2] = -2(M_1 \Gamma_\mu - \overline{\Gamma}_\mu), \\ \{\Gamma_5 M_1\} = \{\overline{\Gamma}_5 M_1\} = [\Gamma_5 M_2] = [\overline{\Gamma}_5 M_2] = 0. \end{gathered} \tag{8}$$

With the help of these equations one can obtain the commutation rules between the Γ_μ and R:

$$[\Gamma_\mu R] = (1/2)^3 (I - R_5 + M_1)(\Gamma_\mu - \overline{\Gamma}_\mu), \tag{9}$$

and with the help of the latter one can prove that the vector operator

$$\begin{gathered} k_0^{-1} A_\mu \Psi = \psi_\mu, \\ A_\mu = (1/2)^4 (I - R_\mu) \prod_{\nu\neq\mu=1}^{4} (I + R_\nu) \end{gathered} \tag{10}$$

is equal to

$$A_\mu = \Gamma_\mu R \Gamma_\mu \quad \text{(n.s.)} \tag{11}$$

By performing the Larmor transformation on the A_μ, we obtain the pseudovector operators. These operators have the obvious properties:

$$A_\mu^n = A_\mu, \quad \widetilde{A}_\mu^n = \widetilde{A}_\mu, \quad \mathrm{Sp}\, A_\mu = \mathrm{Sp}\, \widetilde{A}_\mu = 1. \tag{12}$$

The rules for commuting A_μ, $\widetilde{A}_\mu$ with the Γ_μ and with each other, and also the analogous relations for the tensor projection operators can easily be worked out from the formulae given above.

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[1] A. A Borgardt, J. Exptl. Theoret. Phys. (U.S.S.R.) **24**, 24 (1953).

[2] N. Kemmer, Proc. Roy. Soc. (London) **173A** 94 (1939).

* "n. s." means no summation.

2. TRANSITION TO MIXTURES AND PURE FIELDS

It follows from (11) that the sum $\Gamma_\mu \vec{R} \Gamma_\mu$ extracts from the undor* the matrix vector $k_0(\psi, \psi_0)$. If we express the operator $\sum_{\mu=1}^{4} A_\mu$ in terms of the matrices β_μ we obtain the quantity β, which was at first introduced by Harish-Chandra[4]. Now it is easy to extract from the matrix equations in terms of the Γ_μ, the part relating, for example, to a pure scalar field.

If we construct the 16-dimensional unit matrix I from the scalar, vector, pseudovector, and pseudoscalar unit matrices, $I(s)$, $I(v)$, $I(\tilde{v})$ and $I(\tilde{s})$, $I(s)$, we see that the extraction of the pure scalar field from the Ψ field is accomplished by the operator $I(s)$

$$I(s) = R + \sum_{\mu=1}^{4} \Gamma_\mu R \Gamma_\mu. \tag{13}$$

The addition of $\tilde{R} + I(s)$ to the 16-dimensional unit matrix obviously extracts the pseudovector field

$$I(\tilde{v}) = I - I(s) - \tilde{R}. \tag{14}$$

The pseudoscalar and vector unit operators are obtained from $I(s)$ and $I(v)$ by a Larmor transformation.

The equations derived above can be used to obtain from the general equations

$$(\Gamma_\lambda \partial / \partial x_\lambda + k_0) \Psi = Q \tag{15}$$

for example, the equation of the pseudoscalar field

$$(\Gamma_\lambda \partial / \partial x_\lambda - k_0) \tilde{R}\Psi + k_0 I(\tilde{s}) \Psi = I(\tilde{s}) Q - \tilde{R} Q. \tag{16}$$

In the equations for mixed fields we can use at the same time both sets of anticommuting matrices Γ_μ and $\bar{\Gamma}_\mu$, and construct from them two reducible representations of the Kemmer algebra

$$\beta_\mu = 1/2 (\Gamma_\mu + \bar{\Gamma}_\mu), \quad \beta'_\mu = 1/2 (\Gamma_\mu - \bar{\Gamma}_\mu), \tag{17}$$

where $[\Gamma_\mu \bar{\Gamma}_\nu] = 0$. In this way we are led to wave equations of the type

$$[1/2 (\Gamma_\nu \pm \bar{\Gamma}_\lambda) \partial / \partial x_\lambda + k_0] \Psi = Q. \tag{18}$$

* Here and in the following we use the term "undor" to denote a matrix which has as elements a scalar, a vector and the components of all completely antisymmetric tensors in four-dimensional space (total number 16) (cf. Ref. 3).

[3] F. Belinfante, Physica **6**, 849, 870 (1939).

[4] Harish-Chandra, Proc. Roy. Soc. (London) **184A**, 215 (1946).

3. COVARIENT FORM OF THE HAMILTONIAN

The use of Hamiltonian equations, instead of relativistically symmetric ones like (15), in explicit calculations has a number of substantial advantages. However, this method does not satisfy the covarience requirement of modern theory[5], because of the way the time coordinate is singled out. We introduce therefore a three-dimensional hypersurface σ with a normal unit vector n_μ, $n_\lambda^2 = -I$. Any four vector A_μ can then be resolved into a longitudinal part A_μ^L and a transverse part A_μ^T by the relations

$$A_\mu^L = -n_\mu n_\lambda A_\lambda, \quad A_\mu^T = -A_{[\mu} n_{\lambda]} n_\lambda, \tag{19}$$

One verifies immediately that this resolution is correct and unique, since

$$A_\mu = A_\mu^L + A_\mu^T, \quad A^L = n_\lambda A_\lambda^L, \quad n_\lambda A_\lambda^T = 0.$$

Similarly we can form the matrix vectors Γ_μ and R_μ the purely longitudinal quantities

$$\Gamma_L = n_\lambda \Gamma_\lambda, \quad R_L = 2\beta_L^2 - I. \tag{20}$$

The properties of these matrices follow from their definition:

$$\{\Gamma_L \Gamma_\mu\} = 2n_\mu, \quad \{\Gamma_L R_5\} = 0, \tag{21}$$

$$\{\Gamma_L R_\mu\} = 2n_\mu \bar{\Gamma}_\mu.$$

The matrix R_L performs the transformation of the undor Ψ which represents a reversal of the direction of the normal to the hypersurfaces, and the operator $1/2 (I + R_L)$ is therefore related to the separation of the field quantities into longitudinal and transverse parts. It may be calculated in the following way. We introduce for the β_μ the representation $1/2 (\Gamma_\mu + \Gamma_\nu)$; then

$$\beta_L^2 = 1/2 (1 - n_\nu n_\sigma \hat{T}_{\lambda\sigma}), \tag{22}$$

where

$$\hat{T}_{\mu\nu} = 1/2 (\bar{\Gamma}_\mu \Gamma_\nu + \Gamma_\mu \bar{\Gamma}_\nu). \tag{23}$$

From the definition of R_L we find that

$$1/2 (I \pm R_L) = 1/2 (I \mp n_\nu n_\sigma \hat{T}_{\lambda\sigma}). \tag{24}$$

[5] J. Schwinger, Phys. Rev. **74**, 1439 (1948).

The commutation relations and some other necessary equations are given below:

$$[\hat{T}_{\mu\nu}\hat{T}_{\rho\tau}] = {}^1/_2\,(-(\Gamma_\rho\Gamma_\mu - \overline{\Gamma}_\mu\overline{\Gamma}_\rho)\,\delta_{\nu\tau} - (\Gamma_\tau\Gamma_\nu - \overline{\Gamma}_\nu\overline{\Gamma}_\tau)\,\delta_{\mu\rho} \tag{25}$$
$$+ (\Gamma_\nu\Gamma_\rho - \overline{\Gamma}_\rho\overline{\Gamma}_\nu)\,\delta_{\mu\tau} + (\Gamma_\mu\Gamma_\tau - \overline{\Gamma}_\tau\overline{\Gamma}_\mu)\,\delta_{\nu\rho}), \tag{26}$$
$$\{\Gamma_\mu\hat{T}_{\nu\rho}\} = \Gamma_\nu\delta_{\mu\rho} + \overline{\Gamma}_\mu\delta_{\mu\nu},\quad \{\overline{\Gamma}_\mu\hat{T}_{\nu\rho}\} = \overline{\Gamma}_\nu\delta_{\mu\rho} + \Gamma_\rho\delta_{\mu\nu}, \tag{27}$$
$$[R_5\hat{T}_{\mu\nu}] = [R_\mu\hat{T}_{\nu\rho}]_{\mu\neq\nu,\,\rho} = \{R_\mu\hat{T}_{\mu\rho}\}_{\mu\neq\rho} = \{\Gamma_5\hat{T}_{\mu\nu}\} = 0, \tag{28}$$
$$\hat{T}_{\mu\lambda}\Gamma_\lambda = {}^1/_2\,(6\overline{\Gamma}_\mu - M_1\Gamma_\mu),\qquad \hat{T}_{\mu\lambda}\overline{\Gamma}_\lambda = {}^1/_2\,(6\Gamma_\mu - M_1\overline{\Gamma}_\mu).$$

The symmetric tensor operator $\hat{T}_{\mu\nu}$ forms a density, which is conserved in the absence of sources for the field. Indeed, the equations

$$(\Gamma_\lambda\partial/\partial x_\lambda + k_0)\,\Psi = 0,$$
$$(\overline{\Gamma}_\lambda\partial/\partial x_\lambda + k_0)\,\Psi = 0, \tag{29}$$
$$(-\Gamma_\lambda^T\partial/\partial x_\lambda + k_0)\,\Psi^+ = 0,$$
$$(-\overline{\Gamma}_\lambda^T\partial/\partial x_\lambda + k_0)\,\Psi = 0$$

lead to the continuity equation

$$\partial\,(\Psi^+(\overline{\Gamma}_\mu\Gamma_\lambda + \Gamma_\mu\overline{\Gamma}_\lambda)\,\Psi)/\partial x_\lambda = 0, \tag{30}$$

i.e., the conservation law for the energy-momentum tensor of the field described by eqations (29). The complete Ψ field, with the anticommuting velocity matrices Γ_μ, possesses an energy-momentum density operator of the form $\Gamma_\mu\Gamma_\nu$. The operator $\hat{T}_{\mu\nu}$ enters also into the commutation laws for the components of Ψ. The commutator must be a solution of the wave equations of first order. In the case of the field described by the equations

$$[{}^1/_2\,(\Gamma_\lambda \pm \overline{\Gamma}_\lambda)\,\partial/\partial x_\lambda + k_0]\,\Psi = 0,$$

this requirement is satisfied by the undor

$$S^{(\pm)}(x) = (1/2\,k_0)\,[k_0^2 \pm \hat{T}_{\lambda\sigma}\,\partial^2/\partial x_\lambda\,\partial x_\sigma - k_0\,(\Gamma_\lambda \pm \overline{\Gamma}_\lambda)\,\partial/\partial x_\lambda]\,\Delta(x), \tag{31}$$

where

$$\Delta(x) = (2\pi)^{-3}\int e^{i\mathbf{k},x}\,\frac{\sin\left(c\sqrt{\mathbf{k}^2+k_0^2}\right)t}{c\sqrt{\mathbf{k}^2+k_0^2}}\,(d\mathbf{k}). \tag{32}$$

For a field with an anticommutive algebra we obtain from (31), as expected,

$$S(x) = (-\Gamma_\lambda\,\partial/\partial x_\lambda + k_0)\,\Delta(x). \tag{33}$$

It is convenient to express the commutative laws by resolving Ψ into a potential Ψ^{I} and a tension Ψ^{II}. They then take the form

$$[\Psi_\alpha^{\mp\mathrm{II}}(x'),\ \Psi_\beta^{\mathrm{II}}(x)] = -(i\hbar c^2/2)\,(k_0^2\delta_{\alpha\beta} \mp \hat{T}_{\lambda\sigma}^{\alpha,\,\beta}\,\partial^2/\partial x_\lambda\,\partial x_\sigma)\,\Delta(x-x'), \tag{34}$$
$$[\Psi_\alpha^{\mp\mathrm{I}}(x'),\ \Psi_\beta^{\mathrm{I}}(x)] = -(i\hbar c^2/2)\,(\delta_{\alpha\beta} \mp k_0^{-2}\hat{T}_{\lambda\sigma}^{\alpha,\,\beta}\,\partial^2/\partial x_\lambda\,\partial x_\sigma)\,\Delta(x-x'),$$
$$[\Psi_\alpha^{\mp\mathrm{II}}(x'),\ \Psi_\beta^{\mathrm{I}}(x)] = (i\hbar c^2/2)\,(\Gamma_\lambda \pm \overline{\Gamma}_\lambda)^{\alpha\beta}\,(\partial/\partial x_\lambda)\,\Delta(x-x').$$

The transition to a field with a purely anticommutive algebra can be carried out by omitting the terms with the alternative sign and doubling the even ones.

Because of the covariance of the division of the operator $\partial/\partial x_\mu$ into a longitudinal and a transverse part, the covarient form of the Hamiltonian equations is

$$((\hbar c/i)\,\partial/\partial x_L + H_{0L})\,\Psi = Q_L,$$
$$((\hbar c/i)\,\partial/\partial x_L + H_{0L}^+)\,\Psi^+ = Q_L^+, \tag{35}$$

where

$$H_{0L} = -(i\hbar c/2)\,[\Gamma_\lambda\Gamma_\sigma]\,n_{[\lambda}\partial/\partial x_{\sigma]} - \mathcal{E}_0\Gamma_L,\qquad Q_L = -\Gamma_L Q, \tag{36}$$
$$H_{0L}^+ = -(i\hbar c/2)\,[\Gamma_\lambda\Gamma_\sigma]^T\,n_{[\lambda}\partial/\partial x_{\sigma]} + \mathcal{E}_0\Gamma_L^T,\qquad Q_L^+ = +\Gamma_L^T Q^+. \tag{37}$$

In the equations of mixed Kemmer fields the Γ_μ must, as usual, be replaced by $1/2\,(\Gamma_\mu \pm \bar{\Gamma}_\mu)$.

4. CONNECTION BETWEEN Γ_μ AND FOUR-DIMENSIONAL QUATERNIONS

The separation of the wave function into a potential and a pressure by means of the operator $1/2(I \pm R_5)$ allows, besides the transition to photon theory, also the introduction of contracted, eight-row anticommuting matrices . Strictly speaking, there is no need to number the components of Ψ^{I} and Ψ^{II} differently in the equations

$$\Gamma_\lambda \partial\Psi^{I}/\partial x_\lambda + \Psi^{II} = Q^{II},$$
$$\Gamma_\lambda \partial\Psi^{II}/\partial x_\lambda + k_0^2\Psi^{I} = Q^{I}, \quad (38)$$

because of the difference in the superscripts. We may lable the components of Ψ^{I} and Ψ^{II} by the same set of numbers, if we change from Ψ^{I} to a new function $\bar{\Psi}^{I} = (1/i)\,\Gamma_4\Psi^{I}$. It is natural that the matrix Γ_4 has to be used for this purpose, since this is just the matrix which interchanges the canonically conjugate variables Ψ^{I} and Ψ^{II} ($\Psi^{+}R_5\Gamma_4\Psi$ is an action density, i.e., an invarient product of canonical coordinates and momenta). Equation (38) can now be put into the form*

$$\partial\bar{\Psi}^{I}/c\partial t + \vec{\Gamma}'\nabla\bar{\Psi}^{I} + \Psi^{II} = Q^{II},$$
$$-\partial\Psi^{II}/cdt + \vec{\Gamma}'\nabla\Psi^{II} + k_0^2\bar{\Psi}^{I} = \bar{Q}^{I}, \quad (39)$$

where $\bar{Q}^{I} = (1/i)\,\Gamma_4 Q^{I}$, and the eight-row matrices Γ' are defined by the relations

$$\vec{\Gamma}' = (1/i)\,\Gamma_4\vec{\Gamma}\cdot I(8) \quad (40)$$

and satisfy the anticommutive algebra

$$1/2\,\{\Gamma'_i\Gamma'_k\} - \delta_{ik}\cdot I = 0. \quad (41)$$

The product of all three matrices Γ'_i gives a matrix which commutes with the whole three-dimensional group G_8 formed by I, the Γ'_k and their products

$$\Gamma_0 = (1/i)\,\Gamma'_1\Gamma'_2\Gamma'_3. \quad (42)$$

From the definition of Γ_0 follows its main property:

$$1/2\,[\Gamma'_i\Gamma'_k] = i\Gamma_0\Gamma'_l. \quad (43)$$

(i, k, l being a permutation of 1, 2, 3). The group $\bar{G}_8$ ($[\bar{\Gamma}_i,\ \Gamma_k] = 0$) can be obtained by forming in analogy with (40), the eight-row matrices $\bar{\Gamma}'_k$ from the $\bar{\Gamma}_4\bar{\Gamma}_k$. The reflection matrices R_k are formed in the usual way

$$R_k = \bar{\Gamma}'_k\Gamma'_k \quad \text{(n.s.)} \quad (44)$$

their properties in relation to G_8, $\bar{G}_8$ being the same as in the 16-dimensional representation, except for the following new relations, which are absent in the latter case

$$R_iR_k = R_l, \quad M_1 = M_2, \quad (45)$$
$$M_3 = R_1R_2R_3 = I.$$

The scalar and pseudoscalar projection operators now cannot be separated (note that the Larmor transformation matrix Γ_5 commutes with Γ'_k!) and both occur in the form of the operator

$$R = R^n = (1/2)^3 \prod_{k=1}^{3} (I + R_k) = 1/4\,(I + M_1) \quad (46)$$

(the operator $\prod_{k=1}^{3} (I - R_k) \equiv 0$ vanishing identically). The vector projection operators are found as usual: $A_k = \Gamma'_k R\Gamma'_k$ (n.s.), but the vectors and pseudovectors are again undistinguishable.

The eight-dimensional representation Γ'_k is reducible, and we can extract from the contents of G_8 three anticommuting four-row matrices, which belong to the group G_{16} of the Dirac matrices of electron theory.

It would now be possible to go directly from equation (39) to a pure quaternion notation, but we shall make this transition by starting directly from the original wave equation

$$(\Gamma_\lambda\,\partial/\partial x_\lambda + k_0)\,\Psi = Q. \quad (47)$$

We now introduce a wave function which is complex even for a neutral field, letting $\bar{\Gamma}_s$ play the part of $\sqrt{-1}$:

$$\Phi = (I - \bar{\Gamma}_5)\,\Psi. \quad (48)$$

The equation

$$\Gamma_5\Phi = -\Phi, \; \Gamma_5\Phi = -R_5\Phi, \quad (49)$$

* In order to make the numbering of the elements of Ψ^{I}, Ψ^{II} continuous, it is better to choose a special representation, in which $\Psi = (\vec{\mathscr{E}},\ \psi,\ \vec{\mathscr{H}},\ \tilde{\psi},\ k_0\vec{\psi},\ k_0\psi_0,\ k_0\overleftarrow{\tilde{\psi}},\ k_0\psi_0)$.

which follows from (48), shows that one half of Φ repeats the other, apart from a sign. To spell this out, split Φ into a potential and a tension

$$\Phi^{I} = (1/2\,k_0)(I - R_5)\,\Phi, \qquad \Phi^{II} = {}^1\!/_2\,(I + R_5)\,\Phi. \tag{50}$$

Making use of (49), we then obtain

$$\Gamma_5\Phi^{I} = \Phi^{I}, \quad \Gamma_5\Phi^{II} = -\,\Phi^{II} \tag{51}$$

If we now also split Φ^{I} and Φ^{II} into parts by means of the relations* (52)

$$^1\!/_2\,(I - R_4)\,\Phi^{I} = \varphi^{I}, \quad {}^1\!/_2\,(I + R_4)\,\Phi^{I} = \varphi^{I'},$$
$$^1\!/_2\,(I + R_4)\,\Phi^{II} = \varphi^{II}, \quad {}^1\!/_2\,(I - R_4)\,\Phi^{II} = \varphi^{II'},$$

we find that $\Gamma_5\varphi^{I} = \varphi^{I'}$ and $\Gamma_5\varphi^{II} = -\,\varphi^{II'}$. Therefore φ^{I}, φ^{II} are equivalent with $\varphi^{I'}$, $\varphi^{II'}$

We now express the wave equation in terms of φ^{I} and φ^{II}:

$$(-\,i/c\Gamma_4\Gamma_5\partial/\partial t + \vec{\Gamma}\,\nabla)\,\varphi^{I} + \varphi^{II} = q^{II},$$
$$(+\,i/c\Gamma_4\Gamma_5\partial/\partial t + \vec{\Gamma}\,\nabla)\,\varphi^{II} + k_0^2\varphi^{II} = q^{I}.$$

We multiply both equations by $\Gamma_4\Gamma_5$ and remember the properties of Γ_4 and Γ_5 when operating on φ^{I} and φ^{II}. We find finally (Γ_4 leads to an identical labelling of the components of φ^{I} and φ^{II})

$$i\,\Box\,\varphi^{I} + \varphi^{II} = q^{II}, \quad i\,\Box^*\,\varphi^{II} + k_0^2\varphi^{I} = q^{I}, \tag{53}$$

where $\Box = \partial/c\partial t + \vec{\gamma}\,\nabla;\ \overline{\Box} = \partial/c\partial t - \vec{\gamma}\,\nabla,$

and $\quad {}^1\!/_2\,\{\gamma_i\gamma_k\} - \delta_{ik}I = 0, \quad \gamma_i\gamma_k = -\,i\gamma_l.$ (54)

The three matrices γ_k are connected with the Dirac matrices in the Kramers representation[6] by the relations

$$\gamma_1 = \beta, \quad \gamma_2 = \alpha_1\alpha_2\alpha_3\beta, \quad \gamma_3 = \alpha_1\alpha_2\alpha_3. \tag{55}$$

Amongst the elements of the same irreducible representation of G_{16} there exist three more matrices which have the same properties as the γ_k, but commute with them; they are

$$\overline{\gamma}_1 = i\alpha_1\alpha_2, \quad \overline{\gamma}_2 = i\alpha_2\alpha_3, \quad \overline{\gamma}_3 = i\alpha_3\alpha_1. \tag{56}$$

By carrying out similar operations on the equations

$$(\overline{\Gamma}_\lambda\,\partial/\partial x_\lambda + k_0)\,\Psi = \overline{Q},$$

we can show that their reduction to a four-row notation gives the equations

$$i\,\overline{\Box}\,\varphi^{I} + \varphi^{II} = \overline{q}^{II}, \quad i\,\overline{\Box}^*\,\varphi^{II} + k_0^2\varphi^{I} = \overline{q}^{I}. \tag{57}$$

Both systems (53) and (57) are equivalent to the corresponding Klein equation by virtue of the identity

$$\Box^*\Box = \overline{\Box}^*\,\overline{\Box} = -\,\nabla^2 + \partial^2/c^2\partial t. \tag{58}$$

Besides the condition of mutual commutability, a connection between $\overline{\gamma}_k$ and γ_k can be established by means of the reflection operators $r_k = \overline{\gamma}_k\gamma_k$ (n.s.). The properties of the r_k are the usual ones, but unlike the situation in the case of the 16-dimensional representation, multiplication of the r_k does not take us out of the framework of the four quantities (I, r_k) because of the relations $\gamma_i\gamma_k = -\,i\gamma_l$, $\overline{\gamma}_i\overline{\gamma}_k = -\,i\overline{\gamma}_l$; Therefore

$$r_i r_k = r_l, \quad r_1r_2r_3 = I, \quad M_1 = M_2. \tag{59}$$

The projection operators have the same properties as the corresponding operators in the 8-dimensional representation (see above).

The elimination of the (non-covariant) longitudinal components of φ^{I} and φ^{II} is done by means of the operator

$$T = T^n = {}^1\!/_2\,(I - R) = \sum_{k=1}^{3}\gamma_k R\gamma_k = {}^1\!/_4\,(3 + M_1), \tag{60}$$

which converts the anticommuting four-row matrices into three-row matrices of the three-dimensional irreducible representation of the Kemmer algebra:

$$T\gamma_k T = \beta_k, \tag{61}$$

$$\beta_i\beta_k\beta_l + \beta_l\beta_k\beta_i - \delta_{ik}\beta_l - \delta_{lk}\beta_i = 0.$$

Since the matrices γ_k satisfy, in addition, the relations $\beta_i\beta_k = -\,i\beta_l$, they can be transformed into Tamm matrices[7].

In this variant of the theory there is no possibility of a Larmor transformation, but here again the anticommuting matrices γ_k, $\overline{\gamma}_k$ may be expressed, similarly to (17) in terms of two orthogonal sets of matrices β_k:

* This division is already not covariant; for covariance one has to use the operator $^1\!/_2\,(I \pm R_L)$, cf. Sec. 3. The further conclusions remain, since $\{\Gamma_5 R_L\} = 0$.

[6] H. A. Kramers, Proc. Roy. Acad. Amsterdam **40**, 814 (1937).

[7] I. Tamm, Dokl. Akad. Nauk SSSR **23**, 551, 1940.

$$\gamma_k = \beta_k + \beta'_k, \quad \bar{\gamma}_k = \beta_k - \beta_k. \qquad (62)$$

It must be pointed out that the four-dimensional quantities of the type $A = A_0 + \vec{\gamma}\mathbf{A}$ are not real quaternions, similar to Hamilton's quaternions in three-dimensional space, because of the special treatment of the time. This is due to the pseudo-Euclidean character of the Minkowski world. For the same reason the result of the multiplication of the Cauchy-Riemann equations (53) is a hyperbolic equation.

In this way the present formulation of the theory may, with good justification, be called a pseudo-quaternion theory, at least in the case of non-static fields.

CONCLUSIONS

1. There exist relations which make it possible to use anticommuting matrices, not only in the theory of mixtures of meson fields, but also for "pure" fields.

2. It is possible to replace the sixteen-dimensional reducible representation of the Dirac algebra by an eight-dimensional one.

3. One may use, as kinematic matrices of photon or vector meson theory, the reflection matrices of electron theory, γ_4, $\gamma_1\gamma_2\gamma_3$ and $\gamma_1\gamma_2\gamma_3\gamma_4$. The connection of these with the Tamm matrices has been found.

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Interrelation between the Anisotropy of the Hall Effect and the Change in Resistance of Metals in a Magnetic Field. I. Investigation of Zinc

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The dependence of the resistance and the Hall effect for zinc on the magnitude of the angle between the axis of symmetry of the sixth order and the magnetic field is investigated for magnetic fields of up to 25,000 oersteds and for temperatures of 4 and 20°K. The possibility of explaining the observed regularities within the framework of present day theory is considered.

INTRODUCTION

A VERY large anisotropy of the resistance in a magnetic field has been observed for a number of metals. The resistance of a single crystal in a transverse magnetic field may change by 15 to 30 times when it is turned about an axis parallel to the current. Such a strong anisotropy, observed for certain orientations of single crystals of gallium, zinc, cadmium and tin[1-5], is unexpected, since in the absence of a magnetic field the anisotropy in the conductivity is small--of the order of tenths of a percent.[6]

The strong anisotropy is manifested by the occurrence of deep narrow minima, which we may call "anomalous minima," in the resistance, as plotted against angle of rotation. The position of the crystal which corresponds to the appearance of such an anomalous minimum is distinguished, as a rule, not only by the magnitude of the resistance, but also by the character of the dependence of the resistance on the magnetic field. In large fields the dependence on the field is not found to be quadratic, but weaker, approximately linear.[2-5]

It has previously been shown[7-9] that the delayed growth of the resistance in a magnetic field is

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