Electromagnetic Properties of a Finely Stratified Medium

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(Submitted to JETP editor June 9, 1954)
J. Exper. Theoret. Phys. USSR **29**, 605-616 (November, 1955)

Media composed of alternate layers of two isotropic materials, when the layers are sufficiently thin, behave on the average with relation to an electromagnetic field as if they were homogeneous but anisotropic (uniaxial crystal). The effective permeability tensors ϵ and μ of such a crystal are obtained, and limiting values are derived for thin layers as functions of the parameters of their materials, and of the frequency. Losses in a finely stratified medium are considered, and also boundary conditions at its surface.

INTRODUCTION

THE propagation of waves in media with a periodic laminated structure has often been considered, but attention has been directed primarily at the case in which the structural period d is comparable with the wavelength[1,2]. The case $d \ll \lambda$ was studied by Tamm and Ginzburg[3] in the application to a laminated core, composed of alternate layers of a dielectric and a metal (iron), on the assumption of a quite strongly developed skin effect in the metal. Levin[4] discussed in detail the propagation of an electromagnetic wave in the direction perpendicular to the layers of the periodic structure, without absorption. His method of calculation and his results for the case of long waves will be used below.

Recently the electrodynamics of stratified media, composed of alternate layers of a metal and a dielectric, have again attracted attention in connection with the following application. It has been found that in a transmission line filled with such a stratified medium, with proper arrangement of the layers and choice of the phase velocity of the wave, there is a possibility of decreasing the losses accompanying the propagation[5]. In the work cited, the calculation was done for the simplest case of a line formed of two conducting planes, the space between which is uniformly or partially filled with the stratified medium. The calculation was carried out both with and without allowance for the thickness of the elementary layers. The amount of the loss, calculated for a coaxial line with the stratified medium, was found to be in satisfactory agreement with experiment[6]. In addition, a detailed theoretical analysis was carried out for this type of transmission line[7,8]*.

In a study of the electromagnetic properties of stratified media, I obtained in 1949 the results reported below. (These results are partly contained in Sec. 3 of a work[10] published later, where they were obtained by another method.)

The subject to be considered is the "macroscopic" electromagnetic properties of a medium that consists of alternate layers of arbitrary homogeneous materials. We suppose that there are alternate plane layers of two substances with complex permeabilities ϵ_1, μ_1 (layer thickness a) and ϵ_2, μ_2 (layer thickness b). We are interested in the field averaged over the period $d = a + b$ of such a stratified structure. The introduction of this average field evidently has significance only under the condition that, along any arbitrary direction of propagation, its change in a distance of order d is sufficiently small. Quantitatively, this condition can be formulated thus:

$$kd\,|n| \ll 1, \qquad (1)$$

where $k = \omega/c = 2\pi/\lambda$, and n is the effective index of refraction of the medium, different for different

* References 5 and 7 are reviewed in *Uspekhi fizicheskikh nauk*[9], but regrettably without mention of reference 3--work done considerably earlier, and based on considerations similar to those enunciated by Academician N. D. Papaleksia with reference to a quasi-stationary system (a laminated core, the dimensions of which are small in comparison with the wavelength in the surrounding space).

1 Rayleigh, Phil. Mag. **24**, 145 (1887); Sci. Pap. III, p. 1.

2 L. Brillouin, *Wave propagation in periodic structures*, McGraw-Hill (1946), Sec. 44.

3 I. E. Tamm and V. L. Ginzburg, Izv. Akad. Nauk SSSR, Ser. Fiz. 7, 30 (1943).

4 M. L. Levin, J. Tech. Phys. USSR **18**, 1399 (1948).

5 A. M. Clogston, Proc. IRE **39**, 767 (1951); Bell Syst. Tech. J. **30**, 491 (1951).

6 H. S. Black, C. O. Mallinckrodt and S. P. Morgan, Proc. IRE **40**, 902 (1952).

7 S. P. Morgan, Bell Syst. Tech. J. **31**, 883, 1121 (1952).

8 E. F. Vaage, Bell Syst. Tech. J. **32**, 695 (1953).

9 Usp. Fiz. Nauk **49**, 325 (1953).

10 T. Sakurai, J. Phys. Soc. Japan **5**, 389 (1950).

directions of propagation and for different polarizations. As is shown below, for sufficiently long waves the effective properties of the medium are especially simple. That is, the inhomogeneous isotropic medium under consideration then behaves, with respect to the average field, like a homogeneous but anisotropic medium, i.e., the effective permeabilities ϵ^e, μ^e are singly degenerate tensors. This limiting transition to sufficiently small d/λ is analogous to the transition from x-rays to ultraviolet and longer waves in the case of crystals. The possibility of artificially constructing anisotropic materials of the indicated type and of varying their electromagnetic properties is likely to be of greater interest for "optical" centimeter waves than is their application merely to the decrease of losses in transmission lines.

From the symmetry of the problem, it is clearly sufficient to consider three cases of wave propagation: propagation in a direction parallel to the layers, for two polarizations (with either the electric or the magnetic vector parallel to the layers), and propagation in a direction perpendicular to the layers.

We take the z axis of a rectangular coordinate system perpendicular to the layers. The intensities of the true (quasi-microscopic) field in the material of the layers we will represent with $\mathbf{e}$ and $\mathbf{h}$, the average over a period d, by a line above.

PROPAGATION ALONG THE x AXIS; e DIRECTED ALONG THE y AXIS

In this case we have the following equation for the nonvanishing components $e_y = e$, h_x, and h_z of the quasi-microscopic field (Fig. 1):

$$\frac{\partial e}{\partial z} = ik\mu h_x, \tag{2}$$

$$\frac{\partial e}{\partial x} = -ik\mu h_z, \quad \frac{\partial h_x}{\partial z} - \frac{\partial h_z}{\partial x} = ik\varepsilon e,$$

where the permeabilities ϵ, μ are periodic functions of z, taking alternately the values ϵ_1, μ_1 and ϵ_2, μ_2. We seek a solution, periodic in z with period d, in the form

$$e = U(z)e^{-iknx}, \quad h_x = V(z)e^{-iknx}, \quad h_z = W(z)e^{-iknx}. \tag{3}$$

Substitution in Eq. (2) gives

$$\frac{dU}{dz} = ik\mu W, \quad nU = \mu W, \tag{4}$$

$$\frac{dV}{dz} + iknW = ik\varepsilon U.$$

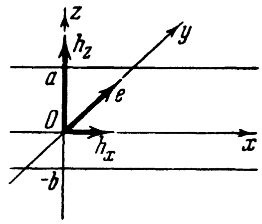

FIG. 1.

If ϵ and μ were continuous functions of z, then it would be possible immediately to apply to the ordinary Eq. (4), in the case of small d in which we are interested, the theorem of Fatou[11] on the approximate solution of an equation with rapidly oscillating periodic coefficients. Since in our problem the conditions of Fatou's theorem are not satisfied, and also to avoid premature commitment to the case of small d, we will start from the exact solution of Eqs. (4):

$$U = A\cos\alpha_1 z + B\sin\alpha_1 z, \qquad z \subset a,$$
$$V = -\frac{\alpha_1}{ik\mu_1}(A\sin\alpha_1 z - B\cos\alpha_1 z),$$
$$\alpha_1 = k\sqrt{n_1^2 - n^2}, \tag{5}$$
$$W = \frac{n}{\mu_1}(A\cos\alpha_1 z + B\sin\alpha_1 z), \quad n_1^2 = \varepsilon_1\mu_1,$$
$$U = C\cos\alpha_2 z + D\sin\alpha_2 z, \qquad z \subset b,$$
$$V = -\frac{\alpha_2}{ik\mu_2}(C\sin\alpha_2 z - D\cos\alpha_2 z),$$
$$\alpha_2 = k\sqrt{n_2^2 - n^2},$$
$$W = \frac{n}{\mu_2}(C\cos\alpha_2 z + D\sin\alpha_2 z), \quad n_2^2 = \varepsilon_2\mu_2.$$

On this solution must be imposed four conditions of continuity and periodicity of e and h_x (i.e., U and V) with respect to z:

$$U(+0) = U(-0), \quad U(a-0) = U(-b+0), \tag{6}$$
$$V(+0) = V(-0), \quad V(a-0) = V(-b+0).$$

On substituting Eq. (5) in Eq. (6), we get for A, B, C and D the four homogeneous equations

$$C = A, \quad C\cos\alpha_2 b - D\sin\alpha_2 b = A\cos\alpha_1 a + B\sin\alpha_1 a, \tag{7}$$
$$D = \varkappa B, \quad C\sin\alpha_2 b + D\cos\alpha_2 b = -\varkappa(A\sin\alpha_1 a - B\cos\alpha_1 a),$$

where

$$\varkappa = \mu_2\alpha_1/\mu_1\alpha_2. \tag{8}$$

[11] P. Fatou, Bull. de la Soc. math. de France 57, 98 (1928); L. Boltzmann, Wiss. Abh. I, 43 (1909).

On equating to zero the determinant of the system of equations (7), we get the following dispersion equation, determining n as a function of k:

$$(1+\varkappa^2)\sin\alpha_1 a\sin\alpha_2 b + 2\varkappa(1-\cos\alpha_1 a\cos\alpha_2 b) = 0. \tag{9}$$

By solving this equation for κ, it is not difficult to reduce it to the following combination of two equations:

$$\frac{\operatorname{tg}\frac{\alpha_2 b}{2}}{\operatorname{tg}\frac{\alpha_1 a}{2}} = \begin{cases} -\varkappa, \\ -1/\varkappa. \end{cases} \tag{10}$$

With the aid of (5) we compute the mean values, over the period d, of U, V and W. If in addition we use the values of the ratios $A:B:C:D$ from (7), and the expression (8), we get the following expressions for the ratios of $\overline{U}$ and $\overline{V}$ to $\overline{W}$:

$$\frac{\overline{U}}{\overline{W}} = \frac{\overline{e}}{\overline{h}_z} = \frac{\mu_1\alpha_2^2-\mu_2\alpha_1^2}{n(\alpha_2^2-\alpha_1^2)}, \tag{11}$$

$$\frac{\overline{V}}{\overline{W}} = \frac{\overline{h}_x}{\overline{h}_z} = \frac{(1/\mu_1)-(1/\mu_2)}{k^2[(1/\alpha_1^2)-(1/\alpha_2^2)]}P,$$

where

$$P = \frac{U(a-0)-U(+0)}{V(a-0)-V(+0)} = -\frac{ik\mu_1}{\alpha_1}\,\frac{1}{\operatorname{tg}(\alpha_2 b/2)}\,\frac{\varkappa\operatorname{tg}(\alpha_1 a/2)+\operatorname{tg}(\alpha_2 b/2)}{\varkappa\operatorname{tg}(\alpha_2 b/2)+\operatorname{tg}(\alpha_1 a/2)}. \tag{12}$$

According to Eq. (10), P is either zero [if the roots of the top equation (10) are taken for n] or infinite (for the roots of the botton equation).

In the first case the structure of the quasi-microscopic field is this: e and h_z are distributed in each layer symmetrically about its middle, i.e., they are even functions of $z - a/2$ in layer I and of $z + b/2$ in layer II. The longitudinal component h_x, however, is odd, so that the mean field is a transverse wave ($\overline{h}_x = 0$) with intensities $\overline{e}$ and $\overline{h}_z$.

In the second case the picture is the direct opposite: h_x is even, e and h_z odd with respect to the middles of the layers; consequently, the mean field has only the component $\overline{h}_x$. It is easy to see, however, that this case is not compatible with the condition of slow change of the mean field over distances of order d. For the space between the middles of the two layers, e.g., between $-b/2$ and $a/2$, can be regarded as bounded by perfectly conducting planes, since on them $e = h_z = 0$. But in such cracks of thickness $d/2$, for arbitrary permeabilities ϵ_1, μ_1 and ϵ_2, μ_2 of the "packings" between the perfectly conducting walls, the whole wave with $\lambda > d/2$ will be extinguished in the x direction in distances at most of order $d/2\pi$. Thus for the mean field in a stratum of the medium, the second case never satisfies condition (1) and therefore is of no interest.

In the first case, in order that the mean field intensities $E_y = \overline{U}e^{-iknx}$ and $H_z = \overline{W}e^{-iknx}$ may satisfy Maxwell's equations

$$\frac{\partial E_y}{\partial x} = -ik\mu^e H_z, \quad \frac{\partial H_z}{\partial x} = -ik\varepsilon^e E_y, \tag{13}$$

it is necessary to define the effective permeabilities ϵ^e and μ^e by the equations

$$n = \sqrt{\varepsilon^e\mu^e}, \quad \frac{E_y}{H_z} = \sqrt{\frac{\mu^e}{\varepsilon^e}} = \frac{\overline{U}}{\overline{W}}.$$

Hence, using (11), we get

$$\varepsilon^e = n^2\frac{\alpha_2^2-\alpha_1^2}{\mu_1\alpha_2^2-\mu_2\alpha_1^2}, \quad \mu^e = \frac{\mu_1\alpha_2^2-\mu_2\alpha_1^2}{\alpha_2^2-\alpha_1^2}, \tag{14}$$

where n is the root of the upper equation (10), i.e., of the equation

$$\frac{\alpha_2}{\mu_2}\operatorname{tg}\frac{\alpha_2 b}{2} = -\frac{\alpha_1}{\mu_1}\operatorname{tg}\frac{\alpha_1 a}{2}. \tag{15}$$

PROPAGATION ALONG THE x AXIS; $\mathbf{h}$ DIRECTED ALONG THE y AXIS

The equations for the quasi-microscopic field, which in this case has components e_x, e_z and $h_y = h$ (Fig. 2), are as follows:

$$\frac{\partial h}{\partial z} = -ik\varepsilon e_x, \quad \frac{\partial h}{\partial x} = -ik\varepsilon e_z,$$

$$\frac{\partial e_x}{\partial z} - \frac{\partial e_z}{\partial x} = -ik\mu h,$$

and the conditions of continuity and periodicity at the boundaries between the layers are imposed on h and e_x. Evidently all the formulas relating to the present case can be obtained directly from the formulas of the preceding paragraph by changing $\mathbf{h}$, $\mathbf{e}$, ϵ, μ to $\mathbf{e}$, $-\mathbf{h}$, μ, ϵ, respectively. Here also there are two possible structures of the quasi-microscopic field. One of these does not give a mean field satisfying condition (1), since the corresponding microfield is rapidly extinguished in equivalent slots between the middles of the layers. Here these middle planes can be regarded as surfaces of an ideal magnet, for on them $h = e_z = 0$. The other structure of the microfield leads to a mean field that forms a transverse wave with intensities $E_z = \overline{e}_z$ and $H_y = \overline{h}$. This mean field satisfies Maxwell's equations:

$$\frac{\partial H_y}{\partial x} = ik\varepsilon^e E_z, \quad \frac{\partial E_z}{\partial x} = ik\mu^e H_y, \tag{16}$$

where ϵ^e and μ^e are given by the formulas

$$\varepsilon^e = \frac{\varepsilon_1\alpha_2^2 - \varepsilon_2\alpha_1^2}{\alpha_2^2 - \alpha_1^2}, \quad \mu^e = n^2 \frac{\alpha_2^2 - \alpha_1^2}{\varepsilon_1\alpha_2^2 - \varepsilon_2\alpha_1^2}, \tag{17}$$

obtained from (14) by the change indicated above. Eq. (15), correspondingly changed, determines the value of n:

$$\frac{\alpha_2}{\varepsilon_2} \operatorname{tg} \frac{\alpha_2 b}{2} = -\frac{\alpha_1}{\varepsilon_1} \operatorname{tg} \frac{\alpha_1 a}{2}. \tag{18}$$

The transcendental equations (15) and (18) in the general case do not permit expression of n in explicit form in terms of the parameters of the problem. Therefore, condition (1), for the slowness of change of the mean field, is also incapable of exhibition in a more explicit form. It must be emphasized, however, that in the two cases considered (propagation along the layers) condition (1) does not necessarily imply that the changes of the quasi-microscopic field across the thickness of the layers are small. The correctness of Eqs. (14), (15) and (17), (18) is by no means connected with this last, more severe condition. To clarify this remark, we apply these formulas to the case considered in reference 3, in which layers of iron (thickness a, $\epsilon_1 = -i4\pi\sigma/\omega$, $n_1^2 = -2i/k^2\delta^2$, where σ is the conductivity and δ is the skin depth) alternate with layers of an ideal dielectric (thickness b, $\mu_2 = 1$, $\epsilon_2 = n_2^2$).

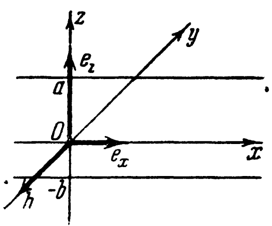

FIG. 2.

If the vector **e** is parallel to the layers [formulas (14) and (15)], then propagation along the layers is impossible, i.e., the microfield, and consequently also the mean field, dies out along x in a distance of order d. For, taking into account that a and $b \gg \delta$, and assuming in advance that $|n| \ll |n_1|$, we have

$$\frac{\alpha_1 a}{2} = \frac{ka}{2}\sqrt{n_1^2 - n^2} \approx \frac{ka}{2} n_1 = \frac{a}{\delta}\sqrt{-\frac{i}{2}},$$

so that

$$\operatorname{tg} \frac{\alpha_1 a}{2} \approx -i. \tag{19}$$

Then from (15)

$$\operatorname{tg} \frac{\alpha_2 b}{2} \approx \frac{\sqrt{2i}}{\mu_1 \alpha_2 \delta},$$

from which it is clear that $\alpha_2 b$ differs from π by a quantity of order $\mu_1 \delta / b$. Neglecting this correction, we set $\alpha_2 b = \pi$, i.e.,

$$\alpha_2 = k\sqrt{n_2^2 - n^2} = \pi / b,$$

and since $n_2^2 = \epsilon_2 \ll \pi^2/k^2b^2$,

$$n \approx -i\pi / kb.$$

Thus the propagation factor has the form $\exp\{-iknx\} = \exp\{-\pi x/b\}$, and condition (1) is not satisfied.

If, however, the vector **h** is parallel to the layers [formulas (17) and (18)], then, using (19), we get from (18)

$$\operatorname{tg} \frac{\alpha_2 b}{2} \approx -\frac{k^2\varepsilon_2\mu_1\delta}{\alpha_2\sqrt{2i}},$$

from which it is evident that $|\alpha_2 b|$ is very small. Therefore, replacing the tangent by its argument, we find

$$\alpha_2^2 = k^2(\varepsilon_2 - n^2) = -\frac{2k^2\varepsilon_2\mu_1\delta}{b\sqrt{2i}},$$

whence

$$n^2 \approx \varepsilon_2\left(1 + \frac{2\mu_1\delta}{b\sqrt{2i}}\right).$$

That is, the mean index of refraction is close to the index of refraction of the dielectric layers, and the extinction of the mean field in the x direction proceeds very slowly (over distances of order $2b/\mu_1\sqrt{\epsilon_2}\,k\delta$), despite the rapid change of the microfield over the thickness of the layers ($\delta \ll a$). Using the expression already written for α_2^2 and n^2 and the expression $\alpha_1^2 = -2i/\delta^2$, we get from (17)

$$\varepsilon^e = \varepsilon_2\left(1 + \frac{2\delta}{b\sqrt{2i}}\right), \quad \mu^e = \frac{\mu_1 + (1+i)(b/2\delta)}{1 + (1+i)(b/2\delta)}.$$

This expression for the effective magnetic permeability agrees with that obtained in reference 3 [Eq. (14)].

We now derive formulas that hold in the case of sufficiently long waves, for which $|\alpha_1 a|$ and $|\alpha_2 b| \ll 1$. Replacing the tangents by their arguments in (15) and (18), we get from (15) and (14)

$$n = \sqrt{\varepsilon^e \mu^e}, \quad \varepsilon^e = \bar{\varepsilon}, \quad \mu^e = \tilde{\mu}, \tag{20}$$

where

$$\bar{\varepsilon} = \frac{a\varepsilon_1 + b\varepsilon_2}{a+b}, \quad \frac{1}{\tilde{\mu}} = \frac{\bar{1}}{\mu} = \frac{(a/\mu_1)+(b/\mu_2)}{a+b}, \tag{21}$$

and from (18) and (17):

$$n = \sqrt{\varepsilon^e \mu^e}, \quad \varepsilon^e = \tilde{\varepsilon}, \quad \mu^e = \bar{\mu}, \tag{22}$$

where

$$\frac{1}{\tilde{\varepsilon}} = \frac{\bar{1}}{\varepsilon} = \frac{(a/\varepsilon_1)+(b/\varepsilon_2)}{a+b}, \quad \bar{\mu} = \frac{a\mu_1 + b\mu_2}{a+b}. \tag{23}$$

A more precise evaluation--with account taken of cubic terms in the expansion of the tangents in (15) and (18)--shows that the correction terms in the effective permeabilities are of order k^2d^2. For example, the expressions for ϵ^e and μ^e in the case of **e** parallel to the layers, with the indicated correction, are

$$\varepsilon^e = \bar{\varepsilon}\left[1 + \frac{k^2a^2b^2}{12d^2}\,\frac{\bar{\mu}\,\tilde{\mu}}{\mu_1\mu_2}(n_1^2 - n_2^2)\,\frac{\varepsilon_1 - \varepsilon_2}{\bar{\varepsilon}}\right], \tag{24}$$

$$\mu^e = \tilde{\mu}\left[1 + \frac{k^2a^2b^2}{12d^2}\,\frac{\bar{\mu}\,\tilde{\mu}^2}{\mu_1^2\mu_2^2}(n_1^2 - n_2^2)(\mu_1 - \mu_2)\right].$$

Replacement of ϵ and μ by μ and ϵ in (24) gives the expressions for ϵ^e and μ^e in the case in which the vector **h** is parallel to the layers.

Formulas (24) are valid provided the correction terms are small; the latter, unlike condition (1), contain differences of parameters describing properties of the layers. Consequently, if the parameters of one of the materials are not vastly different from the parameters of the other, then the applicability of (24) will be subject to less stringent limitations than that of condition (1). A different situation may be encountered in those cases in which the parameters of the two materials are significantly different (in absolute value), as is the case in the example considered of a laminated iron core. Here formulas (24) and analogous formulas for the other polarization will be valid only provided

$$a^2 \ll 6\delta^2,$$

that is, when there is sufficiently small change of the microfield over the thickness of the iron sheets.

PROPAGATION ALONG THE z AXIS

The microfield $e_x = e$, $h_y = h$ (Fig. 3) satisfies the equations

$$\frac{de}{dz} = -ik\mu h, \quad \frac{dh}{dz} = -ik\varepsilon e. \tag{25}$$

With continuous $\epsilon(z)$ and $\mu(z)$, it would be possible to conclude at once, on the basis of Fatou's theorem[11], that for $d \to 0$ the solution of Eqs. (25) would approach the solution of the same equations with average permeabilities $\bar{\epsilon}$ and $\bar{\mu}$*. A rigorous solution shows that this result remains valid also in the case under consideration of discontinuous ϵ and μ. Nevertheless, since we shall be interested in the magnitude of the correction to $\bar{\epsilon}$ and $\bar{\mu}$ in the effective ϵ^e and μ^e, here also we shall carry out an exact elementary calculation, following a method introduced by Levin[4].

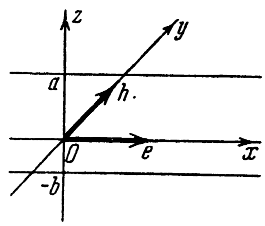

FIG. 3.

According to a theorem of Floquet[12], the solution of Eq. (25) has the form

$$e = U(z)\,e^{-iknz}, \quad h = V(z)\,e^{-iknz}, \tag{26}$$

where U and V are periodic functions of z with period d. If we integrate Eqs. (25) directly for each of the layers and compare the results with (26), we get

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* We notice that, on elimination of h (or e) from (25), we get for e (or h) an equation of second order; applied to it, Fatou's theorem gives for the mean index of refraction the value $n^2 = \overline{\epsilon\mu}$, while from (25) it follows that $n^2 = \bar{\epsilon}\bar{\mu}$. This discrepancy is due to the fact that the transformation to a second-order equation is tied up with a differentiation of one of the equations (25); and the derivative of the approximate solution is not equal to the limit for $d \to 0$ of the derivative of the exact solution. Therefore, for getting an approximate e and h, the transformation to a second-order equation is not permissible.

[12] G. Floquet, Ann. École Norm. 12, 47 (1883); Whittaker and Watson, *Modern Analysis* (4th ed., Cambridge, 1940), p. 412.

$$\begin{aligned} &U = e^{iknz}(Ae^{i\alpha_1 z} + Be^{-i\alpha_1 z}), && z \subset a, \\ &V = -\frac{\alpha_1}{k\mu_1} e^{iknz}(Ae^{i\alpha_1 z} - Be^{-i\alpha_1 z}), && \alpha_1 = kn_1 = k\sqrt{\varepsilon_1\mu_1}, \\ &U = e^{iknz}(Ce^{i\alpha_2 z} + De^{-i\alpha_2 z}), && z \subset b, \\ &V = -\frac{\alpha_2}{k\mu_2} e^{iknz}(Ce^{i\alpha_2 z} - De^{-i\alpha_2 z}), && \alpha_2 = kn_2 = k\sqrt{\varepsilon_2\mu_2}. \end{aligned} \tag{27}$$

Imposing on U and V the conditions of continuity and periodicity, which here also take the form (6), we get four homogeneous equations for the constants of integration:

$$\begin{gathered} C + D = A + B, \\ e^{-iknb}(Ce^{-i\alpha_2 b} + De^{i\alpha_2 b}) = e^{ikna}(Ae^{i\alpha_1 a} + Be^{-i\alpha_1 a}), \\ C - D = \varkappa(A - B), \\ e^{-iknb}(Ce^{-i\alpha_2 b} - De^{i\alpha_2 b}) = \varkappa e^{ikna}(Ae^{i\alpha_1 a} - Be^{-i\alpha_1 a}), \end{gathered} \tag{28}$$

where

$$\varkappa = \frac{\alpha_1\mu_2}{\alpha_2\mu_1} = \sqrt{\frac{\varepsilon_1\mu_2}{\varepsilon_2\mu_1}}. \tag{29}$$

Setting the determinant of the system (28) equal to zero gives the dispersion equation, which determines n:

$$\cos knd = \cos\alpha_1 a \cos\alpha_2 b - \frac{1+\varkappa^2}{2\varkappa}\sin\alpha_1 a \sin\alpha_2 b. \tag{30}$$

In contrast to the two cases considered earlier, now, when the propagation occurs in the direction perpendicular to the layers, slow change of the mean field is possible only on condition that the microfield also changes only slightly over the thickness of the layers. In other words, condition (1) can be realized only when the waves are so long that $|\alpha_1 a|$ and $|\alpha_2 b| \ll 1$. If, on this assumption, we keep terms of no higher than the first order in k in calculating the mean values over a period, we get for the ratio of $\overline{U}$ to $\overline{V}$

$$\frac{\overline{U}}{\overline{V}} = \frac{\overline{e}}{\overline{h}} = \sqrt{\frac{\overline{\mu}}{\overline{\varepsilon}}}\left[1 + \frac{ikab}{4d}\frac{\mu_1\varepsilon_2 - \mu_2\varepsilon_1}{\sqrt{\overline{\varepsilon}\overline{\mu}}}\right], \tag{31}$$

where $\overline{\epsilon}$ and $\overline{\mu}$ have the same meaning as in (21) and (23). To the same degree of exactness, n is obtained from Eq. (30) in the form

$$n = \sqrt{\overline{\varepsilon}\overline{\mu}}, \tag{32}$$

that is, the correction to n is of second order in kd (cf. reference 4).

If we require that Maxwell's equations

$$\frac{\partial E_x}{\partial z} = -ik\mu^e H_y, \quad \frac{\partial H_y}{\partial z} = -ik\varepsilon^e E_x, \tag{33}$$

shall hold for the mean fields $E_x = \overline{e}$ and $H_y = \overline{h}$, we must determine ϵ^e and μ^e from the conditions

$$n = \sqrt{\varepsilon^e\mu^e}, \quad \frac{E_x}{H_y} = \sqrt{\frac{\mu^e}{\varepsilon^e}} = \frac{\overline{U}}{\overline{V}};$$

whence

$$\varepsilon^e = n\overline{V}/\overline{U}, \qquad \mu^e = n\overline{U}/\overline{V}.$$

Substituting here (31) and (32), we get

$$\begin{aligned} \varepsilon^e &= \overline{\varepsilon}\left[1 - \frac{ikab}{4d}\frac{\mu_1\varepsilon_2 - \mu_2\varepsilon_1}{\sqrt{\overline{\varepsilon}\overline{\mu}}}\right], \\ \mu^e &= \overline{\mu}\left[1 + \frac{ikab}{4d}\frac{\mu_1\varepsilon_2 - \mu_2\varepsilon_1}{\sqrt{\overline{\varepsilon}\overline{\mu}}}\right]. \end{aligned} \tag{34}$$

The validity of Fatou's theorem is evident from this: with diminution of kd we arrive at the simple mean of the coefficients in Eqs. (25), i.e.,

$$\varepsilon^e = \overline{\varepsilon}, \qquad \mu^e = \overline{\mu}. \tag{35}$$

THE CASE OF LONG WAVES

Let kd be so small that the correction terms can be neglected both in (24) and in (34). The corresponding conditions take the following form, if we introduce the notation

$$\frac{\overline{\varepsilon}\tilde{\varepsilon}}{\varepsilon_1\varepsilon_2} = \frac{a\varepsilon_1 + b\varepsilon_2}{a\varepsilon_2 + b\varepsilon_1} = P, \qquad \frac{\overline{\mu}\tilde{\mu}}{\mu_1\mu_2} = \frac{a\mu_1 + b\mu_2}{a\mu_2 + b\mu_1} = Q, \qquad kab/d = R. \tag{36}$$

For propagation along the layers with e parallel to the layers:

$$\frac{R^2}{12}\left|Q(n_1^2-n_2^2)\frac{\varepsilon_1-\varepsilon_2}{\bar{\varepsilon}}\right|\ll 1, \qquad (37a)$$

$$\frac{R^2}{12}\left|Q^2(n_1^2-n_2^2)\frac{\mu_1-\mu_2}{\bar{\mu}}\right|\ll 1 .$$

For propagation along the layers with h parallel to the layers:

$$\frac{R^2}{12}\left|P^2(n_1^2-n_2^2)\frac{\varepsilon_1-\varepsilon_2}{\bar{\varepsilon}}\right|\ll 1, \qquad (37b)$$

$$\frac{R^2}{12}\left|P(n_1^2-n_2^2)\frac{\mu_1-\mu_2}{\bar{\mu}}\right|\ll 1.$$

For propagation perpendicular to the layers:

$$\frac{R}{4}\left|\frac{\mu_1\varepsilon_2-\mu_2\varepsilon_1}{\sqrt{\tilde{\varepsilon}\tilde{\mu}}}\right|\ll 1. \qquad (37c)$$

If, in particular, $a = b = d/2$, then $P = Q = 1$, $R = kd/4$, and conditions (37 a, b, c) take the quite simple form

$$\frac{k^2d^2}{96}\left|(n_1^2-n_2^2)\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right|\ll 1, \quad \frac{k^2d^2}{96}\left|(n_1^2-n_2^2)\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right|\ll 1, \qquad (38)$$

$$\frac{kd}{8}\left|\frac{\mu_1\varepsilon_2-\mu_2\varepsilon_1}{\sqrt{(\varepsilon_1+\varepsilon_2)(\mu_1+\mu_2)}}\right|\ll 1.$$

In this special case it is particularly clear that satisfaction of these inequalities still does not imply satisfaction of condition (1), which for the directions of propagation and of polarization considered takes the form:

$$kd\left|\sqrt{\frac{\mu_1\mu_2(\varepsilon_1+\varepsilon_2)}{\mu_1+\mu_2}}\right|\ll 1, \qquad (39)$$

$$kd\left|\sqrt{\frac{\varepsilon_1\varepsilon_2(\mu_1+\mu_2)}{\varepsilon_1+\varepsilon_2}}\right|\ll 1,$$

$$\frac{kd}{2}\left|\sqrt{(\varepsilon_1+\varepsilon_2)(\mu_1+\mu_2)}\right|\ll 1.$$

Considering conditions (37 a, b, c) satisfied and gathering together Eqs. (13), (16), (20), (22), (33) and (35), we get

$$\frac{\partial E_y}{\partial x}=-ik\tilde{\mu}H_z, \qquad \frac{\partial H_z}{\partial x}=-ik\bar{\varepsilon}E_y,$$

$$\frac{\partial H_y}{\partial x}=ik\tilde{\varepsilon}E_z, \qquad \frac{\partial E_z}{\partial x}=ik\bar{\mu}H_y,$$

$$\frac{\partial E_x}{\partial z}=-ik\bar{\mu}H_y, \qquad \frac{\partial H_y}{\partial z}=-ik\bar{\varepsilon}E_x.$$

It is not difficult to see that these equations belong to the general system of Maxwellian equations for a medium whose permeabilities are described by singly degenerate tensors with coincident principal axes and with principal values

$$\varepsilon_1^e=\varepsilon_2^e=\bar{\varepsilon}=\frac{a\varepsilon_1+b\varepsilon_2}{a+b}, \qquad \varepsilon_3^e=\tilde{\varepsilon}=\frac{\varepsilon_1\varepsilon_2(a+b)}{a\varepsilon_2+b\varepsilon_1},$$

$$\mu_1^e=\mu_2^e=\bar{\mu}=\frac{a\mu_1+b\mu_2}{a+b}, \qquad \mu_3^e=\tilde{\mu}=\frac{\mu_1\mu_2(a+b)}{a\mu_2+b\mu_1}. \qquad (40)$$

Thus in its "optical" range, the medium under consideration has the properties of a uniaxial crystal with optic axis perpendicular to the layers, and with normal surface consisting (for real ϵ_1, μ_1, ϵ_2, μ_2) of two ellipsoids of revolution (Fig. 4).

From the expressions (40) it is clear that, depending on the relations between the layer thicknesses a and b and the absolute values of the permeabilities, there are possible very different combinations of values of the effective permeabilities and, correspondingly, different degrees of anisotropy, dichroism, etc.

We shall express ϵ and μ in terms of the real permeabilities ϵ', μ' and the tangents of the loss angles

$$\varepsilon_k=\varepsilon_k'(1-i\,\mathrm{tg}\,\delta_k), \quad \mu_k=\mu_k'(1-i\,\mathrm{tg}\,\gamma_k) \quad (k=1,2),$$

$$\bar{\varepsilon}=\bar{\varepsilon}'(1-i\,\mathrm{tg}\,\bar{\delta}), \quad \bar{\mu}=\bar{\mu}'(1-i\,\mathrm{tg}\,\bar{\gamma}), \qquad (41)$$

$$\tilde{\varepsilon}=\tilde{\varepsilon}'(1-i\,\mathrm{tg}\,\tilde{\delta}), \quad \tilde{\mu}=\tilde{\mu}'(1-i\,\mathrm{tg}\,\tilde{\gamma}).$$

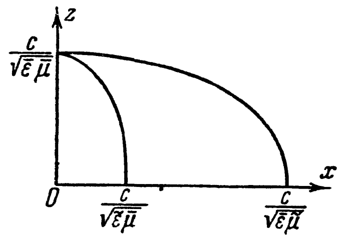

FIG. 4.

Neglecting squares and products of the tangents of the loss angles, we get from (40) and (41)

$$\bar{\varepsilon}' = \frac{a\varepsilon_1' + b\varepsilon_2'}{a+\beta}, \quad \operatorname{tg}\bar{\delta} = \frac{a\varepsilon_1' \operatorname{tg}\delta_1 + b\varepsilon_2' \operatorname{tg}\delta_2}{a\varepsilon_1' + b\varepsilon_2'}, \tag{42}$$

$$\tilde{\varepsilon}' = \frac{\varepsilon_1'\varepsilon_2'(a+b)}{a\varepsilon_2' + b\varepsilon_1'}, \quad \operatorname{tg}\tilde{\delta} = \frac{a\varepsilon_2' \operatorname{tg}\delta_1 + b\varepsilon_1' \operatorname{tg}\delta_2}{a\varepsilon_2' + b\varepsilon_1'}$$

and analogous formulas for the magnetic quantities.

Figure 5 illustrates the behavior of $\bar{\epsilon}'$ and $\tilde{\epsilon}'$ as functions of a. On the graph is indicated the value corresponding to the maximum value of the difference $\bar{\epsilon}' - \tilde{\epsilon}'$, and this latter value itself is given. In Fig. 6 the dependence on a is shown for $\operatorname{tg}\bar{\delta}$ and $\operatorname{tg}\tilde{\delta}$, on the assumption $\epsilon_1' > \epsilon_2'$, for the cases $\operatorname{tg}\delta_2 > \operatorname{tg}\delta_1$ (solid curve) and $\operatorname{tg}\delta_2 < \operatorname{tg}\delta_1$ (dotted curve).

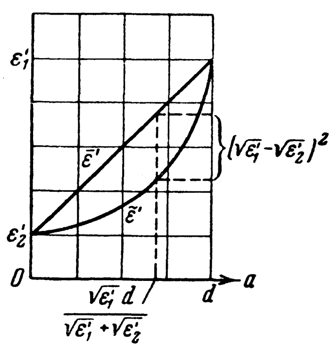

FIG. 5.

To the same degree of approximation as in formulas (42), the following expressions are obtained for the principal indexes of refraction, which we write in the usual form $\sqrt{\epsilon^e \mu^e} = n(1-i\kappa)$. With propagation along the optic axis (the z axis),

$$n = \sqrt{\bar{\varepsilon}'\bar{\mu}'}, \quad \varkappa = \frac{\operatorname{tg}\bar{\delta} + \operatorname{tg}\bar{\gamma}}{2}. \tag{43a}$$

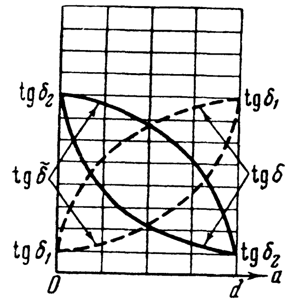

FIG. 6.

With propagation perpendicular to the optic axis: for the ordinary ray (**E** parallel to the layers),

$$n = \sqrt{\bar{\varepsilon}'\tilde{\mu}'}, \quad \varkappa = {}^1/_2\,(\operatorname{tg}\bar{\delta} + \operatorname{tg}\tilde{\gamma}); \tag{43b}$$

for the extraordinary ray (**E** perpendicular to the layers),

$$n = \sqrt{\tilde{\varepsilon}'\bar{\mu}'}, \quad \varkappa = {}^1/_2\,(\operatorname{tg}\tilde{\delta} + \operatorname{tg}\bar{\gamma}). \tag{43c}$$

For illustration we give a numerical example. We suppose that layers of a ceramic or titanate alternate with layers of paraffin. For wavelength $\lambda = 6$ cm we have

Titanate: $\epsilon_1' = 200$; $\operatorname{tg}\delta_1 = 0.006$;

Paraffin: $\epsilon_2' = 2.25$; $\operatorname{tg}\delta_2 = 0.0002$.

Let $a = b = d/2$*. Then the third (most stringent) of the conditions (38) gives

$$kd \ll 0.58, \text{ i.e., } d \ll 0.55 \text{ cm}.$$

Thus when $d = 1$ mm, formula (40) may be considered valid. From (42) we get

$$\bar{\varepsilon}' = 101, \quad \operatorname{tg}\bar{\delta} = 0.006,$$
$$\tilde{\varepsilon}' = 4.5, \quad \operatorname{tg}\tilde{\delta} = 0.0026.$$

From this it follows that for the ordinary wave the index of refraction (uniform in all directions for a magnetically isotropic medium) is $n = 10$, and $\kappa = 0.003$; i.e., the ordinary wave decays by a factor e in a distance $1/k\kappa n = 32$ cm. For the extraordinary wave, the second principal index of refraction is $n = 2.1$, and $\kappa = 0.0013$, which gives $1/k\kappa n = 347$ cm.

* The maximum anisotropy would correspond to $a = 0.87\,d$.

Obviously, in the range of values of the parameters in which Maxwell's equations hold for the mean field, with the effective permeabilities (21) and (23), the losses per unit volume of the medium can be calculated with the usual formula, derived directly from the averaged equations. But if conditions (37 a, b, c) are not satisfied, then the relations between means and mean squares become complicated, and the value of the losses can be found only by averaging of the true expressions for the heat generated, as is done in particular in reference 3. Equations (5) and (27) for the amplitudes of the quasi-microscopic fields make possible the carrying through of this type of calculation.

BOUNDARY CONDITIONS

When conditions (37) are satisfied, the stratified medium obeys Maxwell's equations with the tensor permeabilities ϵ^e and μ^e [cf. Eq. (40)]. In this case the boundary conditions, on a surface arbitrarily oriented with respect to the optic axis, can be obtained from Maxwell's equations themselves by the usual passage to a limit; they consist, as always, in continuity of the tangential components of the intensities **E** and **H** and of the normal components of the inductions **D** and **B**. If we represent by subscripts e and i the external and internal mean fields, and by **n** the unit vector normal to the boundary surface, we get

$$[\mathbf{n}, \mathbf{E}_e] = [\mathbf{n}, \mathbf{E}_i], \quad [\mathbf{n}, \mathbf{H}_e] = [\mathbf{n}, \mathbf{H}_i],$$
$$(\mathbf{n}, \mathbf{D}_e) = (\mathbf{n}, \mathbf{D}_i), \quad (\mathbf{n}, \mathbf{B}_e) = (\mathbf{n}, \mathbf{B}_i),$$

where $\mathbf{D}_i$ and $\mathbf{B}_i$ are related to $\mathbf{E}_i$ and $\mathbf{H}_i$ by the tensors ϵ^e and μ^e:

$$\mathbf{D}_i = \varepsilon^e \mathbf{E}_i, \quad \mathbf{B}_i = \mu^e \mathbf{H}_i.$$

In connection with these conditions, the following doubt may arise. The fields $\mathbf{E}_i$ and $\mathbf{H}_i$ inside the medium were obtained by averaging not all solutions of the equation for the microfield, but only those that lead to a transverse wave. The basis for rejecting the other solutions [for example, those corresponding to the lower Eq. (10)], for **e** parallel to the layers, was that they decay rapidly, in distances of the order of a period of the structure. But in the problem of boundary conditions, when theorems about flux or circulation are applied to a surface or contour arbitrarily close to the boundary surface, the argument mentioned loses its force, and there is no justification for asserting in advance that these decaying fields will not enter into the boundary conditions. That they actually do not enter can be shown as follows.

Any component of induction or of intensity of the microfield, either on the outside or on the inside face of the boundary, must be approximately periodic along the boundary, with the period of the structure on it. Let the boundary be a plane containing the y axis and inclined to a layer at angle φ (Fig. 7). Then along the boundary, i.e., along the η axis, the period will be equal to $d' = d/\sin\varphi$. Thus the component of induction or intensity under consideration, for example e_y, can be expanded in a Fourier series in coordinate η with period d'; the coefficients also vary as functions of η, but slowly - with a period $\gg\lambda$:

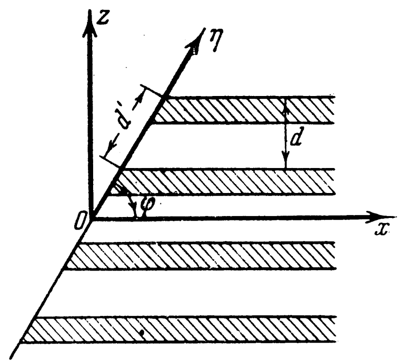

FIG. 7.

$$e_y = a_0 + a_1 \cos\frac{2\pi\eta}{d'} + \ldots + b_1 \sin\frac{2\pi\eta}{d'} + \ldots$$

If d' is small, not only in comparison with λ, but also in comparison with the shortest wavelength in the medium, i.e.,

$$d' = \frac{d}{\sin\varphi} \ll \frac{\lambda}{|n|_{max}}, \quad \text{or} \quad kd\,|n|_{max} \ll \sin\varphi, \tag{45}$$

then averaging over the period d' gives

$$\overline{e}_y = E_y = a_0 + O\left(\frac{d'}{\lambda}\right),$$

that is, upon such averaging of the boundary conditions for the microfield, all rapidly oscillating terms practically drop out. However, decaying fields are excited in the medium, thanks to these same rapidly oscillating terms in the resolution of the field on the boundary. In the outside region this is immediately obvious, since there we have an ordinary homogeneous medium. As for the inside region, if condition (37) is satisfied it likewise behaves like a homogeneous (though anisotropic) medium; and for $d' < \lambda/|n|_{max}$ the boundary field, oscillating with period d', creates decaying waves in it also.

This reasoning and condition (45) show, together with this, that the case of small φ is actually peculiar. In this case, i.e., when $kd|n|_{max} \geq \sin\varphi$, the periodic structure on the boundary represents a coarse "grating", giving not only decaying waves, but diffracted waves that propagate (in the reflected field outside the medium, such waves appear only when $kd \geq \sin\varphi$). As φ decreases, the angles of diffraction, counted from the maxima of order zero (which correspond to regular reflection and refraction), will decrease; and in the limit for $\varphi = 0$ (boundary parallel to a layer), such secondary waves will again not exist. Thus the averaged boundary conditions (44) are useful when condition (45) is satisfied, i.e, when the cut is not too oblique. Practically, such oblique cuts are of scarcely any interest.

I take this occasion to thank Academician M. A. Leontovich for the interest he showed in this work, and for discussion of a number of problems.

Translated by W. F. Brown, Jr.
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