On Longitudinal Vibrations of Plasma, II.

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On the basis of the results of paper I, the dispersion properties of plasma are investigated in various special cases. It is shown that the motion of ions is of essential importance in most cases of propagation of longitudinal waves in discharge tubes. Values are found for the spatial period and logarithmic decrement as functions of the parameters of the discharge.

1. LONGITUDINAL VIBRATIONS OF PLASMA IN THE HIGH FREQUENCY REGION

FOR the case of plasma vibration at high frequency, the "dispersion" equation (57) of paper I simplifies considerably, and is easy to solve, so that the frequency dependence of the wave number and logarithmic decrement are obtained in explicit form. This equation, expressed in terms of dimensionless integration variables, has the form:

$$1 = -\frac{1}{k^2a_1^2} + \frac{1}{k^2a_1^2}\frac{\beta_1}{\sqrt{2\pi}}\int\limits_{C_1}\frac{\exp\{-u^2/2\}\,du}{\beta_1 - u} \quad (1)$$

$$-\frac{1}{k^2a_2^2} + \frac{1}{k^2a_2^2}\frac{\beta_2}{\sqrt{2\pi}}\int\limits_{C_2}\frac{\exp\{-u^2/2\}\,du}{\beta_2 - u},$$

where $a_1 = \sqrt{\Theta_1/4\pi Ne^2}$ is the Debye length for the electrons, $a_2 = \sqrt{\Theta_2/4\pi Ne^2}$ is the Debye length for the ions, $u = \sqrt{m/\Theta_1}(\xi - \xi_{01})$ in the integrals over contour C_1, $u = \sqrt{M/\Theta_2}(\xi - \xi_{02})$ in the integrals over contour C_2:

$$\beta_1 = \sqrt{\frac{m}{\Theta_1}}\left(\frac{\omega}{k} + \frac{i}{\tau_1 k} - \xi_{01}\right), \quad (2)$$

$$\beta_2 = \sqrt{\frac{M}{\Theta_2}}\left(\frac{\omega}{k} + \frac{i}{\tau_2 k} - \xi_{02}\right).$$

We shall look for a solution with small logarithmic decrement, i.e., we shall assume that

$$|\gamma| \ll |\varkappa| \quad (\varkappa + i\gamma = k). \quad (I)$$

In integrating along the contours C_1 and C_2 in Eq. (1), the points β_1 and β_2 are circled from below, so that each of these integrals can be written as the sum of an integral along the real axis between the limits $-\infty$ to $+\infty$, plus the residue of the integrand at the singularity, multiplied by πi [since, according to condition (I)

$$\mathrm{Re}\,\beta_1 \gg \mathrm{Im}\,\beta_1,$$

and

$$\mathrm{Re}\,\beta_2 \gg \mathrm{Im}\,\beta_2,$$

as can be seen from what follows]. Now Eq. (1) is expressed in the form:

$$1 = -\frac{1}{k^2a_1^2} + \frac{1}{k^2a_1^2}\frac{\beta_1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\frac{\exp\{-u^2/2\}\,du}{\beta_1 - u} \quad (3)$$

$$-i\sqrt{\frac{\pi}{2}}\frac{\beta_1}{a_1^2k^2}\exp\{-\beta_1^2/2\}$$

$$-\frac{1}{k^2a_2^2} + \frac{1}{k^2a_2^2}\frac{\beta_2}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\frac{\exp\{-u^2/2\}\,du}{\beta_2 - u}$$

$$-i\sqrt{\frac{\pi}{2}}\frac{\beta_2}{a_2^2k^2}\exp\{-\beta_2^2/2\}.$$

Since we are considering the high frequency case, it is natural to introduce the conditions:

$$1/\tau_1 \ll \omega - \varkappa\xi_{01}, \quad 1/\tau_2 \ll \omega - \varkappa\xi_{02}, \qquad \text{(II)}$$

which mean physically that the frequency, as measured in a coordinate system moving with the electron current, is much greater than the frequency of electron-atom collisions; and the frequency of vibration, measured in a coordinate system moving with the ion current, is much greater than the frequency of ion-atom collisions.

Keeping in mind conditions (I) and (II), we obtain the following values for β_1 and β_2 in zeroth approximation:

$$\beta_1 \approx \sqrt{\frac{m}{\Theta_1}}\left(\frac{\omega}{\varkappa} - \xi_{01}\right), \quad \beta_2 \approx \sqrt{\frac{M}{\Theta_2}}\left(\frac{\omega}{\varkappa} - \xi_{02}\right).$$

We also require that

$$\left|\frac{\omega}{\varkappa} - \xi_{01}\right| \gg \sqrt{\frac{\Theta_1}{m}}, \quad \left|\frac{\omega}{\varkappa} - \xi_{02}\right| \gg \sqrt{\frac{\Theta_2}{M}}, \qquad \text{(III)}$$

i.e., the phase velocity of the wave as measured in a coordinate system moving with the electron current shall be much greater than their mean thermal velocity, and that the phase velocity as measured in a system moving with the ions shall be much greater than the average thermal velocity of the ions. In this case, $|\beta_1| \gg 1$ and $|\beta_2| \gg 1$. The conditions (II) and (III) are a simple generalization of the analogous conditions that are introduced in solving the "dispersion" equation for high frequencies, when one neglects the ions and the drift of the charged particles.

Expanding the factors multiplying the exponentials in the integrands in Eq. (3) in powers of u/β_1 and u/β_2, and limiting ourselves to five terms in the expansion, we obtain:

$$1 = \frac{1}{k^2 a_1^2 \beta_1^2} + \frac{3}{k^2 a_1^2 \beta_1^4} - i\sqrt{\frac{\pi}{2}}\frac{\beta_1}{k^2 a_1^2}\exp\{-\beta_1^2/2\} + \frac{1}{k^2 a_2^2 \beta_2^2} + \frac{3}{k^2 a_2^2 \beta_2^4} - i\sqrt{\frac{\pi}{2}}\frac{\beta_2}{k^2 a_2^2}\exp\{-\beta_2^2/2\}. \qquad (4)$$

Separating into real and imaginary parts, we find in first approximation after a simple calculation, the well known dispersion equation relating the wave number κ to the vibration frequency ω, and the expression for the logarithmic decrement γ:

$$(\omega - \varkappa\xi_{01})^2 = \omega_{10}^2(1 + 3a_1^2\varkappa^2), \qquad (5)$$

$$\gamma = \frac{1}{\tau_1(\xi_{01} + 3a_1^2\omega_{10}\varkappa)} + \sqrt{\frac{\pi}{8}}\frac{\omega_{10}(1 + 3a_1^2\varkappa^2)^2}{\varkappa^3 a_1^3(\xi_{01} + 3a_1^2\omega_{10}\varkappa)} \times \exp\left\{-\frac{m\omega_{10}^2(1 + 3a_1^2\varkappa^2)}{2\Theta_1\varkappa^2}\right\}. \qquad (6)$$

Terms due to the ions will be of at least first order in the small quantity m/M compared to the electronic terms, and have therefore been omitted in deriving the relations (5) and (6).

These last relations were obtained under the assumption that the conditions (I)-(III) were fulfilled. It is easy to see that these conditions are fulfilled for sufficiently long wave lengths ($\kappa a_1 \ll 1$) at pressures sufficiently low so that

$$[\tau_1(\xi_{01} + 3a_1^2\omega_{10}\varkappa)]^{-1} \ll \varkappa.$$

The first term in the expression for the absorption coefficient is due to elastic collisions; the second represents the loss of energy, via Coulomb interaction, from electrons participating in the ordered motion, to electrons in random motion. Both the collisions and the loss of phase relation by the particles lead to damping of the disturbance as we move away from the source. In fact, the expression in the denominator of the right hand side of Eq. (6) is just the group velocity of the wave, in first approximation:

$$v_{gr} = d\omega/d\varkappa = \xi_{01} + 3a_1^2\omega_{10}\varkappa, \qquad (7)$$

which is positive, since its direction coincides with the direction of propagation of the disturbance. Thus, for a wave propagating away from the source, γ is positive, which represents a damping. It is interesting that for very long wave lengths, as follows from (7), the perturbation propagates in the direction of the drift velocity of the electrons.

The linear absorption coefficient γ which we have obtained is related to the value of the time decrement[1] γ_t by:

$$\gamma = \gamma_t / \cdot v_{gr} \quad (8)$$

Relation (8) is also satisfied for all the other cases considered in this paper, which indicates its universality.

2. LONGITUDINAL VIBRATIONS OF PLASMA IN THE LOW FREQUENCY REGION

Let us consider the case of excitation of plasma vibrations by a perturbation of low frequency. We include the limiting case $\omega=0$, which corresponds to the appearance of a stationary spatial stratification in the particle distribution, under the action of a given jump of the potential at the boundary.

In the case we are now considering, where the frequency is low, and the average translational velocities of the electrons, ξ_{01}, and of the ions, ξ_{02}, are arbitrary, the integrals in Eq. (1) cannot be expressed in terms of elementary functions. They can however be expressed in terms of the error integral with complex argument[2].

Using Fok's relation[3]:

$$\frac{1}{\sqrt{2\pi}} \int\limits_C \frac{e^{-u^2/2} du}{\beta - u}$$

$$= \sqrt{2} e^{-\beta^2/2} \int\limits_{-i\infty}^{\beta/\sqrt{2}} e^{z^2} dz - i \sqrt{2\pi} e^{-\beta^2/2}$$

and carrying out some simple transformations, we rewrite Eq. (1) in the form:

$$1 = -\frac{1}{k^2 a_1^2} - \frac{i\sqrt{\pi}}{k^2 a_1^2} (\alpha_1 + i\delta_1)(\alpha_3 + i\delta_3) \quad (9)$$

$$-\frac{1}{k^2 a_2^2} - \frac{i\sqrt{\pi}}{k^2 a_2^2} (\alpha_2 + i\delta_2)(\alpha_4 + i\delta_4),$$

where the following notation is used:

$$\beta_1 / \sqrt{2} = \alpha_1 + i\delta_1, \quad \beta_2 / \sqrt{2} = \alpha_2 + i\delta_2; \quad (10)$$

$$\exp\{-\beta_1^2/2\} \left(1 + \frac{2i}{\sqrt{\pi}} \int\limits_0^{\beta_1/\sqrt{2}} e^{z^2} dz\right) = \alpha_3 + i\delta_3,$$

$$\exp\{-\beta_2^2/2\} \left(1 + \frac{2i}{\sqrt{\pi}} \int\limits_0^{\beta_2/\sqrt{2}} e^{z^2} dz\right) = \alpha_4 + i\delta_4. \quad (11)$$

In Eq. (9) integrals of the type of (11) appear, for which tables are available.

Going over to dimensionless quantities, and choosing the Debye length for electrons, a_1, as unit of length, we will have in place of (9):

$$k^{*2} = -1 - (\Theta_1/\Theta_2) - i\sqrt{\pi}(\alpha_1 + i\delta_1)(\alpha_3 + i\delta_3)$$
$$- (\Theta_1/\Theta_2)\, i\sqrt{\pi}(\alpha_2 + i\delta_2)(\alpha_4 + i\delta_4), \quad (12)$$

where k^* is the dimensionless complex wave number ($k^* = ka_1$). Equation (12) can be solved formally, giving κ^* and γ^* in terms of α_1, δ_1, α_2, δ_2, α_3, δ_3, α_4, δ_4. In fact, setting $\kappa^* = \kappa^* + i\gamma^*$ in Eq. (12) and separating real and imaginary parts, we get a pair of algebraic equations whose solution can be written in the form:

$$\kappa^* = \pm\sqrt{y}, \quad \gamma^* = \pm\sqrt{\bar{y}}, \quad (13)$$

where

$$y = (g_1 \pm \sqrt{g_1^2 + g_2^2})/2, \quad (14)$$

$$\bar{y} = (-g_1 \pm \sqrt{g_1^2 + g_2^2})/2. \quad (15)$$

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[1] G. E. Gordeev, J. Exper. Theoret. Phys. USSR **22**, 230 (1952)

[2] V. N. Faddeeva and N. N. Terentiev, Tables of Values of the Probability Integral for Complex Argument, G.I.T.T.L. (Gov't. Publ. Tech. Lit.), 1954; M. Born, Optics.

[3] V. A. Fok, Diffraction of Radio Waves, Publishing House, Acad. Sci. USSR, 1946.

For given κ^*, the sign of γ^* is uniquely determined by the sign of g_2, since

$$\gamma^* = g_2/2\varkappa^*. \tag{16}$$

The relations (13)-(15) enable us to compute the period of the spatial inhomogeneity in the plasma, $\lambda^*=2\pi/\kappa^*$, if we are given the damping coefficient γ^* and the frequency ω (in the general case, for a given frequency we get different values of the wave length λ^*, depending on the damping coefficient γ^*). For practical purposes, it is more convenient to give α_1 and δ_1 rather than ω and γ^*. Assigning α_1 and δ_1 determines the values of ω, γ^*, κ^* and correspondingly α_2, δ_2, α_3, δ_3, α_4 and δ_4. However, in the general case this dependence is not explicit. The values of α_2, δ_2 can be found immediately for given α_1 and δ_1 only if we neglect collisions (knowing α_2 and δ_2, we can find from the tables[2] values of α_3, δ_3, α_4, δ_4, and corresponding to these κ^*, γ^* and ω). Actually, since β_2 is related to β_1 by the relation

$$\beta_2 = \sqrt{\frac{M}{m}\frac{\Theta_1}{\Theta_2}}\,\beta_1 + \sqrt{\frac{M}{\Theta_2}}\,\frac{i}{k}\left(\frac{1}{\tau_2}-\frac{1}{\tau_1}\right) + \sqrt{\frac{M}{\Theta_2}}\,(\xi_{01}-\xi_{02}), \tag{17}$$

β_1 immediately determines β_2 if $\sqrt{\frac{M}{\Theta_2}}\cdot\frac{i}{k}\left(\frac{1}{\tau_2}-\frac{1}{\tau_1}\right)=0$. Inclusion of this term makes it very difficult to establish the relation between ω, κ^* and γ^*, which we must know in order to match values of these quantities which will satisfy the "dispersion" equation. Consequently, finding an exact solution of the "dispersion" equation in the general case is practically impossible.

The fact that collisions were not considered in using the approximation of the Boltzmann integral to calculate the first approximation for the distribution function, cannot have any decisive effect on the value of the space period of a vibration process in the plasma if that process is the result of Coulomb interaction between the particles.*

* This last does not mean that in the present theory the space period does not depend at all on the number of collisions. This dependence enters implicitly into the expression for the zeroth approximation to the distribution function: the velocity of drift and the concentration of charged particles are determined by it.

However, if we are investigating absorption, the collisions can no longer be ignored, so that, having dropped them, we must renounce any attempt to find the value of the logarithmic decrement. All we can do in this respect is to show the existence of waves with increasing amplitude in the absence of collisions.

In this same way we can show the possibility for occurrence of undamped or weakly damped waves, despite the considerable absorption due to collisions.

We consider in detail the case $\omega=0$. Here all the calculations simplify considerably, since in this case α_1 and δ_1 are uniquely determined by the parameters of the discharge, and we can then find κ^* and determine the sign of γ^* (as we pointed out above, it is meaningless to determine a numerical value for γ^* if we neglect collisions). It is typical that, for the case of $\omega=0$, the values of κ^* and γ^* are determined uniquely, whereas in the general case of $\omega\neq 0$, there is a whole sequence of values of κ^* and γ^* which correspond to a definite frequency.

Assuming that the experimentally observed stratified illumination of the positive column (striations) could be due to the presence of a longitudinal density wave, we can compare the values of $\lambda=\lambda^* a_1$ thus obtained with the experimental values. In this interpretation, the case of $\omega=0$ corresponds to fixed striations, the case of $\omega\neq 0$ to moving ones.

The comparison of theory with experiment is made difficult by the lack of precise information concerning some of the discharge parameters which are used in the theory. Usually in experiments on gas discharges only the following data are given: the electron concentration N, which is approximately equal to the ion concentration, the dimensions of the tube, the current I to the anode (or the current density j), the electron temperature T_1, and the spatial period of striation λ_s.

In our formulas there appear the expressions, not for the currents, but rather for the average velocities of the directed motion of electrons and ions. This velocity for the electrons can be determined from the current density j by the familiar relation:

$$j = \frac{1}{1.36}\,Ne\,\xi_{01}. \tag{18}$$

The order of magnitude of the ionic drift velocity can be determined from the simplest gas-kinetic considerations:

$$\xi_{02} = -\,{}^1\!/_2\sqrt{m/M}\,\xi_{01}. \tag{19}$$

Table 1

Gas	N electrons per cm^3	T_1 °K	Pressure in cm Hg	Radius R or cross section S of tube	j or I
N_2^{14}	2×10^8	3×10^4	0.16	3.9 cm	0.27 mA/cm^2
Ne^{20}	10^{10}	3×10^4	1–2	3.5 "	$\frac{Ne}{1.36}$ 4×10^6"
H_2^4	4×10^9	1.2×10^4	0.63	1.1 "	8mA
$Hg^{200.6}$	1.77×10^{10}	3×10^4	3×10^{-3}	0.6×0.4 cm^2	20mA

Gas	λ_s in cm	λ in cm for $T_2 = 1/10\ T_1$	λ in cm for $T_2 = 1/5\ T_1$	λ in cm for $T_2 = 1/2\ T_1$	λ in cm for $T_2 = 300$°K	Literature references
N_2^{14}	6	1.2	2.9	24	0.12	[7]
Ne^{20}	2.5	0.45	1.1	10.70	0.044	[8-9]
H_2^4	0.9	0.16	0.45	3.20	0.041	[10]
$Hg^{200.6}$	0.2	0.04	0.085	0.56	0.013	[11]

We still need to know the ion temperature, which also is not determined in experiments on striations. Apparently it is impossible to speak of the ion temperature in a plasma without specific information about the current distribution. Thus the ion temperature must be investigated separately in each experiment.

For orientation, we have used those data on ion temperatures which are available in the literature [4-6]. In accordance with this information, we shall compute the periods of striations setting the ion temperature T_2 equal, in turn, to $(1/10)T_1$, $(1/5)\ T_1$, $(1/2)T_1$ and room temperature.

The experimental data[7-11] in Table 1 are given along with the value of the space period computed theoretically for the ion temperatures shown there.

Table 1 shows that, for slightly non-isothermal plasma ($T_1/T_2 \approx 2-10$) the space period of stationary striations as calculated theoretically agrees in order of magnitude with the experimental value. There is of course no expectation of exact agreement of theory and experiment, since our formulation of the problem is highly idealized.

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[5] L. Tonks and I. Langmuir, Phys. Rev. **34**, iuy (1929).
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[9] A. A. Zaitsev and Iu. L. Klimontovich, Vestn. M.G.U. (Moscow St. Univ.) ser. phys. math. and nat. sci., **12**, 59 (1951).
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In particular, the effect of the radius of the tube on the period of striations has not been considered.

Nevertheless, order of magnitude calculations are still of interest, since for striation periods smaller than the tube diameter (which is the case for the experiments considered), we should expect that the tube radius should not seriously affect the space period. Calculations of the space period ommiting the ions show that the contribution of the ions to the value of λ will be the same in order of magnitude as that of the electrons. Consequently in this case the motion of the ions cannot be neglected.

In the case of highly non-isothermal plasma ($T_2 = 300\,^\circ$K), only the ionic vibrations are important. This can be shown by comparing the value of the period computed taking account of the ions with the value for the electrons alone.

We should also point out that in the case of electrons alone, the damping coefficient γ is always positive, ($g_2 > 0$), for positive values of κ, i.e., damping occurs. Since the inclusion of collisions leads to even greater damping, spatial layering is not possible in this case. Joint oscillations of electrons and ions lead to negative values of γ ($g_2 < 0$). This increase in amplitude, compensating the damping caused by collisions, can result in undamped waves.

A special feature of the results obtained is the occurrence of stationary striations for $|\xi_{01}| < 0.93 v_{Te} = 0.93\sqrt{2\Theta_1/m}$, in complete accord with experiment. The point is that, in several papers on the theory of striations, a condition has been presented for the existence of stationary striations

$$\xi_{01} \geqslant 0.93\, v_{Te},$$

supposedly following from the kinetic equations. This condition was first derived by Vlasov[12] from a form of the dispersion equation which was not entirely correct, since it contained a divergent integral. But in a paper of Klimontovich[13], the same condition was obtained on the basis of a correct dispersion equation. By considering an unbounded plasma, and neglecting collisions, Klimontovich arrived at the result that in such a plasma there cannot occur a time-independent distribution of electric field of the form $E = E_0 e^{-px} (p = \gamma + i\kappa)$, for $|\gamma| > |\kappa|$. (The last condition is equivalent to the limitation on the drift velocity mentioned above). However, this condition states only that in an unbounded plasma there can exist no exponentially increasing or decreasing distribution of electron density; this assertion, though certainly correct, has nothing to do with the appearance of striations. First of all, when we take account of the boundary (as is done in this paper), solutions of the type $E = E_0 \exp\{-(\gamma + i\kappa)x\}$ with $|\gamma| > |\kappa|$ do exist and satisfy the dispersion equation. Furthermore the true criterion for occurrence of spatial periodicity will have the form: $|\kappa| > |\gamma|$, where the quantity γ must be calculated by taking account of collisions. This condition gives a relation between the values of the drift velocities of the particles, their thermal velocities, and the frequency of collision of charged particles with neutrals.

We have considered the case of the appearance of spatial periodicity in the distribution of the charge density under the action of a jump in potential at some point of the plasma ($\omega = 0$). If the perturbation on the boundary varies periodically with frequency ω, it will excite in the plasma a traveling wave with phase velocity ω/κ. For small phase velocities, namely for $|\omega/\kappa| \ll \xi_{02}$ | the dispersion equation remains, in first approximation, the same as for case of $\omega = 0$. For such waves, the period of the traveling spatial inhomogeneity will be the same as for stationary striations. This result is in agreement with experiment.

Usually for plasma in gas discharge tubes, $|\xi_{01}| \sim 10^6 - 10^8$ cm/sec, $|\xi_{02}| \sim 10^4 - 10^6$ cm/sec. For the centimeter wave region, the condition imposed on the phase velocity of the wave corresponds to the frequency range $\omega < 10^4 - 10^6$ c.p.s., to which the results of this section are therefore applicable.

3. THE CASE OF HIGH DRIFT VELOCITIES

For high drift velocity of the electrons, the problem of propagation in the plasma of a periodically varying perturbation, including effects of collisions of electrons and ions with gas atoms, can be completely solved by a method of successive approximations (just as in the case of high frequencies). In this case, the "dispersion" equation enables us to give explicitly the dependence of the space period of the inhomogeneity and the damping coefficient on the vibration frequency and the parameters of the discharge.

We shall assume that the drift velocity of the

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12 A. Vlasov, Theory of Many Particles, Govt. Publ. House 1950.

13 Iu. L. Klimontovich, Journal of Experimental and Theoretical Physics., 21, 1292 (1951).

electrons is much greater than their thermal velocity, i.e.,

$$|\xi_{01}| \gg v_{Te} = \sqrt{2\Theta_1/m}. \tag{20}$$

A similar relation will hold for the ions:

$$|\xi_{02}| \gg v_{Ti} = \sqrt{2\Theta_2/M}. \tag{21}$$

We also assume that the following conditions are satisfied:

$$|\omega/\varkappa| \ll |\xi_{02}|, \quad |1/\tau_2\varkappa| \ll |\xi_{02}|. \tag{22}$$

Then, expanding the denominators of the integrands in Eq. (57) of paper I in powers of the small parameters,

$$\alpha = (\xi - \xi_{01})\Big/\left(\xi_{01} - \frac{\omega}{k} - \frac{i}{\tau_1 k}\right)$$

and

$$\beta = (\xi - \xi_{02})\Big/\left(\xi_{02} - \frac{\omega}{k} - \frac{i}{\tau_2 k}\right)$$

and limiting ourselves to the first approximation in computing the integrals, we obtain

$$k^2 = \left[\omega_{10}^2/\xi_{01}^2\left(1 - \frac{\omega}{k\xi_{01}} - \frac{i}{\tau_1 k\xi_{01}}\right)^2\right] \tag{23}$$

$$+ \left[\omega_{20}^2/\xi_{02}^2\left(1 - \frac{\omega}{k\xi_{02}} - \frac{i}{\tau_2 k\xi_{02}}\right)^2\right]$$

$$+ \sqrt{\frac{\pi}{2}}\,\omega_{10}^2\left[\left(\frac{m}{\Theta_1}\right)^{3/2}\left(\xi_{01} - \frac{\omega}{k} - \frac{i}{\tau_1 k}\right)\right.$$

$$\left.\times \exp\left\{-\frac{m}{2\Theta_1}\left(\xi_{01} - \frac{\omega}{k} - \frac{i}{\tau_1 k}\right)^2\right\}\right]$$

$$+ \sqrt{\frac{\pi}{2}}\,\omega_{20}^2\left[\left(\frac{M}{\Theta_2}\right)^{3/2}\left(\xi_{02} - \frac{\omega}{k} - \frac{i}{\tau_2 k}\right)\right.$$

$$\left.\times \exp\left\{-\frac{M}{2\Theta_2}\left(\xi_{02} - \frac{\omega}{k} - \frac{i}{\tau_2 k}\right)^2\right\}\right].$$

The solution of Eq. (23) for positive values of κ, satisfying the condition (22) in the frequency range $\omega \ll \omega_{20}$ and $1/\tau_2 \ll \omega_{20}$, up to the terms of first order in the small quantities $\omega/k\xi_{01}$, $\omega/k\xi_{02}$, $1/\tau_1 k\xi_{01}$ and $1/\tau_2 k\xi_{02}$, has the form:

$$\varkappa = \sqrt{\frac{\omega_{10}^2}{\xi_{01}^2} + \frac{\omega_{20}^2}{\xi_{02}^2}} + \frac{\omega}{\xi_{01}}\,\frac{1}{1 + (\omega_{20}^2/\omega_{10}^2)(\xi_{01}^2/\xi_{02}^2)} \tag{24}$$

$$+ \frac{\omega}{\xi_{02}}\,\frac{1}{1 + (\omega_{10}^2/\omega_{20}^2)(\xi_{02}^2/\xi_{01}^2)}, \tag{25}$$

$$\gamma = \frac{1}{\tau_1\xi_{01}}\,\frac{1}{1 + (\omega_{20}^2/\omega_{10}^2)(\xi_{01}^2/\xi_{02}^2)}$$

$$+ \frac{1}{\tau_2\xi_{02}}\,\frac{1}{(\omega_{10}^2/\omega_{20}^2)(\xi_{02}^2/\xi_{01}^2) + 1}$$

$$+ \sqrt{\frac{\pi}{2}}\,\frac{m\xi_{01}|\xi_{01}|}{2\Theta_1 a_1}\,\frac{\exp\{-m\xi_{01}^2/2\Theta_1\}}{\sqrt{1 + (\omega_{20}^2/\omega_{10}^2)(\xi_{01}^2/\xi_{02}^2)}}$$

$$+ \sqrt{\frac{\pi}{2}}\,\frac{M\xi_{02}|\xi_{02}|}{2\Theta_2 a_2}\,\frac{\exp\{-M\xi_{02}^2/2\Theta_2\}}{\sqrt{(\omega_{10}^2/\omega_{02}^2)(\xi_{02}^2/\xi_{01}^2) + 1}}.$$

From the relations (24), (25), we see that the ions play an essential part in this case of propagation of a perturbation in the plasma. Thus their contribution to value of the space period is the same order of magnitude as the contribution of the electrons. The direction of propagation of the disturbance is determined entirely by the ionic component, and in fact coincides with the direction of drift of the ions; i.e., the disturbance propagates only in the direction of the cathode. This last statement follows from the expression for the group velocity, which is a consequence of Eq. (24):

$$v_{gr} = \left[\frac{1}{\xi_{02}}\,\frac{1}{(\omega_{10}^2/\omega_{20}^2)(\xi_{02}^2/\xi_{01}^2) + 1}\right. \tag{26}$$

$$\left. + \frac{1}{\xi_{01}}\,\frac{1}{1 + (\omega_{20}^2/\omega_{10}^2)(\xi_{01}^2/\xi_{02}^2)}\right]^{-1} \approx {}^5/_4\,\xi_{02}.$$

Let us examine Eq. (25) for the damping coefficient in detail. The first two terms in it are due to elastic collisions of electrons and ions with neutral gas molecules, the remainder are caused by the

Coulomb interaction, with the wave, of randomly moving electrons and ions. Here the collisions of the ions with molecules play a dominant role compared with collisions of electrons with molecules. As we know, collisions lead to damping of the wave. As for the last two terms in the expression for γ, the modulus of their ratio can be greater or less than unity, depending on the temperatures of electrons and ions; thus for $\Theta_1 < 4\Theta_1$, the ionic term exceeds the electric one, while for $\Theta_1 > 4\Theta_2$ the electronic term can far exceed the ionic term, so that the latter can be neglected. The electronic term appears in (25) with a minus sign, i.e., it makes possible an increase in the amplitude of the wave. Consequently, in this last case, because of the Coulomb interaction of the electron and ion currents, there results a decrease in the damping, or even the ocurrence of waves with rising amplitude ($\gamma < 0$). This will occur if:

$$l_e > a_1 \frac{v_{Te}^2}{\xi_{01}^2} \exp\left\{\frac{\xi_{01}^2}{v_{Te}^2}\right\}, \qquad (27)$$

where $l_e = \xi_{01}\tau_1$ is the mean free path of the electrons.

Condition (27) is fulfilled for $l_e \gg a_1$. As l_e decreases, the terms in γ due to collisions increase until, for l_e less than $a_1 (v_{Te}^2/\xi_{01}^2) \exp\{\xi_{01}^2/v_{Te}^2\}$, the damping coefficient is of the same order as the terms due to collisions, and has a positive sign. It is still meaningful to speak of a space-periodic distribution of the charged particles (fixed or moving) in this case, if $l_e \gg \xi_{01}/\omega_{10} > a_1$ since then $\gamma \ll \kappa$.

Thus spatial periodicity in the distribution of particle density, potentials, etc., can exist only for $l_e > a_1$, which coincides with the criterion given by many authors[13,14].

According to the paper of Zaitsev et al[8,9], for certain special cases there exists a region near the anode in which the drift velocity of the electrons is comparable to the thermal velocity or even exceeds it. This effect, which is caused by the large potential gradient at the anode, enables us to compare the results of the present section with experiment. According to Zaitsev's data, for anode striations (i.e. striations moving from the anode to the cathode), $N \approx 10^8 - 10^9$ electrons/cm^3, (in the positive column $N \approx 10^{10}$ electrons/cm^3), $|\xi_{01}| = 10^8$ cm/sec, $T = 3\cdot 10^4$ °K ($|\xi_{01}| > v_{Te}$), $\lambda_s = 2.5$ mm. The calculated value for the period, from formula (24) for $\omega < 10^5 - 10^6$ c.p.s. in zeroth approximation is : $\lambda = 5$ mm for $N = 10^8$ el./cm^3, and $\lambda = 1.6$ mm for $N = 10^9$ el./cm^3. The agreement of theory with experiment is good, considering that the concentration in the experiment is not known exactly. In this frequency region, the period hardly depends on the frequency and is the same as the period of fixed spatial inhomogeneities, as the experiments show.

It is interesting to note that the expression (24) gives the correct qualitative dependence of the space period on pressure. In fact, we have from (24), approximately, $\lambda \approx (2\pi/\sqrt{5})(\xi_{01}/\omega_{10})$. Since ξ_{01} increases with decreasing pressure, while the concentration decreases, λ increases with decreasing pressure.

The decrease of λ with pressure also occurs in the case of $\omega = 0$, discussed in the second section. For $v_{Te} \approx \xi_{01}$ we can obtain from (13) the following expression for the space period:

$$\lambda = A\sqrt{\Theta_1/Ne^2},$$

where A is a numerical factor. So, with increasing N, the space period decreases.

More complicated is the case of $v_{Te} \gg \xi_{01}$. Here one cannot draw any definite conclusions concerning the pressure dependence of λ. The result will depend on the relative rates of change of ξ_{01} and N with pressure.

In addition to considering these special cases (large drift velocity, high and low frequencies of vibration), we also found general conditions under which the motion of the ions can be disregarded completely. These conditions, obtained from analysis of the "dispersion" equation (57) of paper I, have the following form:

$$\omega \gg \omega_{20}, \quad \omega \gg 1/\tau_2; \qquad (28)$$

$$\omega/\varkappa \gg \xi_{02}, \quad \omega/\varkappa \gg v_{Ti}.$$

As expected, the motion of the ions can be neglected for sufficiently high oscillation frequencies.

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