

Fluctuations in Oscillating Systems of the Thomson Type. I

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Fluctuations of phase and amplitude in weakly non-linear oscillating systems amenable, in the periodic mode, to calculation through an expansion in terms of a small parameter are considered. In contrast with previous studies¹⁻⁵, the Einstein-Fokker equations are not used; instead, symbolic differential equations containing random functions which describe the fluctuations are used in conjunction with methods of correlation theory.

The first part of the paper is devoted to a consideration of a system with one degree of freedom; to illustrate the method, the general theory is applied to the case of an isolated system, which has been studied earlier², and to the physical example of a weakly self-excited vacuum-tube oscillator.

1. INTRODUCTION

THE problem of the behavior of dynamic systems in the presence of random effects has been treated in very general form in reference 1, which is undoubtedly fundamental in this particular field. In this paper the statistical approach was that of the Einstein-Fokker transition-probability equation; i.e., it was assumed that the random process taking place in the system is a Markov process.

The general considerations developed in reference 1 were applied further in the study of the natural bandwidth of a vacuum-tube oscillator², in which connection the oscillator was assumed to be a system of the Thomson type; i.e., approximately a conservative harmonic oscillator. The theoretical results were shown to be in good qualitative agreement with experiment³.

Because of the considerable improvement in the sensitivity of receiving and measuring apparatus, fluctuation effects have received a good deal of attention in recent years. In particular, fluctuations in oscillating systems, aside from their theoretical interest, have now acquired direct practical importance, for example in determining the limiting frequency stability which can be achieved in quartz-crystal frequency standards⁶.

In further study of the problems in this field it would seem desirable both to attack new problems other than those considered in reference 2 and reference 3, and to apply other statistical methods. In connection with the first of these approaches

mention should be made of reference 4 in which the Einstein-Fokker equations were used to study the behavior of a vacuum-tube oscillator in which the fluctuations were not assumed to be small and correlation time of which was large compared with the oscillation period, and of reference 5, in which the methods of references 2,3 were used in considering fluctuation in an oscillator with nonlinear inertial properties.

The present paper treats fluctuation in oscillating systems of the Thomson type having one or two degrees of freedom. The statistical approach is one first introduced by Langevin⁷ in the theory of Brownian motion and in general is quite different from those used in the work cited above. In particular, in place of the Einstein-Fokker equations, we consider symbolic differential equations which contain explicitly the random forces acting in the system and determine the random functions themselves rather than the distribution functions. Without attempting a comprehensive comparison between these two methods, we may note here certain features of both.

Using Einstein-Fokker equations, one may easily take into account auxiliary conditions such as the existence of reflecting or absorbing boundaries in considering the region of variation of the random functions; this is done by imposing appropriate boundary conditions. In working with symbolic differential equations, the consideration of such conditions is more complicated. On the other hand, these equations are more general in that they are not limited to Markov processes; the random force

¹ L. Pontriagin, A. Andronov and A. Vitt, *J. Exper. Theoret. Phys. USSR* 3, 165 (1933)

² I. L. Bershtein, *Dokl. Akad. Nauk SSSR* 20, 11 (1938); *J. Exper. Theoret. Phys. USSR* 11, 305 (1941)

³ I. L. Bershtein, *Dokl. Akad. Nauk, SSSR* 68, 469 (1949); *Izv. Akad. Nauk SSSR, Ser. Fiz.* 14, 145 (1950)

⁴ P. I. Kuznetsov, R. L. Stratonovich and V. I. Tikhonov, *Dokl. Akad. Nauk SSSR* 97, 639 (1954)

⁵ M. E. Zhabotinskii, *J. Exper. Theoret. Phys. USSR* 26, 758 (1954)

⁶ A. Blaquièrè, *Ann. de Radioélectrité* 8, No. 31, 36 and No. 32, 153 (1953)

⁷ P. Langevin, *Compt. rend.* 146, 530 (1908)

$F(t)$ need not have δ -correlation. The situation in which $F(t)$ is "absolutely random", i.e.,

$$\overline{F(t)F(t')} = C\delta(t-t') \quad (1.1)$$

and correspondingly, in which the impulse over the time T

$$I(t) = \int_t^{t+T} F(t) dt \quad (1.2)$$

is a non-correlated function, represents an exceptional case. As is well known, it is precisely in this case that the system fluctuations caused by $F(t)$ constitute a Markov process, and the consideration of the problem by correlation-theory methods is equivalent to its solution using transition probabilities. In general, however, the application of symbolic differential equations in conjunction with correlation theory seems to be simpler and more instructive, so that we shall use it here although we shall limit ourselves to random forces having δ -correlation.

First as an illustration of the method, an isolated oscillating system is considered, i.e., the problem solved in reference 2.

In the following paper our method is applied in somewhat more complicated problems: fluctuations in a non-isolated oscillating system with one degree of freedom, namely a vacuum-tube oscillator synchronized by an external driving force and fluctuations in an oscillating system having two degrees of freedom, more specifically, a vacuum-tube oscillator stabilized by coupling to a high-Q circuit.

2. STATEMENT OF THE PROBLEM

Since it is assumed that the systems being investigated are approximately conservative oscillators, it is reasonable to use for the solution of the equation of motion a method involving expansion in a small parameter. This parameter, which we call μ , determines slow variations of both amplitudes and phases of the oscillations at the fundamental frequency and at its harmonics, i.e., it enters not only directly, but also through the "slow" time

$$\tau = \mu t.$$

In other words, the solution will be in the form

$$x = R \cos(t - \varphi) + \mu \sum_n \{P_n \cos n(t - \varphi) + Q_n \sin n(t - \varphi)\},$$

$$R = R(\tau, \mu), \quad \varphi = \varphi(\tau, \mu),$$

$$P_n = P_n(\tau, \mu), \quad Q_n = Q_n(\tau, \mu).$$

The first question which must be settled in applying this treatment of random effects is the choice of the order of magnitude of the random force. Two reasonable choices are possible: first order, i.e., a force $\mu F(t)$, and second order, i.e., a force $\mu^2 F(t)$. In the following it will become apparent that over and above its greater convenience with regard to the numerical values which appear in the dimensionless-parameter equations, the second choice leads to a more consistent perturbation method. Thus, for a system with one degree of freedom the equation of motion will have the following form:

$$\frac{d^2x}{dt^2} + x = \mu f\left(x, \frac{dx}{dt}, t, \mu\right) + \mu^2 F(t). \quad (2.1)$$

The explicit dependence of f on t (which arises in the synchronization problem) is assumed to be periodic with period 2π . In the case of the isolated system the time enters explicitly only through the random force $\mu^2 F(t)$.

As is well known, the equations for the amplitudes and the phase φ are obtained from the requirement that in x there be no terms (in any order in μ) which increase indefinitely with t . This means that, in looking for a solution in the form of a series in μ ($x = x_0 + \mu x_1 + \mu^2 x_2 + \dots$), the right-hand member of the successive approximation equations must not contain resonance harmonics. According to Eq. (2.1), the random force enters in the right-hand member of the equation for x_2 [$\ddot{x}_2 + x_2 = F(t) + \dots$], i.e., speaking formally, it should act as a conservative oscillator — a system with an infinitely large driving force. From this it is clear that in the spectrum of $F(t)$ only the immediate region of the fundamental frequency is of importance and thus $F(t)$ can be written in the form

$$F(t) = F_{\parallel}(\tau) \cos(t - \varphi) - F_{\perp}(\tau) \sin(t - \varphi), \quad (2.2)$$

where F_{\parallel} and F_{\perp} are the components of the force tangent to and normal to the generating circle

(more accurately, to the circle corresponding to the fundamental frequency).

In order to examine the significance of this representation of the force, we write its spectral function

$$F(t) = \int_0^\infty \{U(\omega) \cos \omega t + V(\omega) \sin \omega t\} d\omega. \quad (2.3)$$

Using the Fourier-transform formulas and the correlation function (1.1), it follows that

$$\begin{aligned} \overline{U(\omega) U(\omega')} &= \frac{C}{\pi} \{\delta(\omega - \omega') + \delta(\omega + \omega')\}, \\ \overline{V(\omega) V(\omega')} &= \frac{C}{\pi} \{\delta(\omega - \omega') \\ &\quad - \delta(\omega + \omega')\}, \quad \overline{U(\omega) V(\omega')} = 0. \end{aligned}$$

In the integration over ω and ω' from zero to infinity, terms with $\delta(\omega + \omega')$ always vanish, so that only the first term need be considered:

$$\begin{aligned} \overline{U(\omega) U(\omega')} &= \overline{V(\omega) V(\omega')} \\ &= \frac{C}{\pi} \delta(\omega - \omega'), \quad \overline{U(\omega) V(\omega')} = 0. \end{aligned} \quad (2.4)$$

Comparing Eqs. (2.2) and (2.3), it follows that

$$\begin{aligned} F_{\parallel}(\tau) &= \int_0^\infty \{U(\omega) \cos(\alpha\tau + \varphi) \\ &\quad + V(\omega) \sin(\alpha\tau + \varphi)\} d\omega, \\ F_{\perp}(\tau) &= \int_0^\infty \{U(\omega) \sin(\alpha\tau + \varphi) \\ &\quad - V(\omega) \cos(\alpha\tau + \varphi)\} d\omega, \end{aligned}$$

where

$$\alpha = (\omega - 1) / \mu. \quad (2.5)$$

These expressions are completely rigorous although, strictly speaking, on the left-hand side one cannot write the "slow" time τ as the argument for F_{\parallel} and F_{\perp} . In view of the smallness of μ , however, and in view of the fact that we are interested not in F_{\parallel} and F_{\perp} themselves, but only in their correlation functions, this expression is justified. Thus, for the correlation function of F_{\parallel} , we have

$$\begin{aligned} \overline{F_{\parallel}(\tau) F_{\parallel}(\tau')} &= \\ &= \int_0^\infty \int_0^\infty \overline{\{U \cos(\alpha\tau + \varphi) + V \sin(\alpha\tau + \varphi)\} \{U' \cos(\alpha'\tau' + \varphi') + V' \sin(\alpha'\tau' + \varphi')\}} \times \\ &\quad \times d\omega d\omega'. \end{aligned}$$

In view of Eqs. (2.4) and (2.5), this gives

$$\begin{aligned} \overline{F_{\parallel}(\tau) F_{\parallel}(\tau')} &= \frac{C}{\pi} \int_0^\infty \cos[\alpha(\tau - \tau') + \varphi - \varphi'] d\omega \\ &= \frac{\mu C}{\pi} \cos(\varphi - \varphi') \int_{-1/\mu}^\infty \cos \alpha(\tau - \tau') d\alpha \\ &\quad - \frac{\mu C}{\pi} \sin(\varphi - \varphi') \int_{-1/\mu}^\infty \sin \alpha(\tau - \tau') d\alpha. \end{aligned}$$

It is clear that for $\mu \rightarrow 0$ the integral over $\cos \alpha(\tau - \tau')$ becomes $2\pi \delta(\tau - \tau')$ and the integral over $\sin \alpha(\tau - \tau')$ becomes zero. Thus, within the accuracy of the rapidly oscillating terms, one may use these limiting values of both integrals, and it is in just this sense that we interpret the statement that F_{\parallel} and F_{\perp} depend on the time only through $\tau = \mu t$.

In the factor which multiplies the δ -function one may assume that $\tau' = \tau$ so that $\overline{F_{\parallel}(\tau) F_{\parallel}(\tau')}$

$= 2\mu C \delta(\tau - \tau')$. Incorporating this with the results of similar calculations for the other correlation functions we have finally,

$$\begin{aligned} \overline{F_{\parallel}(\tau) F_{\parallel}(\tau')} &= \overline{F_{\perp}(\tau) F_{\perp}(\tau')} \\ &= 2\mu C \delta(\tau - \tau'), \\ \overline{F_{\parallel}(\tau) F_{\perp}(\tau')} &= 0. \end{aligned} \quad (2.6)$$

We now turn our attention to the solution of Eq. (2.1).

3. SUCCESSIVE APPROXIMATION EQUATIONS

In differentiating any function which not only depends on t explicitly but also through $\tau = \mu t$, we have

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial \tau} = (\cdot) + \mu (\cdot)'. \quad (3.1)$$

For brevity we indicate the respective partial derivatives here and in the following through the dot and the prime-symbol.

In order to simplify the calculations, we make the assumption that the system being studied has cubic non-linearity only. This allows us to neglect the even harmonics, i.e., to look for a solution in the form

$$x = R \cos(t - \varphi) + \mu \{P \cos 3(t - \varphi) + Q \sin 3(t - \varphi) + \dots\}. \tag{3.2}$$

Calculating $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ in accordance with Eq. (3.1), after substitution in Eq. (2.1), we get

$$\begin{aligned} & [-2\mu R' + \mu^2 (R\varphi'' + 2R'\varphi')] \sin(t - \varphi) + [2\mu R\varphi' \\ & + \mu^2 (R'' - R\varphi'^2)] \cos(t - \varphi) + [-8\mu Q + \mu^2 (18Q\varphi' - 6P')] \sin 3(t - \varphi) \\ & + [-8\mu P + \mu^2 (18P\varphi' + 6Q')] \cos 3(t - \varphi) + \dots = \\ & = \mu f(x, \frac{dx}{dt}, t, \mu) + \mu^2 [F_{\parallel}(\tau) \cos(t - \varphi) - F_{\perp}(\tau) \sin(t - \varphi)]. \end{aligned} \tag{3.3}$$

Now we expand f in a Fourier series with respect to $t - \varphi$ (keeping in mind the fact that the explicit dependence of f on t is assumed to be periodic with period 2π):

$$\begin{aligned} f(x, \frac{dx}{dt}, t, \mu) &= \sum_k \{ \Phi_k \sin k(t - \varphi) + \Psi_k \cos k(t - \varphi) \}, \\ \text{where } (k > 0) & \\ \left. \begin{aligned} \Phi_k \} &= \frac{1}{\pi} \int_{-\pi}^{\pi} f \{ R \cos u + \mu (P \cos 3u \\ \Psi_k \} &+ Q \sin 3u) + \dots, -R \sin u + \mu (R\varphi' \sin u \\ &+ R' \cos u - 3P \sin 3u + \dots), \\ & \left. \begin{aligned} u + \varphi, \mu \} & \frac{\sin ku}{\cos ku} du. \end{aligned} \right\} \end{aligned} \tag{3.4}$$

Now equating the coefficients of $\sin(t - \varphi)$, $\sin 3(t - \varphi)$ and $\cos(t - \varphi)$, $\cos 3(t - \varphi)$ to zero, we find

$$\begin{aligned} 2R' + \Phi_1 - \mu (R\varphi'' + 2R'\varphi' + F_{\perp}) &= 0, \\ 2R\varphi' - \Psi_1 + \mu (R'' - R\varphi'^2 - F_{\parallel}) &= 0, \\ 8Q + \Phi_3 - \mu (18Q\varphi' - 6P') &= 0, \\ 8P + \Psi_3 + \mu (18P\varphi' + 6Q') &= 0, \\ \dots \dots \dots \end{aligned}$$

The successive approximation equations are obtained from the substitution of a power series in μ :

$$R = R_0 + \mu R_1 + \dots, \tag{3.5}$$

$$P = P_0 + \mu P_1 + \dots,$$

$$\varphi = \varphi_0 + \mu \varphi_1 + \dots, \quad Q = Q_0 + \mu Q_1 + \dots,$$

in which Φ_k and Ψ_k are also resolved in terms of μ

$$\begin{aligned} \Phi_k &= \Phi_{k0} + \mu \Phi_{k1} + \dots, \\ \Psi_k &= \Psi_{k0} + \mu \Psi_{k1} + \dots \end{aligned}$$

In accordance with Eqs. (3.4) and (3.5)

$$\begin{aligned} \left. \begin{aligned} \Phi_{k0} \} &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(R_0 \cos u, \\ \Psi_{k0} \} &- R_0 \sin u, \quad u + \varphi_0, \quad 0) \frac{\sin ku}{\cos ku} du, \end{aligned} \right\} \end{aligned} \tag{3.6}$$

$$\Phi_{k1} = R_1 \frac{\partial \Phi_{k0}}{\partial R_0} + \varphi_1 \frac{\partial \Phi_{k0}}{\partial \varphi_0} + A_k,$$

$$\Psi_{k1} = R_1 \frac{\partial \Psi_{k0}}{\partial R_0} + \varphi_1 \frac{\partial \Psi_{k0}}{\partial \varphi_0} + B_k,$$

where

$$\begin{aligned} \left. \begin{aligned} A_k \} &= \frac{1}{\pi} \int_{-\pi}^{\pi} \{ f'_{x0} (R_0 \varphi'_0 \sin u + R'_0 \cos u) \\ B_k \} &+ (f'_{x0} P_0 + 3f'_{x0} Q_0) \cos 3u + (f'_{x0} Q_0 \\ &- 3f'_{x0} P_0) \sin 3u + f'_{\mu 0} \} \frac{\sin ku}{\cos ku} du. \end{aligned} \right\} \end{aligned} \tag{3.7}$$

Equating the coefficients of the same order in μ to zero, we find the equation for the first approximation

$$\begin{aligned} 2R'_0 + \Phi_{10} &= 0, & 8Q_0 + \Phi_{30} &= 0, \\ 2R_0\varphi'_0 - \Psi_{10} &= 0, & 8P_0 + \Psi_{30} &= 0, \end{aligned} \quad (3.8)$$

and the equation for the second approximation

$$\begin{aligned} 2R'_1 + \frac{\partial\Phi_{10}}{\partial R_0} R_1 + \frac{\partial\Phi_{10}}{\partial\varphi_0} \varphi_1 \\ = R_0\varphi_0'' + 2R'_0\varphi'_0 - A_1 + F_{\perp}, \end{aligned} \quad (3.9)$$

$$\begin{aligned} 2R_0\varphi'_1 + \left(2\varphi'_0 - \frac{\partial\Psi_{10}}{\partial R_0}\right) R_1 \\ - \frac{\partial\Psi_{10}}{\partial\varphi_0} \varphi_1 = R_0\varphi_0'^2 - R_0'' + B_1 + F_{\parallel}; \\ 8Q_1 + 6P'_0 - 18Q_0\varphi'_0 + \Phi_{31} = 0, \\ 8P_1 - 6Q'_0 - 18P_0\varphi'_0 + \Psi_{31} = 0. \end{aligned} \quad (3.10)$$

The first two equations of (3.8) are the so-called reduced equations. We note that for the regular method of successive approximations being used here the basis of this designation is lost. We do not require the "rejection of quickly oscillating terms" nor averaging over phase, i.e., the usual procedures which are employed in the derivation of the first two equations of (3.8).

Equations (3.9), which are linear in R_1 and φ_1 permit us to express R_1 and φ_1 in terms of the fluctuation forces F_{\parallel} and F_{\perp} whose correlation functions are known, i.e., they permit us to find the correlation functions for the amplitude fluctuations and for the phase fluctuations, and eventually to determine their mean-square values. Thus, the problem of finding the statistical properties of the random process which takes place in the system being investigated is completely solved.

If in Eq. (2.1) the fluctuation force had been introduced in first order in μ , then in place of Eq. (3.8) we would have obtained the following equation for the first-order approximation:

$$2R'_0 + \Phi_{10} = F_{\perp}, \quad 2R_0\varphi'_0 - \Psi_{10} = F_{\parallel}. \quad (3.11)$$

Bershtein² started from these equations but in forming the corresponding Einstein-Fokker equations he was still forced to resort to perturbation methods, i.e., to the linearization of Eqs. (3.11). Taking R_0 and φ_0 to be the solutions of Eqs. (3.11) for $F_{\perp} = F_{\parallel} = 0$, and taking R_1 and φ_1 to be the small deviations caused by the fluctuation forces; resolving Eqs. (3.11) in powers of R_1 and φ_1 and limiting ourselves to linear terms, we have

$$\begin{aligned} 2R'_1 + \frac{\partial\Phi_{10}}{\partial R_0} R_1 + \frac{\partial\Phi_{10}}{\partial\varphi_0} \varphi_1 &= F_{\perp}, \\ 2R_0\varphi'_1 + \left(2\varphi'_0 - \frac{\partial\Psi_{10}}{\partial R_0}\right) R_1 - \frac{\partial\Psi_{10}}{\partial\varphi_0} \varphi_1 &= F_{\parallel}. \end{aligned} \quad (3.12)$$

The linearization of Eqs. (3.11) indicates that the fluctuation force is influenced by the behavior of higher order terms as is shown by a comparison of (3.9) and (3.12). In using the successive-approximation method this effect is introduced naturally as was done above. Furthermore, we see that the linearization of Eqs. (3.11) yields a result which coincides with that of Eq. (3.9) only in those cases for which all additional terms in Eq. (3.9) are either absent or can be set to zero. If one or the other of these conditions is not fulfilled (and this is entirely possible) then Eq. (3.12) does not take into account regular corrections to the amplitude and phase. It is to be understood that the presence of regular additional terms in the right-hand member of Eq. (3.9) is not essential to the fluctuations.

We now consider the case of fluctuations in an isolated oscillating system.

4. ISOLATED OSCILLATING SYSTEM

Since the time t does not appear in f explicitly in this case, the quantities Φ_{k0} , Ψ_{k0} , A_k and B_k do not depend on φ_0 as is seen from Eqs. (3.6) and (3.7). Hence in an isolated system, Eqs. (3.8) and (3.9) assume the form

$$2R'_0 + \Phi_{10}(R_0) = 0, \quad 8Q_0 + \Phi_{30}(R_0) = 0, \quad (4.1)$$

$$2R_0\varphi'_0 - \Psi_{10}(R_0) = 0, \quad 8P_0 + \Psi_{30}(R_0) = 0;$$

$$2R'_1 + \frac{\partial\Phi_{10}}{\partial R_0} R_1 = R_0\varphi_0'' + 2R'_0\varphi'_0 \quad (4.2)$$

$$- A_1(R_0) + F_{\perp},$$

$$2R_0\varphi'_1 + \left(2\varphi'_0 - \frac{\partial\Psi_{10}}{\partial R_0}\right) R_1$$

$$= R_0\varphi_0'^2 - R_0'' + B_1(R_0) + F_{\parallel}.$$

We are interested only in steady-state oscillations, so that $R' = 0$. The first equation of (4.1),

$$\Phi_{10}(R_0) = 0$$

determines only the constant (independent of τ) values of the radii of the generating circles in the zeroth approximation.

If $\Psi_{10}(R_0) \neq 0$, then, from the second equation of (4.1), we have

$$\varphi_0 = \delta\tau + \alpha,$$

where α is an arbitrary constant and the quantity

$$\delta = \varphi'_0 = \frac{1}{2R_0} \Psi'_{10}(R_0)$$

gives the first-order correction to the frequency.

Taking into account the constant R_0 (also P_0 , Q_0) and φ'_0 , we get from Eq. (4.2)

$$2R'_1 + \frac{\partial\Phi_{10}}{\partial R_0} R_1 = -A_1(R_0) + F_{\perp}, \quad (4.3)$$

$$2R_0\varphi'_1 + \left(2\delta - \frac{\partial\Psi_{10}}{\partial R_0}\right) R_1 = R_0\delta^2 - B_1(R_0) + F_{\parallel}.$$

Separating the regular components r_1 and δ_1 in R_1 and φ_1 , i.e., taking

$$R_1 = r_1 + \rho(\tau), \quad \varphi_1 = \delta_1\tau + \chi(\tau), \quad (4.4)$$

we get, upon substitution of Eq. (4.4) in (4.3), the following values of the corrections to the radius of the generating circle r_1 and the second-order correction to the frequency δ_1 :

$$r_1 = -\frac{A_1(R_0)}{\partial\Phi_{10}/\partial R_0},$$

$$\delta_1 = \frac{1}{2R_0} \left[R_0\delta^2 - B_1(R_0) - \left(2\delta - \frac{\partial\Psi_{10}}{\partial R_0}\right) r_1 \right],$$

while the equations for the amplitude fluctuation ρ and the phase fluctuation χ assume the form

$$\rho' + p_1\rho = F_{\perp}(\tau)/2, \quad \chi' + q_1\chi = F_{\parallel}(\tau)/2R_0, \quad (4.5)$$

where

$$p_1 = \frac{1}{2} \frac{\partial\Phi_{10}}{\partial R_0},$$

$$q_1 = \frac{1}{2R_0} \left(2\delta - \frac{\partial\Psi_{10}}{\partial R_0} \right) = -\frac{1}{2} \frac{\partial}{\partial R_0} \left(\frac{\Psi_{10}}{R_0} \right).$$

The quantity p_1 , the increment of the system, characterizes the "stability" of the generating circle with respect to radial departures (for a stable circle $p_1 > 0$). For ρ we have an equation of the relaxation type with a characteristic relaxation time $1/p_1$ in terms of τ (i.e., $1/\mu p_1$ in terms of t). The tracing point, after being displaced from the generating circle by the fluctuation force, is returned to it as though it were attached by a spring and moving in a viscous medium. The existence of a quasi-elastic force $-p_1\rho$ implies the stationary properties of the amplitude fluctuations and the existence of a finite steady-state value of ρ^2 .

In calculating the steady-state correlation function, one may use the solution of the first equation of Eq. (4.5) under initial conditions corresponding to the vanishing of the quantities of interest at $\tau = -\infty$.

$$\rho(\tau) = \frac{e^{-p_1\tau}}{2} \int_{-\infty}^{\tau} e^{p_1\theta} F_{\perp}(\theta) d\theta.$$

Using Eq. (2.4), we find

$$\overline{\rho(\tau)\rho(\tau')} = \frac{\exp\{-p_1(\tau + \tau')\}}{4} \int_{-\infty}^{\tau} e^{p_1\theta_1} d\theta_1 \int_{-\infty}^{\tau'} e^{p_1\theta_2} \overline{F_{\perp}(\theta_1)F_{\perp}(\theta_2)} d\theta_2$$

$$= \frac{\mu C}{2} \exp\{-p_1(\tau + \tau')\} \int_{-\infty}^{\tau} e^{p_1\theta_1} d\theta_1 \int_{-\infty}^{\tau'} e^{p_1\theta_2} \delta(\theta_2 - \theta_1) d\theta_2 = \frac{\mu C}{4p_1} \exp\{-p_1|\tau' - \tau|\}.$$
(4.6)

In particular, the mean-square of ρ is

$$\overline{\rho^2} = \mu C/4p_1. \quad (4.7)$$

Because the system being considered is an isolated one, there is no steady state as far as phase fluctuations are concerned. If we take the phase $\chi = 0$ at time $\tau = 0$, i.e., if an ensemble of systems with this initial value of χ is considered, then, in the course of time, the tracing points spread apart on the generating circle on both sides of the regular ("dynamic") tracing points and the mean-

square value of χ will increase in accordance with a diffusion law (proportional to τ). In reference 1 this situation was aptly described (in terms of the phase plane x_0, x_0) as the "motion of an intoxicated person moving in a channel in which there is a steady current". In the plane which describes the slowly varying amplitudes, one in which the regular rotation along the generating circle is not shown, the indicated spreading is similar to the usual one-dimensional motion of a Brownian particle in a motionless medium except that the

fluctuations in χ depend not only on the direct effect of the random force F_{\parallel} but also on the amplitude fluctuations. The effect of the latter, which are expressed by the term $q_1 \rho$ in the second equation of (4.5) arises when $q_1 \neq 0$, i.e., either in the presence of a first-order frequency correction ($\Psi_{10} \neq 0$) or in the absence of isochronism in the neighborhood of the generating circle ($\partial \Psi_{10} / \partial R_0 \neq 0$), or as a result of both (if they do not cancel; this occurs when $\Psi_{10} \sim R_0$).

Taking into account the absence of a steady-state for χ , we take the solution of the second equation of (4.5) which corresponds to the initial conditions $\chi = 0$ for $\tau = 0$.

$$\chi(\tau) = -q_1 \int_0^\tau \rho(\theta) d\theta + \frac{1}{2R_0} \int_0^\tau F_{\parallel}(\theta) d\theta.$$

Since, in this section, we wish only to illustrate the application of symbolic equations and correlation theory in problems which have already been solved, we will limit ourselves to the calculation of the mean-square value of χ . Assuming that the fluctuations of amplitude have already been determined, we make use of the correlation function (4.6) for ρ .

Since $\rho(\tau)$ and $F_{\parallel}(\tau)$ are not cross-correlated, we have

$$\begin{aligned} \overline{\chi^2(\tau)} &= q_1^2 \int_0^\tau d\theta_1 \int_0^\tau \overline{\rho(\theta_1) \rho(\theta_2)} d\theta_2 \\ &+ \frac{1}{4R_0^2} \int_0^\tau d\theta_1 \int_0^\tau \overline{F_{\parallel}(\theta_1) F_{\parallel}(\theta_2)} d\theta_2. \end{aligned}$$

Substituting Eq. (4.6) in (2.6), we have

$$\overline{\chi^2(\tau)} = \frac{\mu C}{2} \left\{ \frac{q_1^2}{p_1^2} (p_1 \tau + e^{-p_1 \tau} - 1) + \frac{\tau}{R_0^2} \right\}.$$

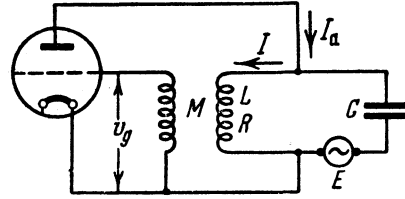
The first term depends on the amplitude fluctuation, the second on the direct effect of the pulses. On the same basis, the second term increases in accordance with a diffusion law; the first, however, is subject to such a law only when $p_1 \tau \gg 1$ and then

$$\overline{\chi^2(\tau)} = \frac{\mu C}{2} \left(\frac{q_1^2}{p_1^2} + \frac{1}{R_0^2} \right) \tau. \quad (4.8)$$

If $p_1 \tau \ll 1$ (and also if τ is much larger than the time between pulses, as is required for the formulation of the problem in terms of symbolic

equations) then* $\overline{\chi^2(\tau)} = \frac{\mu C}{2} \left(\frac{q_1^2 \tau^2}{2p_1} + \frac{\tau}{R_0^2} \right)$.

5. VACUUM-TUBE OSCILLATOR



As an example, which will also be needed for what follows, we consider the vacuum-tube oscillator whose circuit is shown in the Figure. Using the symbols given in the Figure, we have the equation

$$\begin{aligned} L \frac{dl}{dt_1} + RI &= \frac{1}{C} \int (I_a - I) dt_1 \\ &+ \mathcal{E}_0 \sin \omega t_1, \quad v_g = M \frac{dl}{dt_1} \end{aligned} \quad (5.1)$$

(The time is indicated by t_1 while the symbol t is reserved for the dimensionless time). In the following paper, this same example will be considered in connection with synchronization; hence we introduce here a sinusoidal emf, which is included in the condenser branch to simplify the equations.

The plate current is given by the expression

$$I_a = S v_g \left(1 - \frac{v_g^2}{3V^2} \right) + I_{a\text{-shot}}(t_1), \quad (5.2)$$

where S and V are the usual parameters associated with the cubic characteristics of the tube and $I_{a\text{-shot}}$ is the random part of the plate arising from the shot effect**.

Introducing the dimensionless time $t = \omega t_1$ and the dimensionless current $x = I/I_0$, where

$$I_0^2 = V^2 (MS - RC) / M^2 S \omega, \quad (5.3)$$

and using the notation

* Note added in proof.— In a recent paper by Gonorovskii [Dokl. Akad. Nauk SSSR 101, 657 (1955)] it is found that for small τ the mean-square value of the random deviation of the phase, in general, does not contain terms to the first power in τ as though there were no direct effect of the pulses. We propose to examine the reasons for this disagreement elsewhere.

** For the present, in the interest of simplicity we omit the thermal emf in the R branch (cf. Sect. 6 of the following paper).

$$\omega_0^2 = \frac{1}{LC}, \quad \frac{\omega_0^2}{\omega^2} = 1 - \mu\Delta, \\ \frac{(MS - RC)\omega_0^2}{\omega} = \mu, \quad \frac{\omega_0^2 I_{ar}(t_1)}{\omega^2 I_0} = \mu^2 F(t), \quad (5.4) \\ \frac{\mathcal{E}_0}{L\omega I_0} = 2\mu H,$$

from Eqs. (5.1) and (5.2) we get equations in the form of Eq. (2.1) in which

$$f(x, \frac{dx}{dt}, t, \mu) \\ = \frac{dx}{dt} \left[1 - \frac{1}{3} \left(\frac{dx}{dt} \right)^2 \right] + \Delta x + 2H \cos t. \quad (5.5)$$

In considering the case which is of interest to us at this time, the isolated system, it suffices to take $H = 0$ and $\Delta = 0$, i.e., to take $\omega = \omega_0$ everywhere in Eq. (5.4). Then

$$f(x, \frac{dx}{dt}, t, \mu) = \frac{dx}{dt} \left[1 - \frac{1}{3} \left(\frac{dx}{dt} \right)^2 \right]. \quad (5.6)$$

Assuming the shot-effect to be δ -correlated,

$$\overline{I_{a-shot}(t_1) I_{a-shot}(t'_1)} = C_0 \delta(t'_1 - t_1) \quad (5.7)$$

($C_0 = \bar{I}_a e$, e is the charge of the electron), and making use of the relations (5.4) between $F(t)$ and

$$\frac{I_{a-shot}(t_1)}{I_{a-shot}(t'_1)} = \mu^4 I_0^2 \overline{F(t) F(t')} \\ = \mu^4 I_0^2 C \frac{\delta(t'_1 - t_1)}{\omega_0} \quad (\omega = \omega_0),$$

we get

$$\mu^4 C = C_0 \omega_0 / I_0^2. \quad (5.8)$$

Omitting the intermediate steps, we can write directly Eq. (4.1) for the case (5.6)

$$2R'_0 - R_0 \left(1 - \frac{R_0^2}{4} \right) = 0, \quad Q_0 = \frac{R_0^3}{96}, \\ 2R_0 \varphi'_0 = 0, \quad P_0 = 0.$$

Thus $\Psi_{10} = 0$, i.e., there is no first-order frequency correction, the system is completely isochronous and $R_0 = 2$. In Eqs. (4.3), $A_1 = 0$,

since r_1 (the correction to the radius of the generating circle) does not appear and $B_1 = R_0^5 / 128 = 1/4$. Thus, taking into account second order corrections, the fundamental frequency is found to be $n = 1 - (\mu^2 / 16)$ and Eqs. (4.5) for ρ and χ are:

$$\rho' + \rho = F_{\perp} / 2, \quad \chi' = F_{\parallel} / 4 \quad (p_1 = 1, q_1 = 0).$$

Expressions (4.7) and (4.8) for the mean-square values of ρ and χ assume the form

$$\overline{\rho^2} = \frac{\mu C}{4}, \quad \overline{\chi^2} = \frac{\mu C}{8} \tau \quad (5.9)$$

We revert to the initial (physical) parameters. The amplitude fluctuations of the current in the circuit are $\Delta I = I_0 \mu \rho$ and the phase fluctuations $\Delta \varphi = \mu \chi$. From Eqs. (5.3), (5.4), (5.8) and (5.9), we get, therefore,

$$\overline{\Delta I^2} = \frac{I_0^2 \mu^3 C}{4} = \frac{C_0 \omega_0}{4\mu} = \frac{C_0}{4(MS - RC)}, \quad (5.10)$$

$$\overline{\Delta \varphi^2} = \frac{\mu^3 C \tau}{8} = \frac{C_0 \omega_0^3 t_1}{8I_0^2} = \frac{C_0 \omega_0^4 M^3 S}{8V^2 (MS - RC)} t_1,$$

where $C_0 = \bar{I}_a e$ is the constant which appears in the correlation function of the shot-noise (5.7).

Close to the limit of self-excitation ($MS - RC \rightarrow 0$), i.e., close to the limit of stability of the oscillating mode, the "stability" of the generating circle tends toward zero. In this case there occurs an unlimited increase of the intensity of the amplitude fluctuations and of the diffusion coefficient of the phase fluctuations. Actually, the increase in the fluctuations is not unlimited; nevertheless it is large enough to invalidate the order-of-magnitude assumptions with regard to $\mu\rho$ and $\mu\chi$ upon which this analysis is based. The question of fluctuations close to the limit of self-excitation, in which the random effects in the generating circle become comparable to its radius, requires special attention.

Translated by H. Lashinsky