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# Construction of a Distribution Function by the Method of Quasi - Fields

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The method of quasi - fields is developed, by means of which an expression can be constructed for the distribution function. It is shown that the distribution function obtained in this fashion is the same as that of the ordinary theory. A closed expression for the distribution function is given in the form of an infinite-multiple integral.

## INTRODUCTION

An outstanding peculiarity of the present state of the quantum theory of wave fields is the excellent agreement between the theory of quantum electrodynamics with experimental data, while at the same time the results of meson theory (by meson theory we mean the present theory of the interaction of $\pi$-mesons with the field of the nucleus) have only the most general and qualitative character, and cannot be brought into any satisfactory quantitative agreement with experiment. There are two basic viewpoints relative to the origin of this failure of meson theory. The first is that meson theory, based on a formal analogy with electrodynamics, is not adequate for the physical facts, and the simple transposition of electrodynamic concepts into the field of meson phenomena is not adequate; and accordingly for construction of a proper meson theory, a new method is necessary, based on a fundamental reconstruction of our ideas as to the nature of the interaction between mesons and nuclei.

Such a viewpoint seems to us to be very probably true; however, there is also a second possibility. All mathematical methods of modern electrodynamics use, to some degree or other, an analysis of physical quantities in a power series in the charge $e$. This might be done directly in the form of a perturbation theory[1-3] or by means of a summing of several parts of a number of perturbation theories with the aid of solutions of integral equations[4,5] in all cases neglecting magnitudes of the order of some power of $e$. The success of electrodynamics at present is based on the smallness of the constant $e$, and as a result, on the possibility of confining oneself to a small number of approximations for obtaining very good quantitative agreement with experiment. In meson theory, in view of the large value of the binding constant, such a method may not be altogether applicable, or at least applicable only within a limited region (for example, very small meson energies). Hence for meson theory, the problem of first order importance is the search for "precise" solutions, that is, solutions not based on an assumption as to the smallness of the binding constant. The efforts of a large number of theoreticians have been concentrated in this direction; however, a solution of the problem encounters formidable mathematical difficulties and at present is far from satisfactory. Thus the second viewpoint as to the source of the failure of meson theory is essentially that the theory is basically correct, and that the cause of the divergence of theoretical predictions from experiment comes from inadequate mathematical methods.

As long as it is not possible to make a correct approach to the problem of a "precise" solution, and therefore not possible to distinguish the results which stem from the basic theory, from those which are introduced because of approximate mathematical methods, it will be very difficult to make a final pronouncement on the degree of correctness of meson theory.

In this work the attempt is made to construct a new procedure in quantum field theory. The physical ideas basic to this procedure are certain general conceptions regarding the state of free particles. As was done in Feynman's method[2] for "virtual" particles, we shall describe the state of a free particle by a 4-vector of energy-impulse, $p_\mu$, the components of which are not connected by a relation of the type $p^2 = m^2$. The method of Feynman leads to considerable simpli-

[1] J. Schwinger, Phys. Rev. 74, 1439 (1948); 75, 651 (1949); 76, 790 (1949)

[2] R. P. Feynman, Phys. Rev. 76, 749, 769 (1949)

[3] F. J. Dyson, Phys. Rev. 75, 1736 (1949)

[4] S. F. Edwards, Phys. Rev. 90, 284 (1953)

[5] L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov, Doklady Akad. Nauk SSSR 95, 497, 773, 1177 (1954)

fication of the results in comparison with the non-covariant theory of excitation. However, his theory has an essential inadequacy in that it is based on the model of a "one-particle" rather than a secondary-quantized theory, and hence leads to complex expressions for processes with a large number of real or virtual particles. In contrast with Feynman, we construct a secondary-quantized scheme, an advantage of which is that it makes possible a simple and uniform consideration of the properties of a system with an arbitrary number of particles. The resulting formalism is altogether relativistically invariant and very simple in form.

The theory is constructed according to the following pattern: The formal structure of the theory is developed (the quasi-field framework). Then, by use of Hamilton's method, an expression is set up for the fundamental-operator $S$, by means of which the distribution function is determined for a system of interacting particles. It is shown that the distribution functions obtained by this method are identical with those of the ordinary theory.

Because of the simplicity of the formalism, the theory goes further into the computation of the distribution functions, and finally yields expressions for them in the form of infinite-multiple integrals. Analogous expressions, including infinite-multiple integrals, were obtained in the solution of the equations of Schwinger[6], in the work of Gel'fand and Minlos[7] and Fradkin[8].

The appearance in the theory of infinite-multiple integrals obviously comes from the essential problem of field theory. Hence one of the central problems in the further development of the theory is the investigation of the properties of such integrals and approximate methods for their evaluation.

So far no considerable results have been achieved along this line. Infinite-multiple integrals comprise in themselves a new mathematical concept, and the possibility of using them effectively is of necessity connected with a major upheaval in mathematics. Thus, for the solution of the problem of quantum theory of fields, it is necessary to take a new and possibly very difficult step.

## 1. THE CONCEPT OF THE STATE OF A PARTICLE

For clarity we will speak below of the pseudo-scalar meson theory with a pseudo-scalar symmetrical variant of interaction. However, as will be evident later, the method being elaborated is not limited to the given variation of the meson theory, but is obviously applicable for all kinds of other variations of interaction both in Fermi fields and Bose fields, as well as in the case of the interaction of several fields.

We will use a system of units in which $\hbar = c = 1$. For the Dirac matrix, we use the form of Feynman[2]. The 4-vector $a_\mu$ we shall fix by the four actual components, and the scalar product of two vectors we write in the form

$$ab = a_\mu b_\mu = a_4 b_4 - a_1 b_1 - a_2 b_2 - a_3 b_3.$$

We indicate the scalar product of the vector $a_\mu$ with the matrix vector $\gamma_\mu$ by the symbol

$$\hat{a} = \gamma_\mu a_\mu.$$

The point of departure of the development of the method is the general concept of the state of a free particle. The state of a nucleon is specified by the following physical quantities: the 4-vector of the energy-impulse $p_\mu$, the spin variable $\alpha$ and the variable isotopic spin $\rho$. Further, the components of the vector $p_\mu$ are not connected by the relation $p^2 = m^2$, and can take arbitrary values independently of one another. The varying quantity $\alpha$ takes four independent values rather than 2 as in the ordinary theory. The variable $\rho$ takes two values, corresponding to the proton and neutron states. The states of the antinucleons are determined in exactly the same way as those of the nucleons. The state of a meson is determined by the 4-vector of energy-impulse $k_\mu$ and variable isotopic spin $r$. The components of the vector $k_\mu$ are not connected by a relation $k^2 = \mu^2$ and can take independently arbitrary values. The variable $r$ can have three values corresponding to charge states of the $\pi$-meson.

Creation and annihilation operators are introduced for construction of the secondary-quantized scheme. For nucleons two kinds of creation operators are necessary: $a^+_{\alpha\rho}(p)$, $b^+_{\alpha\rho}(p)$, and, corresponding to them, annihilation operators $a_{\alpha\rho}(p)$, $b_{\alpha\rho}(p)$. Operators for creation and annihilation of mesons are designated $c^+_r(k)$ and $c_r(k)$. We note that the operators $a^+$ and $a$, etc., are not assumed Hermitian conjugates. More than this, in the following we will not be concerned with Hermitian properties in the operators and therefore will not introduce the idea of Hermitian conjugates. In relation to the Lorentz transformation, the operators $a$ and $b^+$ behave like the Dirac bi-spinor $\psi$, and the operators $a$ and $b$ like the bi-spinor $\bar{\psi}$, operators $c^+$ and $c$ are pseudo-scalar. In rotation in isotopic space $a^+$, $a$, $b^+$ and $b$ are transformed like spinors, and $c^+$ and $c$, like

[6] J. Schwinger, Proc. Nat. Acad. Sci. 37, 452 (1951)

[7] I. M. Gel'fand and R. A. Minlos, Doklady Akad. Nauk SSSR 97, 209 (1954)

[8] E. S. Fradkin, Doklady Akad. Nauk SSSR 98, 47 (1954)

vectors. The operators of creation and annihilation obey the following commutation conditions:

$$[a_{\alpha\rho}(p),\ a^{+}_{\beta\sigma}(q)]_{+} = i\delta_{\alpha\beta}\delta_{\rho\sigma}\delta(p-q), \tag{1.1}$$
$$[b_{\alpha\rho}(p),\ b^{+}_{\beta\sigma}(q)]_{+} = i\delta_{\alpha\beta}\delta_{\rho\sigma}\delta(p-q),$$
$$[c_r(k),\ c_s^{+}(l)] = i\delta_{rs}\delta(k-l).$$

Here only those commutation brackets are written that differ from zero. In the remaining cases, the nucleon operators anti-commute among themselves and commute with the meson operators, while the meson operators commute with one another.

Now that the operators of creation and annihilation have been determined, we can fix the state of a vacuum $\Phi_0$ as the state in which there is no particle. The mathematical property of a vacuum is fixed by the equations

$$a\Phi_0 = b\Phi_0 = c\Phi_0 = 0. \tag{1.2}$$

Different products of operators of creation, acting on a vacuum, generate a state with the specified number of particles. As is commonly done, we assume the state resulting from this arrangement as a basis in space for all states of a system of nucleon and meson fields.

The mean value over a vacuum of a certain operator $A$ is determined in a natural manner, as expressed through operators of creation and annihilation. This mean value we will designate by the symbol $<A>_0$. In computing the mean in a vacuum, the normalization condition must be used

$$(\Phi_0,\ \Phi_0) = \langle 1\rangle_0 = 1.$$

## 2. THE QUASI-FIELD FRAMEWORK

We now specifiy operators playing in our scheme the role of field operators of the ordinary theory. Although hereafter we will be concerned only with quasi-fields, we reserve for them the designation used in the ordinary formalism for field operators. Operators of the quasi-field are fixed by the following Fourier integrals*.

The nucleon operators of the quasi-field are

$$\psi_\rho(x) = \frac{1}{(2\pi)^2}\int\{(\hat{p}-m)^{-1/2}a_\rho(p)e^{-ipx} + (\hat{p}+m)^{-1/2}b_\rho^{+}(p)e^{ipx}\}\,dp, \tag{2.1}$$

$$\bar{\psi}_\rho(x) = \frac{1}{(2\pi)^2}\int\{a_\rho^{+}(p)(\hat{p}-m)^{-1/2}e^{ipx} + b_\rho(p)(\hat{p}+m)^{-1/2}e^{-ipx}\}\,dp.$$

The quasi-field meson operator is

$$\varphi_r(x) = \frac{1}{(2\pi)^2}\int(k^2-\mu^2)^{-1/2}\{c_r(k)e^{-ikx} + c_r^{+}(k)e^{ikx}\}\,dk. \tag{2.2}$$

In Eqs. (2.1 and (2.2) the integration is carried out throughout the entire 4-dimensional momentum space. Inasmuch as the integrands have singularities at points lying on the surfaces $p^2 = m^2$ and $k^2 = \mu^2$ ($m$ and $\mu$ are the masses of nucleon and meson respectively), for removal of duality of mass values we assume a difference in mass in the form of infinitesimally small negative imaginary particles.

The square roots of the Dirac matrices appearing in Eq. (2.1) are fixed by the equations:

$$(\hat{p}-m)^{-1/2} = \frac{\hat{p}+m-i\sqrt{p^2-m^2}}{\sqrt{2(m-i\sqrt{p^2-m^2})(p^2-m^2)}},$$
$$(\hat{p}+m)^{-1/2} = i\,\frac{\hat{p}-m+i\sqrt{p^2-m^2}}{\sqrt{2(m-i\sqrt{p^2-m^2})(p^2-m^2)}}.$$

The identity of mass values in the determination of the square root will hereafter play no part.

We notice that the operators of the quasi-field [Eqs. (2.1) and (2.2)] do not satisfy any kind of differential equation, and the equations of motion for the operators are nowhere used in the scheme developed here. In the following we depend on the commutation properties of the operators of the quasi-field (only the non-trivial commutation brackets are written out):

$$[\psi_{\alpha\rho}(x),\ \bar{\psi}_{\beta\sigma}(y)]_{+} = 0, \tag{2.3}$$
$$[\varphi_r(x),\ \varphi_s(y)] = 0.$$

The remaining commutation brackets rotate to 0 in a trivial manner. In this way all the operators of the quasi-field at two arbitrary 4-parameter points $x$ and $y$ either commute or anti-commute among themselves.

The meaning of the operators of the quasi-field appears on computation of the vacuum averages of the product of two operators (only the magnitudes different from 0 are written):

$$\langle\bar{\psi}_{\alpha\rho}(x)\,\psi_{\beta\sigma}(y)\rangle_0 = -\langle\psi_{\beta\sigma}(y)\,\bar{\psi}_{\alpha\rho}(x)\rangle_0 = {}^1/_2\,\delta_{\rho\sigma}S^F(y-x), \tag{2.4}$$

* After sending the paper to the editor, the author found works in which analogous operators were considered [e.g., see Iu. V. Novozhilov, Doklady Akad. Nauk SSSR 99, 533, 723 (1954) and S. Coester, Phys. Rev. 95, 1318 (1954)].

$$\langle \varphi_r(x)\, \varphi_s(y) \rangle_0 = {}^1/_2\, \delta_{rs} D^F(x-y).$$

Here $S^F$ and $D^F$ are the well known Feynman functions (see, for example, reference 9). Equations (2.3) and (2.4) follow readily from Eqs. (2.1), (2.2) and (1.1).

Relations (2.4) serve as a connecting link between the formalism of the quasi-field and the ordinary formalism. Actually, Eq. (2.4) leads to the known relations of field theory, if the symbols $\psi$ and $\phi$ stand for the ordinary field operators in the interaction considered, and in place of the product, if the $T$-product of the operators is used.

By virtue of Eq. (2.3), the $T$-product for the quasi-field coincides with the ordinary product. It is possible to eliminate from the theory the idea of the $T$- product, which leads to considerable simplification.

### 3. THE FUNDAMENTAL OPERATOR

We formulate in terms of the quasi-field the analogue of the $S$- matrix, written in the form due to Dyson[3]. For this we determine the interaction operator of the field, which for a symmetrical pseudo-scalar meson theory, has the form

$$K(x) = ig\bar{\psi}(x)\, \gamma_5 \tau_r \psi(x)\, \varphi_r(x). \qquad (3.1)$$

The fundamental operator $S$ is determined by the equation

$$S = \exp\left\{-i \int (x)\, dx\right\}. \qquad (3.2)$$

We notice certain peculiarities of Eq. (3.2). First, by virtue of Eq. (2.3), the quantities $K(x)$ at different points of space commute with one another. Hence, in Eq. (3.2), only quantities are involved that commute, so that the exponential function can be considered in the algebraic sense rather than the symbolic sense, as is done in the references 10 and 11. Second, the integral in the exponent of Eq. (3.2) is distributed in all of 4-dimensional space. This fact insures the validity of the conservation of the 4-momentum in each elementary interaction. If, in place of the quasi-field, one should set up the ordinary field operators, the corresponding integral would transform to 0 by virtue of the existence of relations of the type $p^2 = m^2$, which naturally does not occur in our case.

In what follows, it will be appropriate for us to make the transition from coordinate representation to momentum. Also we introduce into the space of the 4-momentum a cubic lattice with 4-dimensional volume $(2\pi)^4/\Omega$ and will refer all quantities to the nodes of the lattice. To shorten the writing, we will use the complex indices

$$\lambda = (p, \alpha, \rho), \quad l = (k, r),$$

and agree that symbols of the type $-\gamma$ and $-l$ stand for the quantities

$$-\lambda = (-p, \alpha, \rho), \quad -l = (-k, r).$$

We replace the operators of creation and annihilation, introduced in Sec. 1, by the operators

$$a_\lambda^+ = \frac{(2\pi)^2}{\sqrt{\Omega}}\, a_{\alpha\rho}^+(p) \qquad \text{etc.},$$

satisfying, by virtue of Eq. (1.1), the commutation relations

$$[a_\lambda, a_\mu^+]_+ = [b_\lambda, b_\mu^+]_+ = i\delta_{\mu\nu}, \qquad (3.3)$$
$$[c_l, c_m^+] = i\delta_{lm}.$$

After obvious development , we obtain the following expression for $S$:

$$S = e^K, \qquad (3.4)$$

$$\times\, \dot{K} = i \sum_{\mu\nu l} (a_\mu^+ - ib_{-\mu})\, \Gamma_{\mu\nu}^l (b_{-\nu}^+ + ia_\nu) \qquad (3.5)$$

$$(c_l^+ + c_{-l}),$$

$$\Gamma_{\mu\nu}^l = -\frac{g}{\sqrt{\Omega}} (\hat{p} - m)^{-1/2} \gamma_5 (\hat{q} - m)^{-1/2} \qquad (3.6)$$
$$\tau_r (k^2 - \mu^2)^{-1/2}\, \delta_{p-q+k} \quad \mu = (p, \alpha, \rho), \quad \nu = (q, \beta, \sigma),$$
$$l = (k, r).$$

In Eq. (3.6) the matrix indices of the Dirac matrix, and the matrix of the isotopic spin $\tau_r$ are not written out explicitly. We notice that only Eq. (3.6) is specific for a given variant of the meson theory. In what follows we will not depend on the concrete form of $\Gamma$, and hence our derivation will hold for all other variants of the theory.

### 4. DISTRIBUTION FUNCTION

Through the fundamental operator $S$ the distribution function is determined for an arbitrary

[9] A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics*, Moscow (1953)

[10] R. P. Feynman, Phys. Rev. **84**, 108 (1951)

[11] I. Fujiwara, Progr. Theor. Phys. 7, 433 (1952)

system of interacting particles. To be concrete, we consider the particular case in which both initially and finally the states of the system consist of mesons and nucleons; however, all the deliberations clearly carry over to the general case. We determine the distribution function for our case by the expression

$$K(x_1'x_2';\ x_1x_2) = \langle \varphi(x_1')\psi(x_2') \times S\varphi(x_1)\overline{\psi}(x_2)\rangle_0. \quad (4.1)$$

We show that Eq. (4.1) is in identical agreement with the corresponding expression of the ordinary theory. For this, we consider the $N$th term of the expansion of Eq. (4.1) in a series in powers of $g$. According to Eqs. (3.1) and (3.2) this term is equal to

$$K_N(x_1', x_2';\ x_1x_2) = \frac{g^N}{N!}\int dt_1 \ldots \int dt_N \langle \varphi(x_1')\psi(x_2')\overline{\psi}(t_1)\gamma_5\tau_{r_1}\psi(t_1)\varphi_{r_1}(t_1)\ldots \overline{\psi}(t_N)\gamma_5\tau_{r_N}\psi(t_N)\varphi_{r_N}(t_N)\varphi(x_1)\overline{\psi}(x_2)\rangle_0. \quad (4.2)$$

The vacuum average, appearing under the integral sign, in conformity with Wick's theorem[12] (this theorem is valid also for the quasi-field), is equal to the sum of all possible components, in which all operators are connected in pairs. Yet, on the one hand, every system of connection corresponds to some Feynman diagram, and conversely; and, on the other hand, in conformity with Eq. (2.4), the equation for connection of operators of the quasi-field agree with the corresponding expressions for ordinary fields. From this it follows that in Eq. (4.2) contributions are summed of all the Feynman diagrams of the $N$th order, and in this fashion our assertion is demonstrated.

Points $x_i$ and $x_i'$ correspond to the free ends of the outer lines of the diagram, the points $t_n$ correspond to the ends of the inner lines, and the integration is carried out over them.

By means of the propagation function $K(x_1'x_2';\ x_1x_2)$ the elements of the $S$-matrix can be directly computed by Feynman's rule[2] for the process of scattering of mesons by nucleons.

For later applications it is rather convenient to transform Eq. (4.1). We take advantage of the fact that, by virtue of Eqs. (3.1), (3.2) and (2.3), the operator $S$ commutes with all the operators of the quasi-field; and we transpose the operators $\phi(x_1)$ and $\psi(x_2)$ over to the left of $S$. Then

$$K(x_1'x_2';\ x_1x_2) = \langle\varphi(x_1')\varphi(x_1)\psi(x_2')\overline{\psi}(x_2)S\rangle_0 = (\overline{\Phi}_0|\varphi(x_1')\varphi(x_1)\psi(x_2')\overline{\psi}(x_2)|\Psi_0), \quad (4.3)$$

where
$$\Psi_0 = S\Phi_0 \quad (4.4).$$

In this manner the determination of the propagation function boils down to the computation of the vector $\Psi_0$, after which the problem takes on an algebraic character. Essentially the vector $\Psi_0$ does not depend on the specific process to which the given propagation function is related.

### 5. THE RESOLVING OPERATION

The vector $\Psi_0$, determined in the foregoing section, as well as all vectors in state space, can be put in the form

$$\Psi_0 = U\Phi_0, \quad (5.1)$$

where the operator $U$ is expressed only through the operators of particle creation. If we compare Eqs. (4.4) and (5.1), we see that to the operator $S$, which is expressed in terms of operators of creation as well as operators of annihilation, there corresponds the operator $U$ which contains only operators of creation. The transformation from the operator $S$ to the operator $U$, we will call the resolution of operator $S$ and define the transformation symbol $U = S > 0$. A further problem will be to devise certain general methods of resolution.

On the basis of relations which are valid for arbitrary operators $x$ and $a$:

$$e^x a e^{-x} = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\overset{n}{x},\ a\right]. \quad (5.2)$$

[for proof of Eq. (5.2) see Appendix I] and the corresponding commutation Eq. (3.3), it is easy to establish the validity of the following equations:

$$e^P a_\lambda^+ e^{-P} = a_\lambda^+ - ib_{-\lambda},\ e^P b_{-\lambda}^+ e^{-P} = b_{-\lambda}^+ + ia_{-\lambda},\ e^Q c_l^+ e^{-Q} = c_l^+ + c_{-l} \quad (5.3)$$

where

$$P = \sum_\mu a_\mu b_{-\mu}, \quad (5.4)$$

[12] G. C. Wick, Phys. Rev. 80, 268 (1950)

$$Q = -\frac{i}{2}\sum_m c_m c_{-m}.$$

From Eqs. (5.3), (3.4) and (3.5) it follows that the operator $S$ can be expressed in the form

$$S = e^{P+Q}\, e^{K_0}\, e^{-(P+Q)}, \tag{5.5}$$

where the operator

$$K_0 = i\sum_{\mu\nu l} a_\mu^+ \Gamma_{\mu\nu}^l\, b_{-\nu}^+ c_l^+ \tag{5.6}$$

contains only the operators of creation.

We introduce the symbol

$$\sum_{\mu\nu} a_\mu^+ \Lambda_{\mu\nu}\, b_{-\nu}^+ = a^+\Lambda b^+,$$

where $\Lambda$ is a matrix. Then $K_0$ can be written in the two equivalent forms

$$K_0 = i\sum_l x_l c_l^+ = a^+\Gamma\,(c^+)\, b^+, \tag{5.7}$$

where

$$x_l = a^+\Gamma^l b^+, \tag{5.8}$$

$$\Gamma\,(c^+) = i\sum_l \Gamma^l c_l^+.$$

From the properties of the vacuum [Eq. (1.2)] it follows that in resolution, one can reject terms in which an operator of annihilation is found on the right side. Hence, from Eqs. (5.5) and (5.4) comes the relation

$$S\rangle_0 = e^{P+Q}\, e^{K_0}\rangle_0. \tag{5.9}$$

## 6. "DIFFUSION" EQUATIONS

We have for resolution expressions of the type in Eq. (5.9) for which it is characteristic that all operators of annihilation are to the left of the creation operators. Let $F(c^+)$ be some function of the meson creation operators. Then

$$QF\rangle_0 = -\frac{i}{2}\sum_m [c_m,\ [c_{-m},\ F]]$$

$$= \frac{i}{2}\sum_m \frac{\partial^2 F}{\partial c_m^+\,\partial c_{-m}^+},$$

that is, the resolution operator $Q$, standing on the left of the function depending on $c^+$, acts like the differential operator

$$Q = \frac{i}{2}\sum_m \frac{\partial^2}{\partial c_m^+\,\partial c_{-m}^+}. \tag{6.1}$$

Likewise, if $F(a^+,\ b^+)$ is a function of nucleon operators of creation, in the resolution of which the overall number of operators $a^+$ and $b^+$ s even, then

$$PF\rangle_0 = \sum_\mu [a_\mu,\ [b_{-\mu},\ F]]_+, \tag{6.2}$$

that is, $P$ can also be considered a differential operator, but with this difference, that differentiation is carried out with respect to the anti-commuting variables* $a^+$ and $b^+$.

Hence, in view of Eq. (5.9),

$$U = S\rangle_0 = e^P e^Q e^{K_0}, \tag{6.3}$$

where $P$ and $Q$ are operators whose actions are determined by Eqs. (6.1) and (6.2). For evaluating expressions of the type of Eq. (6.3) we use the following relation. If $F_0$ is some function, and $L$ is a linear operator, the function

$$F = e^L F_0$$

can be considered a solution of the equation

$$\partial F/\partial z = LF \qquad \text{for} \qquad z = 0,\ F = F_0 \tag{6.4}$$

if we set $z = 1$. Equation (6.4) has the character of a "diffusion" equation. Analogous equations are used for computing $S$-matrices in references 12 and 13.

In light of the simple character of the operator $Q$, the computation of $e^Q e^{K_0}$ can be carried out directly. We consider the case $L = P$. In this we are limited to a class of functions $F$ depending on the variables

$$x_n = a^+\Lambda^n b^+ \quad (n = 1,\ 2,\ \ldots).$$

Operator $P$ does not develop a function of this class, and Eq. (6.4) takes the form of Eq. (6.5). [The notation is introduced $\lambda_n = \mathrm{Sp}\,\Lambda^n = \sum_\mu (\Lambda^n)_{\mu\mu}$. The derivation of Eq. (6.5) is shown in Appendix II.]

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* We will not introduce any special symbolism for this differentiation.

[13] S. Hori, Progr. Theor. Phys. 7, 578 (1952)

$$\frac{\partial F}{\partial z} = -\sum_{m,n=1}^{\infty} x_{m+n}\frac{\partial^2 F}{\partial x_m \partial x_n} + \sum_{n=1}^{\infty} \lambda_n \frac{\partial F}{\partial x_n}. \quad (6.5)$$

The solution of Eq. (6.5) for $F_0 = e^{x_1}$ is equal to

$$F(z) = \exp\left\{\sum_{n=1}^{\infty}(-1)^{n-1}\left(x_n z^{n-1} + \lambda_n \frac{z^n}{n}\right)\right\}. \quad (6.6)$$

## 7. THE EXPRESSION FOR THE OPERATOR $U$

Taking into account Eqs. (5.7), (5.8) and (6.1), it is easy to establish the equality

$$e^Q e^{K_0} \equiv e^Q \exp\left\{i\sum_l x_l c_l^+\right\} = \exp\left\{i\sum_l (x_l c_l^+ - {}^1/_2\, x_l x_{-l})\right\}. \quad (7.1)$$

Further, from Eq. (6.6), for $z = 1$ and $\Lambda = \Gamma(c^+)$, it follows that

$$e^P e^{K_0} \equiv e^P \exp\{a^+\Gamma(c^+)b^+\} = \exp\left\{\sum_{n=1}^{\infty}(-1)^{n-1}\left[a^+\Gamma^n(c^+)b^+ + \frac{1}{n}\operatorname{Sp}\Gamma^n(c^+)\right]\right\}. \quad (7.2)$$

Results analogous to Eqs. (7.1) and (7.2) and corresponding to resolution of one of the fields are achieved by the methods of ordinary theory in references 10 and 14.

Simultaneous resolution of two fields is performed as follows: We consider the formal expansion of $e^{K_0}$ in an infinite-product Fourier integral

$$e^{K_0} \equiv \exp\left\{i\sum_l x_l c_l^+\right\} = \int \exp\left\{i\sum_l x_l s_l\right\}\exp\left\{i\sum_l c_l^+ t_l\right\} \times \exp\left\{-i\sum_l s_l t_l\right\} ds\,dt = \int \exp\{a^+\Gamma(s)b^+\}\exp\left\{i\sum_l c_l^+ t_l\right\} \times \exp\left\{-i\sum_l s_l t_l\right\} ds\,dt, \quad (7.3)$$

where

$$ds = \prod_l \frac{ds_l}{\sqrt{2\pi}}, \qquad dt = \prod_l \frac{dt_l}{\sqrt{2\pi}}.$$

In Eq. (7.3) variables pertaining to nucleons and mesons are separated. Applying the results of Eqs. (7.1) and (7.2), we obtain, in conformity with Eq. (6.3),

$$U = e^{P+Q}e^{K_0} = \int \exp\left\{\sum_{n=1}^{\infty}(-1)^{n-1}\left[a^+\Gamma^n(s)b^+ + \frac{1}{n}\operatorname{Sp}\Gamma^n(s)\right]\right\} \times \exp\left\{i\sum_l c_l^+ t_l - s_l t_l - \frac{1}{2} t_l t_l\right\} ds\,dt.$$

Integration with respect to $t$ can be carried out directly inasmuch as the infinite-product integral is broken down into the product of double integrals. As a result we obtain the following infinite product expression

$$U = \int \exp\left\{\sum_{n=1}^{\infty}(-1)^{n-1}\left[a^+\Gamma^n(s)b^+ + \frac{1}{n}\operatorname{Sp}\Gamma^n(s)\right]\right\} \times \exp\left\{\frac{i}{2}\sum_l (c_l^+ - s_l)(c_{-l}^+ - s_{-l})\right\} ds. \quad (7.4)$$

By means of Eqs. (7.4) and (5.1) it is easy to obtain the distribution function as was shown in Sec. 4. Analogous expressions for the distribution functions were obtained by Fradkin[8] from Schwinger's theory[6].

In conclusion the author expresses his deep gratitude to Academician I. E. Tamm for his great influence on this work and for general review of the results.

## APPENDIX

### I. DEMONSTRATION OF EQ. (5.2)

In Eq. (5.2), the repeated commutator $\left[\overset{n}{x}, a\right]$ is determined by the recurrent relations

$$\left[\overset{0}{x}, a\right] = a, \quad \left[\overset{n+1}{x}, a\right] = \left[x, \left[\overset{n}{x}, a\right]\right].$$

The formula is easily demonstrated by the induction method

[14] K. Yamazaki, Progr. Theor. Phys. 7, 449 (1952)

$$\left[\overset{n}{x},\ a\right] = \sum_{p+q=n} (-1)^q \binom{n}{q} x^p a x^q.$$

Hence

$$e^x a e^{-x} = \sum_{p,\ q=0}^{\infty} \frac{(-1)^q}{p!\ q!} x^p a x^q$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{p+q=n} (-1)^q \binom{n}{q} x^p a x^q$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\overset{n}{x},\ a\right].$$

## II. DERIVATION OF EQ. (6.5)

Let the operator $a$ have the property that the commutator $[a, x_n]$ commute with operators $x_m$ for all $m$ and $n$. Then the following relation holds*

$$[a,\ F(x_1,\ x_2,\ \ldots)] = \sum [a,\ x_n] \frac{\partial F}{\partial x_n}, \tag{A}$$

proof of which is almost obvious.

Expression (A), in particular, is applicable for $a = a_\mu$ and $a = b_{-\mu}$. Inasmuch as the operators $a^+$ and $b^+$ occur in pairs in the expressions for $x_n$, the action of the operator $P$ on the function $F$ is determined by Eq. (6.2).

For direct computation, use is made of (A) and the relation:

$$[a,\ bc]_+ = [a,\ b]_+ c - b\,[a,\ c],$$
$$[a,\ bc] = [a,\ b]_+ c - b\,[a,\ c]_+,$$

We then get

$$PF = -\sum_{m,\ n} a^+ \Lambda^{m+n} b^+ \frac{\partial^2 F}{\partial x_m \partial x_n} + \sum_n \mathrm{Sp}\,\Lambda^n \frac{\partial F}{\partial x_n},$$

which agrees with the right side of Eq. (6.5).

* The derivatives $\partial F/\partial x_n$ have the ordinary meaning since the operators $x_n$ commute with one another.

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23