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The Effect of Electrical Fluctuations on a Vacuum Tube Oscillator

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The effect of "slowly varying" normally distributed random fluctuations on a vacuum tube oscillator is investigated. Expressions are obtained for the one-dimensional probability density functions of amplitude and phase. An approximation method is shown for the determination of the correlation functions of amplitude and phase.

RECENTLY great interest has been shown in the behavior of a vacuum tube oscillator subjected to electrical fluctuations originating either within the oscillator itself ("internal fluctuations"), or caused by external random occurrences ("external fluctuations"). These latter are sometimes the result of irregular voltage fluctuations of the power supply and occasionally they may be caused by the modulating effect of the oscillator voltage. Frequently these external fluctuations are characterized by a correlation time which is much longer than the corresponding time constants of the oscillator, and therefore, the fluctuations can be regarded as varying slowly compared with the natural oscillations of the oscillator in the absence of these fluctuations.

In the present paper, a general method will be shown for predicting the behavior of the oscillator under the influence of slowly varying electrical fluctuations. The method is based on the generalized equation of Einstein-Fokker [1] and differs slightly from the method given in a previous paper[2-4] for the analysis of high frequency internal fluctuations, provided they are of small order of magnitude.

1. It is well known[5] that a harmonic vacuum tube oscillator can be regarded as a quasi-linear self-oscillatory conservative system with one degree of freedom. The behavior of such a system in presence of an external random disturbance $\xi(t)$ can be generally described by the following equations:

$$\frac{dx_1}{dt} = f_1(x_1, y_1, \xi), \qquad \frac{dy_1}{dt} = g_1(x_1, y_1, \xi), \tag{1}$$

wherein x_1 and y_1 are characteristic variables describing the state of the system, e.g., current and voltage.

Separating in these equations the constant and linear parts and expressing the time derivative by a dot over the variables, we can write,

$$\dot{x}_1 = a_0 + a_1 x_1 + a_2 y_1 + f(x_1, y_1, \xi), \qquad \dot{y}_1 = b_0 + b_1 x_1 + b_2 y_1 + g(x_1, y_1, \xi), \tag{1a}$$

where the constants (a_0, b_0) and the coefficients of x_1 and y_1 depend on $\xi(t)$. The nonlinear terms f and g can be of the form

$$f = a_3 x_1^2 + a_4 x_1 y_1 + a_5 y_1^2 + a_6 x_1^3 + \ldots,$$
$$g = b_3 x_1^2 + b_4 x_1 y_1 + b_5 y_1^2 + b_6 x_1^3 + \ldots$$

Introducing new variables

$$x = x_1 - \alpha, \quad y = y_1 - \beta,$$

with

$$\alpha = \frac{b_0 a_2 - a_0 b_2}{a_1 b_2 - b_1 a_2}, \quad \beta = \frac{a_0 b_1 - a_1 b_0}{a_1 b_2 - b_1 a_2},$$

Eqs. (1a) become

$$\dot{x} = a_1 x + a_2 y + f(x+\alpha, y+\beta, \xi) - \dot{\alpha}, \qquad \dot{y} = b_1 x + b_2 y + g(x+\alpha, y+\beta, \xi) - \dot{\beta}. \tag{2}$$

We shall consider first a linear system described by the differential equations of the first order:

$$\dot{x} = a_1 x + a_2 y, \; \dot{y} = b_1 x + b_2 y.$$

As known, the solution of this system is

$$x = A_0 e^{-pt}[n \sin(\omega t + \varphi_0) + m \cos(\omega t + \varphi_0)],$$
$$y = A_0 e^{-pt} \sin(\omega t + \varphi_0),$$

[1] P. I. Kuznetsov, R. L. Stratonovich, V. I. Tikhonov, J. Exper. Theoret. Phys. USSR **26**, 189 (1954)

[2] L. Pontriagin, A. Andronov and A. Vitt, J. Exper. Theoret. Phys. USSR **3**, 165 (1933)

[3] I.L. Bershtein, Zh. Tekhn. Fiz. **11**, 305 (1941)

[4] I. L. Bershtein, Izv. Akad. Nauk SSSR, Ser. Fiz. **14**, 1 (1950)

[5] A. A. Andronov, S. E. Khaikin, *Theory of Oscillations*, State Technical Publishing House, 1937

wherein A_0 and ϕ_0 are integration constants determined by the initial conditions, and

$$\omega = \sqrt{-a_2 b_1 - \left(\frac{a_1 - b_2}{2}\right)^2}, \quad p = -\frac{a_1 + b_2}{2}, \quad n = \frac{a_1 - b_2}{2b_1}, \quad m = \frac{\omega}{b_1}. \tag{3}$$

Going back to the original Eqs. (2), we notice that, in the absence of the random disturbance $\xi(t)$, we have the ordinary case of self-oscillatory operating condition of the oscillator. The consideration of the nonlinear terms will usually cause only a slight change in the frequency ω of the oscillations and will allow the determination of its stable amplitude. The presence of the random disturbance $\xi(t)$ at the stationary operating condition will cause random fluctuations of amplitude and phase about a certain mean.

Rewriting Eq. (2) in polar-coordinates

$$y = A \sin \vartheta, \quad x = A(n \sin \vartheta + m \cos \vartheta).$$

and taking Eq. (3) into consideration, we obtain

$$\dot{A} = F(A, \vartheta, \xi) - \dot{\gamma} \cos \vartheta - \dot{\beta} \sin \vartheta, \tag{4}$$

$$\dot{\vartheta} = -\omega + G(A, \vartheta, \xi) - (1/A)(\dot{\beta} \cos \vartheta + \dot{\gamma} \sin \vartheta),$$

wherein

$$F(A, \vartheta, \xi) = -nA + \frac{f - ng}{m} \cos \vartheta + g \sin \vartheta,$$

$$G(A, \vartheta, \xi) = \frac{1}{A}\left(g \cos \vartheta + \frac{f - ng}{m} \sin \vartheta\right),$$

$$\gamma = (\alpha - n\beta) / m.$$

According to the method of Bogoliubov[5,6] (of which the well known method of van der Pol is a special case) we can introduce into these equations the mean of all terms over θ. This is justified even when the random disturbances $\xi(t)$ are of a higher order of magnitude, under the assumption that the time correlation of the random function $\xi(t)$ is very high as compared to the period T of the oscillations in the absence of $\xi(t)$:

$$\tau_{corr} \gg 2\pi / \omega = T, \tag{5}$$

Under this condition $\xi(t)$ will not change noticebly during one period (corresponding to one passage of the descriptive point through its cycle, i.e., the change of θ by 2π), so that it can be regarded substantially as constant.

Averaging, we obtain

$$\dot{A} = \Phi(A, \xi) + M, \quad \dot{\vartheta} = -\omega + \Psi(A, \xi) + N, \tag{6}$$

wherein

$$\Phi(A, \xi) = \frac{1}{2\pi} \int_0^{2\pi} F(A, \vartheta, \xi)\, d\vartheta, \tag{7}$$

$$\Psi(A, \xi) = \frac{1}{2\pi} \int_0^{2\pi} G(A, \vartheta, \xi)\, d\vartheta,$$

$$M = -\frac{1}{2\pi} \int_0^{2\pi} (\dot{\gamma} \cos \vartheta + \dot{\beta} \sin \vartheta)\, d\vartheta,$$

$$N = -\frac{1}{2\pi A} \int_0^{2\pi} (\dot{\beta} \cos \vartheta + \dot{\gamma} \sin \vartheta)\, d\vartheta.$$

As β and γ are changing slowly with time, their derivatives with respect to time, which are contained in the expressions for M and N, will be of small order of magnitude, also very slowly varying with time. The quantities M and N can therefore be neglected in the expressions for $\dot{A}$ and $\dot{\theta}$; we then obtain

$$\dot{A} = \Phi(A, \xi), \quad \dot{\vartheta} = -\omega + \Psi(A, \xi). \tag{6a}$$

2. For every given case Eq. (7) allows determination of the function Φ and Ψ which appear in the Eqs. (6a). Herein the first equation determines the fluctuations of the amplitude and the second the fluctuations of the phase. As the first of these equations is independent of θ it can be solved by itself. For the solution the generalized equation of Einstein-Fokker can be applied, which, under the condition

$$\tau_{corr} \ll \tau_1, \tag{8}$$

where τ_1 is the relaxation time of the amplitude, given approximately by

$$\tau_1 \approx \left| \left(\frac{\partial \Phi}{\partial A} \right)^{-1} \right|, \tag{9}$$

takes the form:

$$\frac{\partial w(A, t)}{\partial t} = \sum_{s=1}^{\infty} \frac{(-1)^s}{s!} \frac{\partial}{\partial A^s} [K_s(A)\, w(A, t)]. \tag{10}$$

In this equation, $w(A, t)$ is the one-dimensional probability density and $K_s(A)$ is the system function (sometimes called the g-function) related for stationary processes to the correlation function of s^{th} order by:

$$K_s(A) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} k_{(s)\,\Phi}(\tau_1, \ldots, \tau_{s-1})\, d\tau_1 \ldots d\tau_{s-1}, \tag{11}$$

[6] H. Bogoliubov, *Certain Statistical Methods in Mathematical Physics*, 1945

$$K_1 = k_{\Phi}.$$

As the function Φ is known, the characteristic functions $K_s(A)$ can be determined, provided the correlation functions $k_{(s)\xi}$ for $\xi(t)$ are known. Thereby the problem is reduced to the solving of Eq. (10). If the terms of order higher than 2 can be neglected, this equation reduces to the special form of the Einstein-Fokker equation

$$\frac{\partial w(A,t)}{\partial t} = -\frac{\partial}{\partial A}[K_1(A)w(A,t)] + \frac{1}{2}\frac{\partial^2}{\partial A^2}[K_2(A)w(A,t)]. \quad (12)$$

with

$$K_1(A) = \overline{\Phi(A,\xi)}, \quad (11a)$$

$$K_2(A) = \int_{-\infty}^{\infty}\{\overline{\Phi(A,\xi)\Phi(A,\xi_\tau)} - [\overline{\Phi(A,\xi)}]^2\}\,d\tau.$$

where, inthe determination of the mean values, A is to be regarded as a parameter and not as a random variable.

From Eq. (12) it is not difficult to find the expression for the one-dimensional stationary probability density function directly by quadrature. Actually, the stationary probability density function must be independent of time, i.e., as t approaches infinity the system has come to statistical equilibrium; $\partial w/\partial t$ must vanish:

$$\partial w/\partial t = 0.$$

and Eq. (12) becomes

$$K_1(A)w(A) - {}^1/_2(d/dA)[K_2(A)w(A)] = \text{const}. \quad (12a)$$

The expression

$$j(A) = K_1(A)w(A) - {}^1/_2(d/dA)[K_2(A)w(A)]$$

represents the "probability current" or, with statistical interpretation of probability, the rate of flow of the descriptive points. In problems of this type, where the descriptive point does not leave the phase space (i.e., it cannot enter from one direction and leave through the other), the "probability current" should be assumed zero (since probability is conserved, i.e., it can neither disappear nor be created). Therefore, the solution of Eq. (12 a) may be written as

$$w(A) = \frac{C_0}{K_2(A)}\exp\left[2\int\frac{K_1(A)}{K_2(A)}dA\right], \quad (13)$$

where the integration constant C_0 is given by the normalizing condition:

$$\int_0^\infty w(A)\,dA = 1. \quad (14)$$

3. For the solution of many problems it is not enough to know only the one-dimensional distribution of the random function, but it is also necessary to determine the correlation function. For the determination of the latter, we will show a method based on the following formula:

$$\frac{d}{d\tau}\overline{F(x_0,x_\tau)} = \overline{K_1(x_\tau)\frac{\partial}{\partial x_\tau}F(x_0,x_\tau)} + \overline{\frac{1}{2}K_2(x_\tau)\frac{\partial^2}{\partial x_\tau^2}F(x_0,x_\tau)}, \quad (15)$$

wherein F is an arbitrary function, $x_0 = A(t)$, $x_\tau = A(t+\tau)$. To derive this formula let $w_0(x_0)$ be the probability density distribution function of the random variable x_0, and let $w(x_0, x_\tau, \tau)$ give the probability $w_1(x_0,x_\tau,\tau)dx_\tau$ of the transition of a point, originally at x_0 in the interval $[x_\tau, x_\tau + dx_\tau]$ in time τ. The distribution density function $w_1(x_0,x_\tau,\tau)$ satisfies Eq. (12):

$$\frac{\partial w_1}{\partial\tau} = -\frac{\partial}{\partial x_\tau}[K_1(x_\tau)w_1(x_0,x_\tau,\tau)] + \frac{1}{2}\frac{\partial^2}{\partial x_\tau^2}[K_2(x_\tau)w_1(x_0,x_\tau,\tau)].$$

Inserting this expression into the relation

$$\frac{d}{d\tau}\overline{F(x_0,x_\tau)} = \int\int_{-\infty}^{\infty} F(x_0,x_\tau)w_0(x_0)\frac{\partial w_1(x_0,x_\tau,\tau)}{d\tau}dx_0dx_\tau$$

and integrating the result by parts leads to Eq. (15) (considering that,by the normalizing condition, the difference of the probability currents $j(\infty) - j(-\infty)$ must vanish, and also that the probability density by itself is zero as $x_\tau \to \pm\infty$). In our special case where A is a non-negative function as the integration is carried out in the limits from 0 to ∞, Eq. (15) still holds, since

$$w_1(x_0,x_\tau,\tau) = 0 \quad \text{and} \quad x_\tau \to \infty,$$
$$K_2(0) = 0.$$

As the one-dimensional probability density distribution $w(x)$ is given by Eq. (13), we can calculate

$$\overline{F(x_0, x_\tau)}\Big|_{\tau=0} = \overline{F(x_0, x_0)}. \tag{16}$$

If we assign for the function F in Eq. (15) the value

$$F_1 = K_1(x_\tau)\frac{\partial}{\partial x_\tau}F(x_0, x_\tau),$$
$$F_2 = K_2(x_\tau)\frac{\partial^2}{\partial x_\tau^2}F(x_0, x_\tau), \tag{17}$$

it is possible to obtain relations which will determine the time variations of $\overline{F_1}$ and $\overline{F_2}$, if their initial values, corresponding to $\tau = 0$, are also known.

Continuing this process, we will obtain a system of linear equations (generally infinite in number) which can be solved by known methods. An example with the application of this method will be considered below.

4. As an illustration, and to clarify the conditions under which the Einstein-Fokker method may be applied, we will consider the following typical example. Let stationary Gauss fluctuations be present in the grid circuit of a vacuum tube oscillator with a resonant-circuit in the anode loop. (See Fig. 1). For such a system, under the usual assumptions (no grid current, neglecting the effect of the loaded anode, and so on), and approximating the anode-grid characteristic of the tube by a polynomial of the third degree

$$i_a = \varphi(u_c) = i_0 + \alpha u_c + \beta u_c^2 - \gamma u_c^3 \quad (\alpha > 0,\ \gamma > 0),$$

the equation system (1) becomes

$$\dot{x} = -\frac{r}{L}x_1 + \omega_0 y_1,$$
$$y_1 = -\omega_0 x_1 + \frac{1}{C}\varphi\left[\frac{M}{L}(y_1 - \delta x_1) + \xi(t)\right]. \tag{18}$$

wherein

$$x_1 = i\rho, \quad y_1 = u, \quad \omega_0 = 1/\sqrt{LC},$$
$$\delta = r/\rho, \quad \rho = \sqrt{L/C}, \tag{19}$$

and i and u are current and voltage respectively in the resonant circuit, the meaning of the other parameters being shown in Fig. 1.

Separating the constant parts and those linear in x_1 and y_1, Eq. (18) can be rewritten as

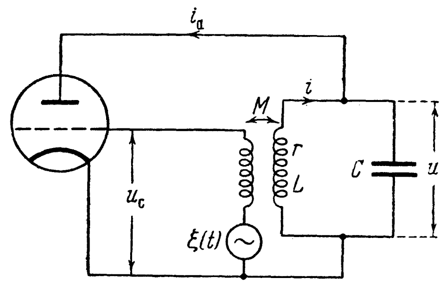

FIG. 1. Schematic diagram of the vacuum tube oscillator

$$\dot{x}_1 = -(r/L)x_1 + \omega_0 y_1, \tag{18a}$$
$$\dot{y}_1 = (1/C)\varphi(\xi) - \omega_0(1 + \omega_0\alpha M\delta)x_1 + \omega_0^2\alpha M y_1 + \Theta(x_1, y_1, \xi),$$

with

$$\Theta(x_1, y_1, \xi) = (\beta/C)\left[k^2(y_1 - \delta x_1)^2 + 2k\xi(y_1 - \delta x_1)\right] - (\gamma/C)\left[k^3(y_1 - \delta x_1)^3 + 3k^2\xi(y_1 - \delta x_1)^2 + 3k\xi^2(y_1 - \delta x_1)\right],$$
$$k = M/L.$$

Introducing new variables

$$x = x_1 - \rho\varphi(\xi), \quad y = y_1 - r\varphi(\xi), \tag{20}$$

we obtain

$$\dot{x} = -(r/L)x + \omega_0 y, \tag{18b}$$
$$\dot{y} = -\omega_0(1 + \omega_0\alpha M\delta)x + \omega_0^2\alpha M y + \Theta(x, y, \xi).$$

Inserting the individual terms into Eq. (3), and using the condition of self-excitation of the oscillator, which is for $\xi(t) \equiv 0$ given approximately by

$$M > rC/\alpha, \tag{21}$$

leads to

$$n \approx \delta \approx 0, \quad m \approx 1/(1 + \delta^2) \approx 1,$$

since usually $\delta \ll 1$. Therefore, the transition to polar-coordinates can be made by

$$x = A\cos\vartheta, \quad y = A\sin\vartheta. \tag{22}$$

Before performing this transition, we notice, however, that in the neighborhood of self-excitation

$$\omega_0\alpha M\delta \approx \delta^2 \lll 1.$$

and, therefore, we can neglect the second coefficient of x in the second of equations (18b), [as is usually done in the analysis of a vacuum-tube oscillator in the absence of the random disturbances $\xi(t)$].

Now, changing to polar-coordinates [Eq. (22)] and introducing the mean over θ the Eqs. (6a) become

$$\dot{A} = {}^1/_2\,\omega_0^2\,[(\alpha M - rC + 2\beta M\xi - 3\gamma M\xi^2)\,A \quad (23)$$
$$- {}^3/_4\,\gamma M(M/L)^2 A^3],$$

$$\dot{\vartheta} = -\omega_0 + \frac{\delta\omega_0^2 M}{2}\left[\frac{3\gamma}{4}\left(\frac{M}{L}\right)^2 A^2 + (3\gamma\xi^2 - 2\beta\xi)\right].$$

Setting $\xi(t) \equiv 0$, the first of these equations gives the condition (21) for self-excitation of the oscillator and also for the radius of the maximum stable cycle:

$$A_0 = 2\frac{L}{M}\sqrt{\frac{\alpha M - rC}{3\gamma M}}. \quad (24)$$

It reaches its largest value

$$A_{0\,\mathrm{MAX}} \approx 0.14\,L/rC\sqrt{\gamma}$$

at

$$M = 1.5\,r\,C/\alpha.$$

5. As was shown before, the basic equation (12) is correct only if

$$T_0 \ll \tau_{\mathrm{corr}} \ll \tau_1. \quad (25)$$

To estimate the order of magnitude of τ_{corr}, we shall assume the following numerical values for the circuit parameters of Fig. 1:

$$f_0 = 2\pi/\omega_0 = 10^7\ \mathrm{cps}, \quad r = 10\,\Omega,$$

$$\rho = 6\cdot 10^3\,\Omega, \quad M = 1.1\,rC/\alpha, \quad A_0 = 10\ \mathrm{volts}.$$

Differentiating Eq. (23) over A gives

$$\partial\Phi/\partial A = {}^1/_2\omega_0^2\,[(M\alpha - rC + 2\beta M\xi - 3\gamma M\xi^2)$$
$$-{}^9/_4\gamma M(M/L)^2 A^2].$$

We will now evaluate $\partial\Phi/\partial A$ at $A = A_0$. Inserting the value of A_0 from Eq. (24), we find

$$\left.\frac{\partial\Phi}{\partial A}\right|_{A=A_0}$$
$$= -\frac{1}{2}\omega_0^2\left[0{,}2rC + \frac{0{,}33\alpha^2\rho^3}{\omega_0 r A_0^2}\left(1 - \frac{2\beta}{3\gamma\xi}\right)\xi^2\right].$$

The inequality (25) must be satisfied for all values of ξ with nonnegligible probability, and, in particular, it must also be satisfied in the case $\xi = 0$. Thereby,

$$\tau_1 = \left|\left(\frac{\partial\Phi}{\partial A}\right)^{-1}\right| \approx \frac{2}{0.2\,\omega_0^2 rC} = \frac{10\rho}{\omega_0 r} = 10^{-4}\ \mathrm{sec}$$

Thus the basic Eq. (12) will hold for all practical purposes; if the correlation time of the random function $\xi(t)$ is

$$\tau_{\mathrm{corr}} \approx (10^{-6} \div 10^{-5})\ \mathrm{sec} \quad (25a)$$

These conditions will be fulfilled in most cases. For the transition to spectral quantities we shall assume that the correlation function for "low-frequency" fluctuations $\xi(t)$ is of the form

$$k_2(\tau) = c_0 e^{-\alpha|\tau|}$$

and, correspondingly, the spectral density of the fluctuations will be given by

$$S(f) = c_0\frac{\alpha}{\alpha^2 + (2\pi f)^2}.$$

In this case the correlation time is

$$\tau_{\mathrm{corr}} = 2\int_0^\infty e^{-\alpha\tau}d\tau = \frac{2}{\alpha}.$$

In order that condition (25a) is fulfilled for this specific example, it is necessary that

$$\alpha \approx 2\,(10^6 \div 10^5)\ \mathrm{sec}^{-1}.$$

Let

$$\tau_{\mathrm{corr}} = 0.2\cdot 10^{-5}\ \mathrm{sec}$$

If, for practical purposes, we neglect all spectral densities at frequencies for which

$$\frac{S(f)}{S(0)} = \frac{1}{1 + (2\pi f\ \alpha)^2} \leqslant 5\,\%,$$

then, the bandwidth of the fluctuations for which this method can be applied will be

$$f_{\mathrm{MAX}} \approx 0.7\ \times 10^6\ \mathrm{cps}$$

It should be noted, however, that because of the high values of ρ/r for high frequency resonant circuit, the range of τ_{corr} for which the method of Einstein-Fokker is valid is considerably larger at high frequency oscillations.

6. We shall now evaluate the one dimensional stationary probability density function $w(A)$ for the fluctuations of the amplitude. In accordance with Eq. (11a) we can find the system functions from the first of the Eqs. (23). Indeed, they can be found very easily if $\xi(t)$ are normally distributed fluctuations with a mean of zero, i.e.,

$$w_2(\xi, \xi_\tau) = \frac{1}{2\pi\sigma^2 \sqrt{1 - R^2(\tau)}} \times \exp\left[-\frac{\xi^2 + \xi_\tau^2 - 2R(\tau)\xi\xi_\tau}{2\sigma^2(1 - R^2(\tau))}\right], \tag{26}$$

wherein σ^2 is the dispersion and $R(\tau)$ is the correlation coefficient.

Actually, let, for example,

$$\eta(\tau) = a + b\xi(t) + c\xi^2(t).$$

In this case, as all odd moments of $\xi(t)$ vanish, Eq. (11a) becomes

$$K_{(2)\eta} = K_{(2)\,b\xi + c\xi^2} = b^2 K_{(2)\xi} + c^2 K_{(2)\xi^2},$$

where

$$K_{(2)\xi} = \int_{-\infty}^{\infty} k_{(2)\xi}(\tau)\,d\tau = \sigma^2 \int_{-\infty}^{\infty} R(\tau)\,d\tau, \tag{27}$$

$$K_{(2)\xi^2} = \int_{-\infty}^{\infty} k_{(2)\xi^2}(\tau)\,d\tau.$$

But, in case of stationary processes the correlation function $k_{(2)\xi^2}$ of $\xi^2(t)$ can be expressed either by the moments or by the correlation functions of $\xi(t)$:

$$k_{(2)\xi^2}(\tau) = m_{(2)\xi^2}(\tau) - (m_{\xi^2})^2 = m_{(4)\xi}(t, t, t+\tau, t+\tau) - \sigma^4.$$

Moreover, in accordance with the general relations between the moments and correlation functions, we have, if $m = 0$,

$$m_4(t_1, t_2, t_3, t_4) = k_2(t_1, t_2)\,k_2(t_3, t_4) + k_2(t_1, t_3)\,k_2(t_2, t_4) + k_2(t_1, t_4)\,k_2(t_2, t_3),$$

and, consequently,

$$m_{(4)\xi}(t, t, t+\tau, t+\tau) = k^2_{(2)\xi}(0) + 2k^2_{(2)\xi}(\tau) = \sigma^4[1 + 2R^2(\tau)].$$

Hence,

$$K_{(2)\xi^2} = 2\sigma^4 \int_{-\infty}^{\infty} R^2(\tau)\,d\tau. \tag{28}$$

Using this relation, and performing all necessary operations, we obtain

$$K_1(A) = \mu A - \nu A^3, \quad K_2(A) = \lambda A^2, \tag{29}$$

where
$$\mu = {}^1/_2\,\omega_0^2(\alpha M - rC - 3\gamma M\sigma^2), \tag{30}$$

$$\nu = {}^3/_8\,\omega_0^2\gamma M(M/L)^2,$$

$$\lambda = (\omega_0^2 M\sigma)^2(\beta^2\tau_{corr} + {}^9/_2\,\gamma^2\sigma^2\tau_{k2}),$$

$$\tau_{corr} = \int_{-\infty}^{\infty} R(\tau)\,d\tau, \quad \tau_{k2} = \int_{-\infty}^{\infty} R^2(\tau)\,d\tau.$$

Substituting the values of the system functions into Eq. (13), we obtain

$$w(A) = \frac{C_0}{\lambda} A^{2(\mu/\lambda - 1)} \exp\left(-\frac{\nu}{\lambda}A^2\right), \tag{31}$$

This equation shows that for $\mu/\lambda < 1$ the probability density $w(A)$ becomes infinite at $A = 0$. However, it should be remembered that this result is obtained because $K_2(0) = 0$ at $A = 0$, and that this latter value of K_2 is not an exact one, due to the approximation by which Eq. (29) was obtained.

To find a more exact expression for $K_2(0)$ we will not neglect the term M in the first of Eqs. (6). Owing to the fact that the descriptive point actually moves in the phase plane, not on a circle (ellipse), but along a spiral with a variable radius

$$A(t) = A(t_0) + \dot{A}(t_0)(t - t_0) \approx A(t_0)\left[1 + \frac{\Phi(A, 0)}{A(t_0)}(t - t_0)\right],$$

it will be more accurate to write the expression for the mean M in the form:

$$M = -\frac{1}{2\pi}\int_0^{2\pi}\left[1 + \frac{\Phi(A, 0)}{A(t_0)}(t - t_0)\right] \left[\frac{d\gamma}{d\vartheta}\cos\vartheta + \frac{d\beta}{d\vartheta}\sin\vartheta\right]\frac{d\vartheta}{dt}\,d\vartheta,$$

wherein the time function can be transformed into function of θ by the substitution

$$t - t_0 = \vartheta/\omega.$$

Substitution in this equation of the approximate expression for θ from Eq. (6a) gives

$$M = -\frac{1}{2\pi}\int_0^{2\pi}\left[1 + \frac{\Phi(A, 0)}{A(t_0)}(t - t_0)\right] \times\left[\frac{d\gamma}{d\vartheta}\cos\vartheta + \frac{d\beta}{d\vartheta}\sin\vartheta\right]\times[-\omega + \Psi(A, \xi)]\,d\vartheta. \tag{32}$$

It can easily be shown that $K_2(0)$ is equal to the system function of this expression with A equal to zero. Generally $K_2(0)$ will not be zero.

If we assume

$$K_2(A) = \rho + \lambda A^2,$$

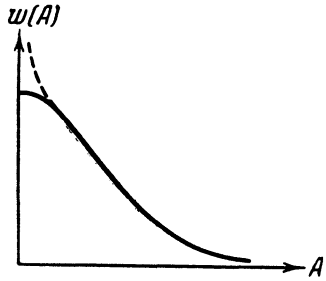

FIG. 2. Condition of not developed oscillations

where ρ is a small quantity, we will obtain a probability density function $w(A)$ which is finite and which can be integrated over all possible values of A. Furthermore, we will have

$$\frac{dw}{dA}\bigg|_{A=0} = 0,$$

which is in conformity with the stipulation of conservation of the descriptive point. However, it should be pointed out that Eq. (30) gives the correct slope of the curve $w(A)$ even for $\mu/\lambda < 1$ as long as A is not too small. Nevertheless, for the determination of the integration constant C_0 the more exact equation (32) should be used.

From the qualitative point of view, the operating condition of the oscillator can be classified as follows:

1. $\mu < \lambda$; condition of not developed oscillations. The descriptive point is located mainly in the neighborhood of the equilibrium (see Fig. 2.).

2. $\lambda < \mu < {}^3/_2\,\lambda$; condition of not fully developed oscillations. The amplitude of the oscillations is scattered on a large zone and may be close to zero (See Fig. 3).

3. $\mu > {}^3/_2\,\lambda$;the most interesting case-condition of fully developed oscillations - when the descriptive point is located mainly in the neighborhood of the critical cycle (see Fig. 4). In the Figs. 2 and 3, the dotted line in the neighborhood of $A = 0$ shows the slope of $w(A)$ according to Eq. (31), the solid line – the corrected value.

We will now determine the integration constant C_0 , and also the moments of the random variable A. It is

$$\overline{A^m} = \frac{C_0}{\lambda}\int_0^\infty A^{2s+m}\exp\left(-\frac{\nu}{\lambda}A^2\right)dA$$

$$= \frac{C_0}{2\lambda}\left(\frac{\lambda}{\nu}\right)^{(2s+m+1)/2}\int_0^\infty x^{(2s+m-1)/2}\,e^{-x}dx$$

$$= \frac{C_0}{2\lambda}\left(\frac{\lambda}{\nu}\right)^{(2s+m+1)/2}\Gamma\left(\frac{2s+m+1}{2}\right),$$

where

$$s = (\mu/\lambda) - 1.$$

Assuming here, in accordance with the normalizing condition (14), $\overline{A_0} = 1$, we obtain

$$C_0 = \frac{2\lambda\,(\lambda/\nu)^{-s-1/2}}{\Gamma(s+{}^1/_2)}\,. \tag{33}$$

Consequently,

$$\overline{A^m} = \left(\frac{\lambda}{\nu}\right)^{m/2}\Gamma\left(s+\frac{m+1}{2}\right)\Big/\,\Gamma\left(s+\frac{1}{2}\right). \tag{34}$$

and,for the even moments of A, (34a)

$$\overline{A^{2k}} = (\lambda/\nu)^k\,(s+{}^1/_2)(s+{}^3/_2)\cdots(s+k-{}^1/_2).$$

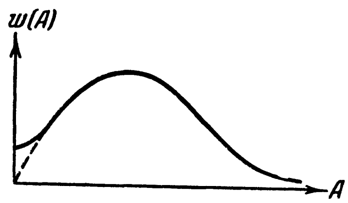

FIG. 3. Condition of not fully developed oscillations

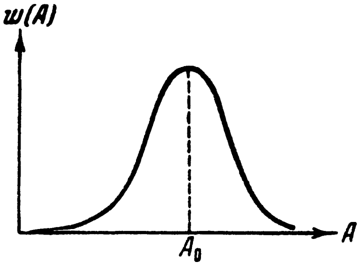

FIG. 4. Condition of fully developed oscillations.

7. We shall now find the correlation functions for $A(t)$. Assuming in Eq. (15)

$$F(x_0,\ x_\tau) = x_0 x_\tau,$$

we will obtain

$$(d/d\tau)\,u_1 = \mu u_1 - \nu u_3, \tag{35}$$

with

$$u_1 = \overline{x_0 x_\tau},\quad u_3 = \overline{x_0 x_\tau^3}.$$

Similarly, for μ_3

$$(d/d\tau)\,u_3 = 3\mu u_3 - 3\nu u_5 + 3\lambda u_3. \tag{36}$$

In general for μ_n, we have

$$\frac{d}{d\tau} u_n = n\left[\mu u_n - \nu u_{n+2} + \frac{n-1}{2}\lambda u_n\right], \quad (37)$$

with

$$u_n = \overline{A(t) A^n(t+\tau)},$$

$$\tau > 0, \quad n = 1, 3, 5, \ldots,$$

and

$$u_n(0) = \overline{A^{n+1}} \quad (38)$$

$$= \left(\frac{\lambda}{\nu}\right)^{(n+1)/2} \Gamma\left(s + \frac{n}{2} + 1\right) / \Gamma\left(s + \frac{1}{2}\right).$$

An approximate solution of this system of equations can be obtained by the following method. Using Eq. (35), we will first determine the coefficients of the Taylor-expansion of $\mu_1(\tau)$:

$$u_1(\tau) = u_1(0) + u_1'(0)\tau + \frac{u_1''(0)}{2!}\tau^2 + \cdots \quad (39)$$

Setting

$$u_n^{(\alpha)}(0) = u_n(0) k_{n,\alpha}, \quad (40)$$

and differentiating Eq. (37) we find the recursion formula for the determination of $k_{n,\alpha}$:

$$k_{n,\alpha+1} = n\left[\left(\mu + \frac{n}{2}\lambda\right)(k_{n,\alpha} - k_{n+2,\alpha}) - \frac{1}{2}\lambda k_{n,\alpha}\right], \quad (41)$$

with

$$k_{n,0} = 1.$$

Hence, from here

$$k_{n,1} = -(n/2)\lambda,$$

and, as

$$k_{n,1} - k_{n+2,1} = \lambda,$$

we have

$$k_{n,2} = n\left[\left(\mu + \frac{n}{2}\lambda\right)\lambda + \frac{1}{4}n\lambda^2\right] = n\mu\lambda + \frac{3}{4}n^2\lambda^2.$$

Subsequently, as

$$k_{n,2} - k_{n+2,2} = -2\mu\lambda - 3\lambda^2(n+1),$$

we obtain

$$k_{n,3} = -2n\mu^2\lambda - n\lambda^2\left(\frac{9}{2}n + 3\right) - \frac{n^2}{2}\lambda^3\left(\frac{15}{4}n + 3\right) \quad \text{, etc.}$$

Consequently,

$$k_{1,1} = -\lambda/2, \quad (42)$$
$$k_{1,2} = \mu\lambda + {}^3/_4\lambda^2,$$
$$k_{1,3} = -2\mu^2\lambda - {}^{15}/_2\mu\lambda^2 - {}^{27}/_8\lambda^3,$$

. . . .

Thus, using the first four terms of Eq. (39) we obtain

$$\overline{A(t)A(t+\tau)} = \frac{1}{\nu}\left(\mu - \frac{\lambda}{2}\right)\left[1 - \frac{\lambda}{2}\tau + \left(\mu\lambda + \frac{3}{4}\lambda^2\right)\frac{\tau^2}{2} - \left(2\mu^2\lambda + \frac{15}{2}\mu\lambda^2 + \frac{27}{8}\lambda^3\right)\frac{\tau^3}{6} + \ldots\right].$$

This determines the correlation coefficient

$$R_A(\tau) = \frac{\overline{AA_\tau} - \overline{A}^2}{\overline{A^2} - \overline{A}^2} = 1 - \frac{\lambda}{2}\frac{\tau}{1-B} + \left(\mu\lambda + \frac{3}{4}\lambda^2\right)\frac{\tau^2}{2(1-B)} - \left(2\mu^2\lambda + \frac{15}{2}\mu\lambda^2 + \frac{27}{8}\lambda^3\right)\frac{\tau^3}{6(1-B)}, \quad (43)$$

where

$$B = \Gamma\left(\frac{\mu}{\lambda}\right)^2 / \Gamma\left(\frac{\mu}{\lambda} - \frac{1}{2}\right)\Gamma\left(\frac{\mu}{\lambda} + \frac{1}{2}\right). \quad (44)$$

Using a sufficient number of terms in the expansion Eq. (39), we can attain any required accuracy.

8. We shall now consider the second of the Eqs. (23), which determines the phase change of the oscillations. Since we have assumed $\tau_{corr} \ll \tau_1$ the random processes $\xi(t)$ and $A(t)$ can be regarded as mutually uncorrelated. Therefore determining the probability characteristics of the process $A(t)$ from the first of Eqs. (23), we can basically determine the probability characteristics of the phase $\theta(t)$. The most important of these characteristics is D – the system function for the phase $\theta(t)$. The probability density distribution of the increment $\theta(t_0 + T) - \theta(t_0) = \theta_T - \theta_0$ in the time interval T, which is much longer than the correlation time τ_{corr} of the process $\theta(t)$, is approximately given by

$$w(\vartheta_T - \vartheta_0) = C_0 \exp\left\{-\frac{(\vartheta_T - \vartheta_0 - \omega T)^2}{2DT}\right\}, \quad (45)$$

where

$$\omega = \overline{\dot{\vartheta}} = -\omega_0 + \frac{\delta\omega_0^2 M}{2}\left[\frac{3\gamma}{4}\left(\frac{M}{L}\right)^2\overline{A^2} + 3\gamma\sigma^2\right].$$

Hence, D determines (for sufficiently long T) the rate of increase of the dispersion of the difference $\theta_T - \theta_0$.

Equation (23) can be written in the form

$$\dot{\vartheta} = -\omega_0 + c_1 A^2 + c_2\xi^2 - c_3\xi = N. \quad (46)$$

As there is no correlation between A and ξ the second characteristic function will be given by

$$K_{(2)N} = c_1^2 K_{(2)A^2} + K_{(2)c_2\xi^2 - c_3\xi},$$

where, because all odd moments of ξ vanish

$$K_{(2)c_2\xi^2 - c_3\xi} = c_2^2 K_{(2)\xi^2} + c_3^2 K_{(2)\xi}.$$

and

$$K_{(2)\xi} = \sigma^2 \tau_{corr}$$

$$= \sigma^2 \int_{-\infty}^{\infty} R(\tau)\, d\tau, \quad K_{(2)\xi^2} = 2\sigma^4 \tau_{k2},$$

Consequently, for the determination of

$$D = c_1^2 K_{(2)A^2} + 2c_2^2 \sigma^4 \tau_{k2} + c_3^2 \sigma^2 \tau_{corr} \tag{47}$$

we must find $K_{(2)A^2}$. For this we will apply a method used previously.
Setting

$$v_n(\tau) = \overline{A^2(t)\, A^{2n}(t+\tau)},$$

we obtain a system of equations, similar to that of Eq. (37):

$$\frac{d}{d\tau} v_n(\tau) = n\left[\mu v_n - \nu v_{n+2} + \frac{n-1}{2} \lambda v_n\right], \tag{48}$$

but with different initial conditions

$$v_n(0) = \overline{A^{n+2}}. \tag{49}$$

Assuming, as before,

$$v_n^{(\alpha)}(\tau) = v_n(0)\, l_{n,\alpha}, \tag{50}$$

we obtain

$$l_{n,\alpha+1} = n\left[\left(\mu + \frac{n-1}{2}\lambda\right) l_{n,\alpha} - \nu l_{n+2,\alpha} \frac{\overline{A^{n+4}}}{\overline{A^{n+2}}}\right],$$

with

$$\frac{\overline{A^{n+4}}}{\overline{A^{n+2}}} = \frac{\lambda}{\nu} \frac{\Gamma\left(s + \frac{n+1}{2} + 2\right)}{\Gamma\left(s + \frac{n+1}{2} + 1\right)}$$

$$= \frac{\lambda}{\nu}\left(s + 1 + \frac{n+1}{2}\right) = \frac{1}{\nu}\left(\mu + \frac{n+1}{2}\lambda\right).$$

Consequently,

$$l_{n,\alpha+1} = n\left[\left(\mu + \frac{n+1}{2}\lambda\right) \times (l_{n,\alpha} - l_{n+2,\alpha}) - \lambda l_{n,\alpha}\right], \tag{51}$$

where $l_{n,0} = 1$.
Using this relation, we find

$$l_{2,1} = -2\lambda, \quad l_{2,2} = 4\mu\lambda + 5\lambda^2, \quad l_{2,3} = -2(8\mu^2\lambda + 42\mu\lambda^2 + 44\lambda^3), \ \ldots \tag{52}$$

Therefore,

$$\overline{A^2 A_\tau^2} = \overline{A^4}\left[1 - 2\lambda\tau + (4\mu + 5\lambda)\lambda\frac{\tau^2}{2} - (8\mu^2 + 42\mu\lambda + 44\lambda^2)\lambda\frac{\tau^3}{3} + \ldots\right]. \tag{53}$$

Correspondingly, we obtain for the correlation coefficient

$$R_{A^2}(\tau) = \frac{\overline{A^2 A_\tau^2} - (\overline{A^2})^2}{\overline{A^4} - (\overline{A^2})^2} = 1 - \frac{2\lambda}{1-c}\tau + \frac{4\mu\lambda + 5\lambda^2}{1-c}\frac{\tau^2}{2} - \frac{8\mu^2 + 42\mu\lambda + 44\lambda^2}{1-c}\lambda\frac{\tau^3}{3} + \ldots, \tag{54}$$

with

$$c = \frac{(\overline{A^2})^2}{\overline{A^4}} = \frac{\Gamma\left(s+\frac{3}{2}\right)^2}{\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(s+\frac{5}{2}\right)} = \frac{\Gamma\left(\frac{\mu}{\lambda}+\frac{1}{2}\right)^2}{\Gamma\left(\frac{\mu}{\lambda}-\frac{1}{2}\right)\Gamma\left(\frac{\mu}{\lambda}+\frac{3}{2}\right)}. \tag{55}$$

It is approximately

$$K_{(2)A^2} = \overline{A^4}\int_{-\infty}^{\infty} R_{A^2}(\tau)\, d\tau \tag{56}$$

$$= \left(\frac{\lambda}{\nu}\right)^2\left(s+\frac{1}{2}\right)\left(s+\frac{3}{2}\right)\int_{-\infty}^{\infty} R_{A^2}(\tau)\, d\tau$$

$$= 2\,\frac{\mu^2 - (\lambda^2/4)}{\nu^2}\int_0^{\infty} R_{A^2}(|\tau|)\, d\tau.$$

Confining ourselves to only three terms of the expansion Eq. (54) we can set

$$R_{A^2}(\tau) = e^{-\alpha\tau}\cos\beta\tau. \tag{57}$$

For the determination of α and β we have

$$\alpha = \frac{2\lambda}{1-c}, \quad \alpha^2 - \beta^2 = \frac{4\mu\lambda + 5\lambda^2}{1-c}. \tag{58}$$

which gives

$$\beta = \left[\left(\frac{2\lambda}{1-c}\right)^2 - \frac{4\mu\lambda + 5\lambda^2}{1-c}\right]^{1/2}.$$

Integrating

$$\int_0^{\infty} e^{-\alpha\tau}\cos\beta\tau\, d\tau = \frac{\alpha}{\alpha^2 + \beta^2},$$

and considering the Eqs. (56), (57) and (58), gives

$$K_{(2)A^2} = \frac{4\mu - \lambda^2}{\nu^2} \frac{1-c}{(3+5c)\lambda - 4\mu(1-c)}. \tag{59}$$

Substituting this into Eq. (47) we obtain D.

We should point out, however, that the approximate expression Eq. (57) for $R_{A^2}(\tau)$ is valid if

$$4\mu + 5\lambda \leqslant \frac{4\lambda}{1-c}, \tag{59a}$$

Otherwise we must use a different expression, e.g.,

$$R_{A^2}(\tau) = e^{-\alpha\tau}\left(1 + \frac{\beta\tau^2}{2}\right), \tag{60}$$

where

$$\alpha = \frac{2\lambda}{1-c}, \quad \alpha^2 + \beta^2 = \frac{4\mu\lambda + 5\lambda^2}{1-c}, \tag{61}$$

Integrating

$$\int_0^\infty R_{A^2}(\tau)\, d\tau = \frac{1}{\alpha} + \frac{\beta}{\alpha^3},$$

and subtituting the values of α and β from Eq. (61), we find

$$K_{(2)A^2} = \frac{4\mu^2 - \lambda^2}{(4\lambda\nu)^2}(1-c)^2(4\mu + 5\lambda). \tag{62}$$

Equation (60) can also be used in case condition (59a) is satisfied.

If a closer approximation for $K_{(2)A^2}$ is required, we can set

$$R_{A^2}(\tau) = e^{-\alpha\tau}\left(1 + \frac{\beta_2\tau^2}{2} + \frac{\beta_3\tau^3}{6} + \ldots\right). \tag{63}$$

and we will obtain

$$\int_0^\infty R_{A^2}(\tau)\, d\tau = \frac{1}{\alpha}\left(1 + \frac{\beta_2}{\alpha^2} + \frac{\beta_3}{\alpha^3} + \ldots\right), \tag{64}$$

where β_2, β_3, ...are found from the conditions for the constants at $\tau = 0$.

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